Towards One-Step Multirate Methods in Chip Design

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Towards One-Step Multirate Methods in Chip Design

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1 Introduction

Large integrated electrical networks are usually build up by numerous coupled subcircuits of different functionality. These subcircuits are modelled independently and composed to one macro system by connecting them at the respective terminals, i.e. each pair of connected terminal nodes merge to one node (see Fig. 1, left).

From a modelling point of view, this procedure can be described by introducing virtual voltage sources at the boundary nodes (see Fig. 1, right). This approach preserves the macro circuits block structure and produces additional variables: branch currents through the coupling voltage sources. These currents are determined by the property, that the node potentials of each pair of connected boundary nodes have to coincide.

Regarding r subcircuits, r differential-algebraic systems, coupled by algebraic equations arise:

\[
\begin{align*}
\mathcal{F}_\lambda(x_\lambda, \frac{d}{dt} w_\lambda(x_\lambda), u, t) &= 0, \hspace{1cm} (\lambda = 1, \ldots, r) \\
\mathcal{G}(x_1, \ldots, x_r) &= 0,
\end{align*}
\]

where \(x_\lambda\) describes the node potentials and currents and \(w_\lambda\) the charges and fluxes of the \(\lambda\)-th subcircuit and \(u\) the coupling currents.

As the subcircuits constitute different functional units, the macro system often shows multirate behaviour, i.e. the subcircuits behave on different timescales. Thus multirate methods can be applied, that integrate subsystems

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showing different transient behaviour with different stepsizes adjusted to each subcircuits activity level.

Fig. 1. Coupling: technical and modelling point of view

2 Partitioned Network

Coupled problems that can be described by the abstract model (1a, 1b) also occur in other applications (e.g. multi-body physics). To set up numerical methods that are adapted to simulating electrical networks, a closer look at their special properties is required.

2.1 Network Equations

For circuits that are designed in the described manner, charge oriented modified nodal analysis (MNA) yields network equations of the following form:

\[
0 = A_{C\lambda} \dot{q}_\lambda + A_{R\lambda} r_\lambda (A_{R\lambda} e_\lambda, t) + A_{L\lambda} j_{L\lambda} + A_{V\lambda} j_{V\lambda} + A_{I\lambda} i_\lambda (t) + A_{u\lambda} u,
\]

(2a)

\[
0 = \dot{\phi}_\lambda - A_{L\lambda}^T e_\lambda,
\]

(2b)

\[
0 = A_{V\lambda} e_\lambda - v_\lambda (t),
\]

(2c)

\[
0 = q_\lambda - q_{C\lambda} (A_{C\lambda} e_\lambda, t),
\]

(2d)

\[
0 = \phi_\lambda - \varphi_{L\lambda} (j_{L\lambda}, t)
\]

(2e)

for the \(\lambda\)-th subcircuit \((\lambda = 1, \ldots, r)\) and the overall coupling equation

\[
0 = \sum_{\lambda=1}^{r} A_{u\lambda}^T e_\lambda.
\]

(3)

Here (2a) constitutes the current balance for each node with the element related currents through capacitances, resistances, inductances, voltage and current sources assembled by the incidence matrices \(A_{C\lambda}, A_{R\lambda}, A_{L\lambda}, A_{V\lambda}, A_{I\lambda}\).

The additional (boxed) term \(A_{u\lambda} u\) reflects the coupling currents to adjoined subcircuits, i.e. through the virtual voltage sources. The appropriate incidence matrix \(A_{u\lambda}\) filters out the adequate boundary nodes.
The flux–node potential correlation (2b), the node – source voltage dependency (2c) and the charge and flux defining equations (2d,2e) are not affected by coupling to other subcircuits, as the information exchange is done solely via coupling currents.

The linear coupling equation (3) states, that the potentials at the boundary nodes of connected subcircuits have to coincide.

2.2 Index properties

The overall system (1a,1b) is made up of $r$ subsystems – each with inner variables $x_\lambda (\lambda = 1, \ldots , r)$ – that are coupled by one equation and one variable $u$ respectively. Hence, several index-1 conditions are assumed to be fulfilled, according to the subsystems and the overall system:

(C1) The overall system (1a,1b) has index 1 (with respect to $x_1 , \ldots , x_r, u$).
(C2) All systems (1a) define index-1 systems with respect to $x_\lambda$ (and $u$ given as input).
(C3) For all $\lambda \in \{1, \ldots , r\}$, the overall system (1a,1b) has index-1 with respect to $x_\lambda$ and $u$ (and $x_i, \forall i \neq \lambda$ given as input).

Remark 1. The condition (C3) concerns the index-1 property of the system $F=0, G=0$ and is not implied by (C1) and (C2) in general.

Topological Conditions

In analogy to the procedure described in [6], topological conditions to guarantee the index conditions (C1)-(C3) can be derived. Therefor (1a,1b) is transformed into the semi-explicit systems for $\lambda = 1, \ldots , r$

\[
\dot{y}_\lambda(t) = f_\lambda(z_\lambda,u,t),
\]

\[
0 = h_\lambda(y_\lambda,z_\lambda,u,t) \tag{4a}
\]

coupled by the algebraic equation

\[
0 = g(z_1, \ldots , z_r) \tag{4b}
\]

where $z_\lambda$ identifies the node potentials and inner currents and $y_\lambda$ defines the charges and fluxes (see also [2]).

3 Multirate Methods

The basic idea of multirate methods is to prevent parts to be integrated more often than necessary to guarantee given error tolerances. This is done by using different stepsizes that are suitable for the different levels of activity at each
time. In the case of problems that are already given in the form of coupled subsystems like (1a,1b) it is convenient to assume, that these subsystems have no intrinsic multirate potential.

Remark 2. Multirate methods have to interweave approximations working on different time grids. This causes an overhead that has to be outbalanced by the reduction of computational costs for the discretisation of the less active (latent) parts. Hence systems showing multirate behaviour are said to have multirate potential if the different timescales are widely separated, the latent parts are larger than the active ones and the coupling amongst subsystems representing different activity levels is weak.

3.1 Multirate schemes for ODE systems

The concept of onestep multirate methods can be described with a system of two coupled ODEs:

\begin{align}
\dot{y}_L &= f_L(y_L, y_A), \quad y_L(t_0) = y_{L,0}, \\
\dot{y}_A &= f_A(y_L, y_A), \quad y_A(t_0) = y_{A,0}.
\end{align}

The idea is to compute one macrostep of the latent part (subscript L) with the stepsize \( H_L \), i.e. get an approximation \( y_{L,1} \approx y_L(t_0 + H_L) \) and to perform \( q \) microsteps with stepsizes \( H_{A,\mu} (\mu = 1, \ldots, q) \) for the active part (see Fig. 2). In its most general way this procedure can be defined as follows:

\[ y_{L,1} = y_{L,0} + \sum_{i=1}^{s_L} b^L_i \cdot k^L_i, \]

\[ y_{A,\mu} = y_{A,\mu-1} + \sum_{i=1}^{s_A} b^A_i \cdot k^{A,\mu}_i \quad (\mu = 1, \ldots, q), \]

\[ k^L_i = \Phi_L(h; y_{L,0}, Y^A_i, k^L_1, \ldots, k^L_{s_L}) \quad (i = 1, \ldots, s_L), \]

\[ k^{A,\mu}_i = \Phi_A(h_{A,\mu}; y_{A,\mu-1}, Y^L_i, Y^A_i, k^{A,\mu}_1, \ldots, k^{A,\mu}_{s_A}) \quad (i = 1, \ldots, s_A), \]

where \( \Phi_\ast \) denotes an \( s_\ast \) stage IRK or ROW scheme with coefficients \( \alpha^\ast, \beta^\ast, \gamma^\ast, \nu^\ast \ (\ast \in \{L,A\}) \).

\[ \text{Fig. 2. Macro- and microsteps} \]
As the subsystems are coupled, the computation of the weights for each part depends on information on the other one at some supporting timepoints:

\[ Y_i^A \approx y_A(t_0 + \alpha_i^L H_L) \quad (i = 1, \ldots, s_L), \]

\[ Y_i^{L,\mu} \approx y_L(t_0 + \sum_{\nu=1}^{\mu-1} H_{A,\nu} + \alpha_i^A H_{A,\mu}) \quad (i = 1, \ldots, s_A; \mu = 1, \ldots, q). \]

There are different strategies to compute these values. Explicitly done extra-/interpolation \cite{4} destroys the onestep character of the method. Generalised multirate \cite{5} a RK-based method calculates \( Y_i^{L,\mu} \) and \( Y_i^A \) in RK-like manner using the stage increments \( k_i^L, k_i^{A,\mu} \). Mixed multirate \cite{1}, ROW-based, builds up on generalised multirate. It decomposes the computation of one macrostep with its inner microstep to a so-called “compound step” and “later microsteps”. For the former the incremental formulation of generalise multirate is used. In the latter dense-output is used for the coupling.

### 3.2 Mixed multirate scheme for coupled index-1 DAE systems

To get a mixed multirate scheme for coupled index-1 DAE systems of semi-explicit form (see (4a,4b) with \( r = 2 \),

\[
\begin{align*}
\dot{y}_L &= f_L(z_L, u) \\
0 &= h_L(y_L, z_L, u) \\
0 &= g(z_L, z_A),
\end{align*}
\]

it is natural to assume the coupling variable \( u \) (defined by \( 0 = g \)) to behave latent like \( y_L \) and \( z_L \). With the index assumption (C3) \( [\dot{y}_L = f_L, 0 = h_L, 0 = g] \) is an index-1 system with respect to \( z_L, u \) and \( [\dot{y}_A = f_A, 0 = h_A] \) is of index 1 with respect to \( z_A \) if (C2) holds. The mixed multirate ansatz for ODEs can be brought forward to the coupled semi-explicit problem (7). The compound steps regulations read:

\[
\begin{pmatrix}
y_{L,1} \\
z_{L,1} \\
u_1
\end{pmatrix} =
\begin{pmatrix}
y_{L,0} \\
z_{L,0} \\
u_0
\end{pmatrix} + b_1^L \begin{pmatrix} l_L \\ k_L \\ p \end{pmatrix},
\begin{pmatrix}
y_{A,1} \\
z_{A,1} \\
u_0
\end{pmatrix} =
\begin{pmatrix}
y_{A,0} \\
z_{A,0} \\
u_0
\end{pmatrix} + b_1^A \begin{pmatrix} l_A \\ k_A \\ p \end{pmatrix}
\]

with weights \( b^\lambda := (b_1^\lambda, \ldots, b_s^\lambda)^t \) and increments \( l_1^\lambda, k_1^\lambda, p \) defined by

\[
\begin{pmatrix}
\text{blkdiag}
\left(\begin{pmatrix}
I_{y_L} & -H_A \gamma(\lambda) \frac{\partial f_L}{\partial z_L} \\
-\gamma(\lambda) \frac{\partial h_L}{\partial y_L} & -\gamma(\lambda) \frac{\partial h_L}{\partial z_L}
\end{pmatrix}
\right)_{\lambda=L,A}
\end{pmatrix}
\begin{pmatrix}
-H_L \gamma(L) \frac{\partial f_L}{\partial u}
\\
-\gamma(L) \frac{\partial h_L}{\partial u}
\\
-\gamma(L) \frac{\partial h_A}{\partial u}
\end{pmatrix}
\begin{pmatrix}
-H_A \nu(A,L) \frac{\partial f_A}{\partial u}
\\
-\nu(A,L) \frac{\partial h_A}{\partial u}
\end{pmatrix}
\]
\[
\begin{pmatrix}
l_{L,i} \\
k_{L,i} \\
l_{A,i} \\
k_{A,i} \\
p_i
\end{pmatrix} = \text{Rhs}(m, \mathcal{H}_L, \mathcal{H}_A, l_L, k_L, l_A, k_A, p) \quad (i = 1, \ldots, s) \quad (8b)
\]

with stepsizeratio \( m := \frac{H_L}{H_A} \)

Remark 3. In the later microsteps it remains, to solve the system \( \dot{y}_A = f_{a,0} = h_a \) with respect to \( y_A, z_A \) and \( u(t) \) entering the right-hand-side via dense output: \( u(t_0 + \xi \cdot H_L) \approx u_0 + \sum_{i=1}^s b_{L,i}(\xi) \cdot p_i \) with \( \xi \in (0, 1) \)

Mixed multirate for coupled network (2a-e,3)

The coupled network equations (2a-e,3) are transferred to the semi-explicit formulation (4a,4b). With fixed projectors \( Q_{C_\lambda}, Q_{V-C_\lambda} \), the variables of both formulations identify each other via linear operators \( \hat{\vartheta}_\lambda, \vartheta_\lambda \) where:

\[
\begin{cases}
\vartheta_\lambda = \hat{\vartheta}_\lambda ((e^t_{\lambda}, f^t_{L,\lambda}, f^t_{V,\lambda})^t) \\
\vartheta_\lambda = \vartheta(\vartheta_\lambda)
\end{cases}
\]

The semi-explicit problem (4a,4b) and its associated method (8a,8b) is suitable to derive order conditions to get adequate coefficients, such that \( \|\text{err}^{(se)}_\lambda\| = O(H_\lambda^{p+1}) \).

As the transformation between the two formulations is not invertible, it is not possible to carry forward the attained method to a method that draws directly on the coupled network. To obtain such “network-regulations” with demanded accuracy (\( \|\text{err}^{(mna)}_\lambda\| = O(H_\lambda^{p+1}) \)) in terms of node voltages and currents \( (e, f_L, f_V) \) however, there is another way:

- Based upon the idea of (8a,8b) regulations with an (undefined) coefficient set can be deduced from the network formulation (2a-e,3).
- The same transformation that carries over the network formulation to the semi-explicit one, applied to the above regulations yields instructions that coincide with (8a,8b).
- Regarding (9), it holds that \( \|\text{err}^{(mna)}_\lambda\| \leq \|\vartheta\| \cdot \|\text{err}^{(se)}_\lambda\| \) for the same coefficient set. Hence, if a coefficient set is chosen properly for the semi-explicit formulation, it is also suitable for the network formulation.

Finally a Block-Gaussian elimination and some linear transformation allow to eliminate the charges and fluxes \( q_\lambda, \phi_\lambda \). This guarantees charge- and flux-conservation and enables error-check and stepsize control based directly on the node potentials and currents \( (e_\lambda, f_{L,\lambda}, f_{V,\lambda}) \) (see also [3]).

\(^1\text{err}^{(se)}_\lambda\) denotes the local error after one step for the \( \lambda \)-th subsystem
Hierarchical mixed multirate

The already known multirate schemes deal with two different levels of activity. However coupled problems like (1a,1b) need \( n \)-level-multirate schemes with stepsizes \( H_1 > \ldots > H_n \). Transferring the 2-level-mixed multirate to n-level-schemes in a straight forward way produces a bunch of coupling coefficients. Hierarchical mixed multirate is a new approach in dealing with an arbitrary amount of activity levels and still limits the amount of coupling-factors. The main idea is to nest compound steps and later micro-steps in a way, that at each time merely a two-level multirate scheme is engaged: (see Fig. 3)

(i) Group remaining subsystems in terms of activity level. This yields \( k_i \) virtual blocks consisting of subsystems showing similar behaviour. If \( k_i = 1 \) employ later micro steps – the coupling to other subsystems/blocks is given by dense output – until endpoint is reached, skip to (iv).

(ii) Build up a sorted stack (top down, decreasing stepsizes). Apply a compound step with the stacks top as latent block and its associated stepsize and all the other blocks combined to one as active block with the stepsize associated to the stacks bottom. The coupling to already integrated subsystems is given by dense output.

(iii) Remove the stacks top. The new endpoint is the one reached by the macrostep. Skip to (i).

(iv) Enlarge the set of remaining subsystems by the ones that produced the last endpoint. If the endpoint is the endpoint of integration as demanded it is finished. Else forget the endpoint and skip to (i).

\[
\begin{align*}
\{1\} + \{2,3\} & \quad \{(2,3)\} \quad \{(3)\} \\
\{\{1\} + \{2,3\}\} & \quad \{(2,3)\} \rightarrow [1] & \{(3)\} \rightarrow [1,2] \\
\{\{2,3\}\} & \quad \{(2,3)\} \rightarrow [1] & \{(3)\} \rightarrow [1,2]
\end{align*}
\]

Fig. 3. Hierarchical mixed multirate for three blocks

\( n \leq r \) as some subsystems may show the same activity level.
Numerical Tests

A first hierarchical multirate-method of order 2 has recently been implemented in MATLAB. It can deal with an arbitrary amount of subcircuits with grouping them in terms of activity levels.

First testruns were done with a three-block circuit (with 3/5/3 nodes) “behaving like” \( \sin(\omega t) \) with \( \omega = 1, 10, 100 \) respectively. This yields promising results (see Table 1) as the mid-latent and latent block are calculated ten and hundred times less than in a corresponding singlerate.

| Table 1. absolute distance multirate (0.1, 0.01, 0.001) to singlerate (0.001) |
|----------------------------------|----------------------------------|----------------------------------|
| \(4 \cdot 10^{-13} \) | \(1 \cdot 10^{-2} \) | \(4 \cdot 10^{-11} \) |
| \(1 \cdot 10^{-3} \) | \(1 \cdot 10^{-4} \) | \(1 \cdot 10^{-6} \) |
| \(2 \cdot 10^{-3} \) | \(1 \cdot 10^{-4} \) | \(1 \cdot 10^{-6} \) |
| \(3 \cdot 10^{-3} \) |
| \(5 \cdot 10^{-3} \) |

5 Conclusions

A multirate scheme for circuit simulation that can deal with an arbitrary amount of subsystems has been derived. Now numerical tests have to be done with industry related examples. Furthermore the order of the method has to be enlarged to order three introducing adapted B-series and a stepsizecontrol adapted to coupled problems has to be derived.

References