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Maghnia Hamou Maamar, Matthias Ehrhardt, and Louiza Tabharit

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A Nonstandard Finite Difference Scheme for a Time-Fractional Model of Zika Virus Transmission

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Abstract. In this paper we consider a compartmental model describing the transmission of Zika virus to humans and mosquito populations and an extended model including a second reservoir host of a non-human primate (monkey). This model is later generalized by a fractional time derivative.

To properly simulate the spread of the disease we design for each model a nonstandard finite difference (NSFD) scheme that is able to guarantee the positivity of the solution and exhibits the correct asymptotic behaviour of the solution.

Numerical simulations of the models illustrate these advantages, e.g. the positivity preservation, compared to using standard solver like the Runge-Kutta Fehlberg method `ode45`.

1. Introduction

The Zika virus (ZIKV) is an emerging arbovirus that is transmitted by several so-called vectors, the most important being the Aedes aegypti mosquito. Vectors are living organisms that can transmit infectious pathogens between humans, or from animals to humans. ZIKV was first isolated from a macaca monkey in the Zika forest in Uganda in 1947, giving the virus its name, cf. [**12, 13**].

The first major ZIKV epidemic began 2007 on the Yap archipelago in the Federated States of Micronesia, where a high number of cases were recorded in about 75% of the population within a few months [**15, 22**]. Later, a worldwide epidemic occurred in French Polynesia (2013-2014) with approximately 28,000 cases (about 11%) of the total population [**28**]. In 2015, ZIKV was reported in Brazil via viremic travelers or infected mosquitoes [**45**], it also began to spread in Mexico [**18**]. Messina [**33**] showed that up to 2.17 billion people live in "risk areas" (tropical and subtropical regions).

The ZIKV infection is associated with mild symptoms: Fever, headache, rash, myalgia, and conjunctivitis, similar to other arboviruses (dengue or chikungunya) [**23**] and no deaths have been reported to date. Nevertheless, ZIKV has emerged as a major cause of the development of the Guillain-Barré syndrome [**5**]. Also, there is still uncertainty about the outcome of co-infections with other arboviruses such as

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Dengue fever. Furthermore, there is no available treatment for ZIKV infection. Patient care is based on symptomatic treatment with a combination of acetaminophen and antihistamine medications [**22**].

Several mathematical models have been developed to address different categories in epidemiology, such as prediction of disease outbreaks and evaluation of control strategies [**8, 21, 29, 43**]. The first mathematical epidemic model dates back to Kermack and McKendrick (1927), who were concerned with mass events in the susceptible, infected, and recovered (SIR) disease transmission cycle [**24**]. Manore and Hyman [**30**] proposed a mathematical model for ZIKV representing disease transition and population dynamics Gao [**17**] developed a model of ZIKV transmission through bites of Aedes mosquitoes and also through sexual contact. Lee and Pietz [**27**] developed a mathematical model for Zika virus using logistic growth in human populations. Nishiura et al. [**37**] proposed a mathematical Zika model that exhibits the same dynamics as Dengue fever.

Fractional order approaches were used in COVID-19 transmission models by using fractional order Caputo derivative [**40**], the analysis of semi-analytical solutions of a hepatitis B epidemic model using the Caputo-Fabrizio operator [**2**], the study of stability and Lyapunov functions for HIV/AIDS epidemic models with the Atangana-Baleanu-Caputo derivative [**41**], the mathematical modeling of the measles epidemic with optimized fractional order under the classical Caputo differential operator [**39**].

In this work we derive a new *nonstandard finite difference scheme* (NSFD) for a recent SEIR (susceptible-exposed-infectious-recovered) model [**29**] that describes the spread of the Zika virus using a human-mosquito compartmental model and a human-mosquito-monkey compartmental model. Despite the fact that this NSFD scheme has a nonlinear denominator function, this schemes has a couple of favourable properties: it is explicit and due to its construction it reproduces important properties of the solution, like the number and location of fixed-points, the positivity and certain conservation laws. The goal of this work is to briefly demonstrate, in detail, how the NSFD methodology is to be applied to a system of coupled ordinary differential equations (ODEs), where the discretizations are dynamical consistent with the basic properties of the continuous differential equations, e.g. positivity, asymptotic behaviour, memory effects, etc..

The paper is organized as follows. In Section 2, we formulate the ZIKV transmission models. Section 3 includes the analysis, especially the boundedness of the solution and the stability analysis of the two considered models. In Section 4 we design the nonstandard finite difference method for the two proposed models and show how it can be extended to time-fractional variants of the models using the L1 method. In Section 4 we propose NSFD schemes for the conventional and the time-fractional version of our models. The numerical results of our novel schemes are shown in Section 5. Finally, Section 6 presents the conclusions and some outlook.

2. The ZIKV transmission models

In this section, we will briefly describe the two considered mathematical compartmental models [**29**] to describe the ZIKV transmission.

In areas without nonhuman primates, such as Yap State and French Polynesia, ZIKV is likely maintained in a human-mosquito-human cycle, suggesting that the

virus has adapted to humans as reservoir hosts [**26**]. This setting will lead us the first model, formulated in a SEIR-SEI framework.

Boorman and Porterfield [**7**] showed in a laboratory setting that Monkeys can become infected with ZIKV. However, there is no evidence that ZIKV is transmitted to humans through contact with animals. On the other hand, the presence of specific antiviral antibodies in various nonhuman primates, suggesting that other reservoirs may play a role in the ZIKV transmission cycle, cf. [**11**]. For this reason we also consider a second extended model.

2.1. The Parameters. The human population is divided into four classes (so-called 'compartments'): susceptible, exposed (latently infected), infected, and recovered (individuals who have acquired immunity). We denote the number in each compartment by S_h, E_h, I_h, and R_h. Accordingly, we divide the vector population (adult female mosquitoes) into three compartments: susceptible, exposed, and infected, with the analogous notation S_v, E_v and I_v. Next, we define the total number of populations as

$$N_h = S_h + E_h + I_h + R_h, \qquad N_v = S_v + E_v + I_v. \tag{2.1}$$

Further, let us introduce a couple of parameters, cf. [**29**].

- B is the average number of bites per mosquito per day.
- β_{vh} is the probability rate that a bite from an infectious vector will infect a human, the product $B\beta_{vh}$ is the number of disease-transmitting bites per infectious mosquito per day, and the product $B\beta_{vh}I_v(t)$ is the number of disease-transmitting bites per day in the entire mosquito population at time t (measured in days). However, multiplying $B\beta_{vh}I_v(t)$ by the proportion of susceptible people at time t represents the number of disease-transmitting bites per day by infectious mosquitoes on susceptible people at time t (the daily rate at which susceptible people are exposed).
- The parameter μ_h is the proportion of the human population that dies each day ('human mortality rate').
- ν_h is the daily rate at which exposed people become infected ('human infection rate').
- η_h denotes the daily rate at which infected people become immune. ('human immunity rate').
- The parameter β_{hv} is the probability rate that the bite of an infectious human will infect a mosquito; $B\beta_{hv}$ is the number of disease-transmitting bites per mosquito per day. Thus, the product $B\beta_{hv}S_v(t)$ is the number of bites per day that result in disease being transmitted by susceptible mosquitoes at time t. Multiplying $B\beta_{hv}S_v(t)$ by the proportion of infectious people at time t the complete rate of disease-transmitting bites at time t (the daily rate at which susceptible mosquitoes become infected).
- The parameter μ_v is the proportion of the mosquito population that dies each day ('mosquito mortality rate').
- ν_v denotes the daily rate at which exposed mosquitoes become infected ('mosquito infection rate').

We include a constant system inflow, the per-capita birth rates Λ_h, Λ_v (e.g. birth of new individuals that can get infected, and the natural mortality rates μ_h, μ_v.

2.2. The human-mosquito model. Now we are ready to formulate the first model. The system of ODEs has the following form

$$
\begin{aligned}
\frac{\mathrm{d}S_h(t)}{\mathrm{d}t} &= \Lambda_h - \left(B\beta_{vh}\frac{I_v(t)}{N_v(t)} + \mu_h\right) S_h(t),\\
\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} &= B\beta_{vh}\frac{I_v(t)}{N_v(t)}\, S_h(t) - (\nu_h + \mu_h)E_h(t),\\
\frac{\mathrm{d}I_h(t)}{\mathrm{d}t} &= \nu_h E_h(t) - (\eta_h + \mu_h)I_h(t),\\
\frac{\mathrm{d}R_h(t)}{\mathrm{d}t} &= \eta_h I_h(t) - \mu_h R_h(t),\\
\frac{\mathrm{d}S_v(t)}{\mathrm{d}t} &= \Lambda_v - \left(B\beta_{hv}\frac{I_h(t)}{N_h(t)} + \mu_v\right) S_v(t),\\
\frac{\mathrm{d}E_v(t)}{\mathrm{d}t} &= B\beta_{hv}\frac{I_h(t)}{N_h(t)}\, S_v(t) - (\nu_v + \mu_v)\, E_v(t),\\
\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} &= \nu_v E_v(t) - \mu_v I_v(t).
\end{aligned}
\tag{2.2}
$$

The dynamical system described by (2.2) is depicted in Figure 1. We note that by a convention in epidemiology models all parameters in (2.2) are assumed to be positive.

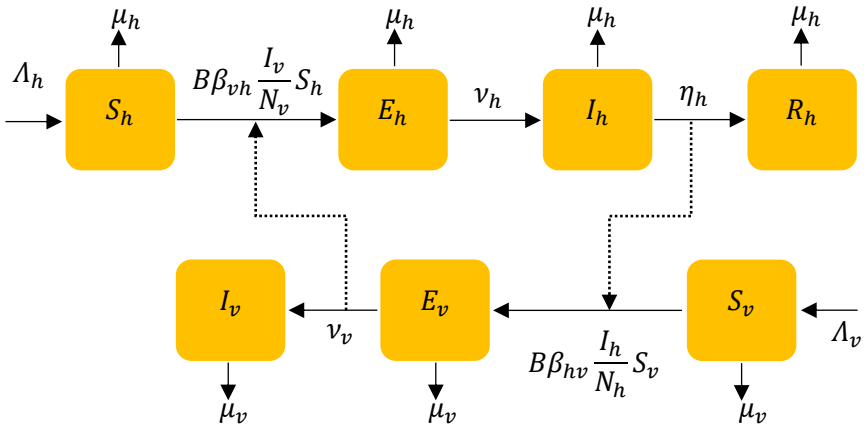

FIGURE 1. A schematic representation of the human-mosquito model (2.2).

Summing up the equations in (2.2) gives immediately the ODE system for the time evolution of the total populations of humans and mosquitos

$$
\begin{aligned}
\frac{\mathrm{d}N_h(t)}{\mathrm{d}t} &= \Lambda_h - \mu_h\, N_h(t),\\
\frac{\mathrm{d}N_v(t)}{\mathrm{d}t} &= \Lambda_v - \mu_v\, N_v(t),
\end{aligned}
\tag{2.3}
$$

that can be solved easily, cf. Section 4.4. Since the Zika virus transmission has a faster dynamic than the human birthrate and the human natural mortality, $N_h(t)$

can be regarded as a conserved quantity of the above ODE system, if we set for simplicity $\Lambda_h = \mu_h = 0$. Note that this is not the case for the vector (mosquito) which has a comparable dynamic and the asymptotic behaviour

$$\lim_{t\to\infty} \mathbf{N}_v(t) = \frac{\Lambda_v}{\mu_v}. \tag{2.4}$$

This well-known limiting behaviour can be exploited for a further simplification of the model (2.2) (so-called 'limiting model') by removing $\mathbf{I}_v$, and thus the remaining vector components $\mathbf{S}_v$, $\mathbf{E}_v$ can be plotted in a 2D phase diagram, cf. [**9**]. We return later to this property when designing the numerical scheme.

2.3. The human-mosquito-monkey model. Accordingly, we define the total monkey population as

$$N_m(t) = S_m(t) + E_m(t) + I_m(t) + R_m(t). \tag{2.5}$$

Next, we introduce similar parameters for the monkey population, cf. [**29**]:

- β_{vm} is the probability rate that a bite from an infectious mosquito will infect a monkey.
- The parameter μ_m is the proportion of the monkey population that dies each day.
- ν_m is the daily rate at which exposed monkeys become infected.
- η_m the daily rate at which infected monkeys become immune.

The corresponding system of ODEs for the temporal evolution of the human, vector and monkey population has the following form

$$\begin{aligned}
\frac{\mathrm{d}S_h(t)}{\mathrm{d}t} &= \Lambda_h - \left(B\beta_{vh}\frac{I_v(t)}{N_v(t)} + \mu_h\right) S_h(t),\\
\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} &= B\beta_{vh}\frac{I_v(t)}{N_v(t)}\, S_h(t) - (\nu_h + \mu_h)E_h(t),\\
\frac{\mathrm{d}I_h(t)}{\mathrm{d}t} &= \nu_h E_h(t) - (\eta_h + \mu_h)I_h(t),\\
\frac{\mathrm{d}R_h(t)}{\mathrm{d}t} &= \eta_h I_h(t) - \mu_h R_h(t),\\
\frac{\mathrm{d}S_v(t)}{\mathrm{d}t} &= \Lambda_v - \left(B\beta_{hv}\,\frac{I_h(t)}{N_h(t)} + B\beta_{mv}\,\frac{I_m(t)}{N_m(t)} + \mu_v\right) S_v(t),\\
\frac{\mathrm{d}E_v(t)}{\mathrm{d}t} &= \left(B\beta_{hv}\,\frac{I_h(t)}{N_h(t)} + B\beta_{mv}\,\frac{I_m(t)}{N_m(t)}\right) S_v(t) - (\nu_v + \mu_v)\, E_v(t),\\
\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} &= \nu_v E_v(t) - \mu_v I_v(t)\\
\frac{\mathrm{d}S_m(t)}{\mathrm{d}t} &= \Lambda_m - \left(B\beta_{vm}\frac{I_v(t)}{N_v(t)} + \mu_m\right) S_m(t),\\
\frac{\mathrm{d}E_m(t)}{\mathrm{d}t} &= B\beta_{vm}\frac{I_v(t)}{N_v(t)}\, S_m(t) - (\nu_m + \mu_m)E_m(t),\\
\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} &= \nu_m E_m(t) - (\eta_m + \mu_m)I_m(t),\\
\frac{\mathrm{d}R_m(t)}{\mathrm{d}t} &= \eta_m I_(t) - \mu_m R_m(t).
\end{aligned} \tag{2.6}$$

The dynamical system described by equations (2.6) is depicted in Figure 2.

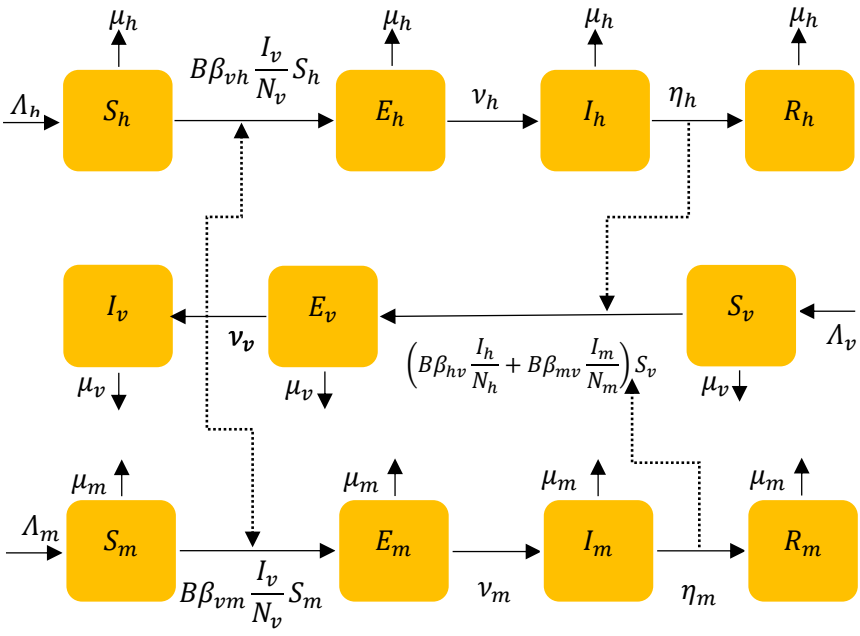

FIGURE 2. A schematic representation of the human-mosquito-monkey model (2.6).

Again, summing up the equations in (2.6) yields for the total populations

$$
\begin{aligned}
\frac{\mathrm{d}N_h(t)}{\mathrm{d}t} &= \Lambda_h - \mu_h\, N_h(t),\\
\frac{\mathrm{d}N_v(t)}{\mathrm{d}t} &= \Lambda_v - \mu_v\, N_v(t),\\
\frac{\mathrm{d}N_m(t)}{\mathrm{d}t} &= \Lambda_m - \mu_m\, N_m(t),
\end{aligned}
\tag{2.7}
$$

with simple exact solutions, see Section 4.4. Analogously, $N_h(t)$ and $N_m(t)$ can be regarded as a conserved quantity of the above ODE system, if we set $\Lambda_h = \mu_h = 0$ and $\Lambda_m = \mu_m = 0$.

Using standard arguments (see e.g. [**36**]) it can be easily shown that both ODE systems (2.2), (2.6) preserve the positivity of the solution. This basic property should be respected by any reasonable numerical method and yields as a byproduct the stability of the scheme.

2.4. A Fractional-order human-vector model. The fractional-order dynamics of the transmission of the Zika virus to human and vector populations is

given by the following system

$$
\begin{cases}
{}^{C}D^{\alpha}S_h(t) &= \Lambda_h^{\alpha} - \Big(B^{\alpha}\beta_{vh}\frac{I_v(t)}{N_{\alpha,v}(t)} + \mu_h^{\alpha}\Big)S_h(t)\\
{}^{C}D^{\alpha}E_h(t) &= B^{\alpha}\beta_{vh}\frac{I_v(t)}{N_{\alpha,v}(t)}S_h(t) - (\nu_h^{\alpha} + \mu_h^{\alpha})E_h(t)\\
{}^{C}D^{\alpha}I_h(t) &= \nu_h^{\alpha}E_h(t) - (\eta_h^{\alpha} + \mu_h^{\alpha})I_h(t)\\
{}^{C}D^{\alpha}R_h(t) &= \eta_h^{\alpha}I_h(t) - \mu_h^{\alpha}R_h(t)\\
{}^{C}D^{\alpha}S_v(t) &= \Lambda_v^{\alpha} - \Big(B^{\alpha}\beta_{hv}\frac{I_h(t)}{N_{\alpha,h}(t)} + \mu_v^{\alpha}\Big)S_v(t)\\
{}^{C}D^{\alpha}E_v(t) &= B^{\alpha}\beta_{hv}\frac{I_h(t)}{N_{\alpha,h}(t)}S_v(t) - (\nu_v^{\alpha} + \mu_v^{\alpha})E_v(t)\\
{}^{C}D^{\alpha}I_v(t) &= \nu_v^{\alpha}E_v(t) - \mu_v^{\alpha}I_v(t),
\end{cases}
\tag{2.8}
$$

with the initial conditions

$$S_h(0),\ E_h(0),\ I_h(0),\ R_h(0),\ S_v(0),\ E_v(0),\ I_v(0) \geq 0,$$

where ${}^{C}D^{\alpha}X(t)$ is the Caputo derivative and it is defined as:

$${}^{C}D^{\alpha}X(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau}(t-\tau)^{-\alpha}\,d\tau, \quad t > 0 \text{ and } 0 < \alpha < 1.$$

Adding the equations of the system (2.8) yields the fractional ODEs

$$
{}^{C}D^{\alpha}N_{\alpha,h}(t) = \Lambda_h^{\alpha} - \mu_h^{\alpha}N_{\alpha,h}(t) \text{ and } {}^{C}D^{\alpha}N_{\alpha,v}(t) = \Lambda_v^{\alpha} - \mu_v^{\alpha}N_{\alpha,v}(t).
\tag{2.9}
$$

In the model given above, we modified the right-hand sides parameters μ_h^{α}, B^{α}, ν_h^{α}, η_h^{α}, μ_v^{α} and ν_v^{α} using the procedure described in Diethelm [**14**] in order to adjust the dimensions because the dimension of the left-hand sides of the equations is $(\text{time})^{-\alpha}$. Note that in the limit case $\alpha \to 1$, the system (2.8) reduces to the classical system given in (2.2).

3. Analysis of the models

3.1. Non-negativity and boundedness of solutions. The positivity and boundedness of the solutions of an epidemiological system are essential properties. Therefore, it is important to prove that all subpopulations in the systems (2.2), (2.6), and (2.8) are non-negative and bounded for all times $t \geq 0$. The following results show how to confirm these two properties.

We now focus on the human-mosquito system (2.2) and prove the following theorem, which confirms the positivity and boundedness of the system.

THEOREM 3.1. *The closed region*

$$\Omega = \Big\{(S_h, E_h, I_h, R_h, S_v, E_v, I_v) \in \mathbb{R}_+^7 : 0 \leq N_h \leq \frac{\Lambda_h}{\mu_h} \text{ and } 0 \leq N_v \leq \frac{\Lambda_v}{\mu_v}\Big\}$$

is a positively invariant set for the system (2.2).

PROOF. Let $S_h(0) > 0$, then

$$
\begin{aligned}
\frac{\mathrm{d}S_h(t)}{\mathrm{d}t} &= \Lambda_h(t) - \Big(B\beta_{vh}\frac{I_v(t)}{N_v(t)} + \mu_h\Big)S_h(t)\\
&\geq -\Big(B\beta_{vh}\frac{I_v(t)}{N_v(t)} + \mu_h\Big)S_h(t).
\end{aligned}
$$

By using the Comparison Lemma [**25**], we have

$$S_h(t) \geq S_h(0) \int_0^t \exp\left(-\left(B\beta_{vh}\frac{I_v(s)}{N_v(s)} + \mu_h\right)\right) ds \geq 0.$$

Similarly, it can be shown that

$$E_h(t) \geq 0,\ I_h(t) \geq 0,\ R_h(t) \geq 0,\ S_v(t) \geq 0,\ E_h(t) \geq 0 \text{ and } I_h(t) \geq 0.$$

From equations (4.4) and (4.5) the quantities $N_h(t)$ and $N_v(t)$ are non-negative for all $t \geq 0$, and

$$\limsup_{t\to\infty} N_h(t) \leq \frac{\Lambda_h}{\mu_h} \quad \text{and} \quad \limsup_{t\to\infty} N_v(t) \leq \frac{\Lambda_v}{\mu_v}.$$

Thus, $S_h(t)$, $E_h(t)$, $I_h(t)$, $R_h(t)$, $S_v(t)$, $E_v(t)$ and $I_v(t)$ are bounded. □

The corresponding proof for the human-mosquito-monkey system (2.6) is analogous.

The following theorem highlights the positivity and boundedness of the fractional-order human-vector model (2.8):

THEOREM 3.2. *The region* $\Omega^\alpha = \left\{(S_h, E_h, I_h, R_h, S_v, E_v, I_v) \in \mathbb{R}_+^7 : 0 \leq N_{\alpha,h} \leq \frac{\Lambda_h^\alpha}{\mu_h^\alpha} \text{ and } 0 \leq N_{\alpha,v} \leq \frac{\Lambda_v^\alpha}{\mu_v^\alpha}\right\}$ *is a non-negative invariant for the model* (2.8) *for* $t \geq 0$.

PROOF. We have

$${}^C D^\alpha N_{\alpha,h}(t) + \mu_h^\alpha N_{\alpha,h}(t) = \Lambda_h^\alpha$$

and using the Laplace transform, we obtain

$$s^\alpha L\big(N_{\alpha,h}(t)\big) - s^{\alpha-1}N_{\alpha,h}(0) + \mu_h^\alpha L\big(N_{\alpha,h}(t)\big) = \frac{\Lambda_h^\alpha}{s}$$

then

$$L\big(N_{\alpha,h}(t)\big) = \frac{s^{\alpha-1}N_{\alpha,h}(0)}{s^\alpha + \mu_h^\alpha} + \frac{\Lambda_h^\alpha s^{-1}}{s^\alpha + \mu_h^\alpha},$$

and applying the inverse Laplace transform, we get

$$N_{\alpha,h}(t) = N_{\alpha,h}(0)E_\alpha\big(-(\mu_h t)^\alpha\big) + \Lambda_h^\alpha t^\alpha E_{\alpha,\alpha+1}\big(-(\mu_h t)^\alpha\big) \tag{3.1}$$

where $E_{\alpha,\alpha+1}$ denotes the Mittag-Leffler function

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \quad \alpha > 0, \beta > 0.$$

Using the well-known recurrence relation for the Mittag-Leffler function [**20**] for $\beta = 1$,

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\beta+\alpha}(z)$$

we may write the equation (3.1) as

$$N_{\alpha,h}(t) = \frac{\Lambda_h^\alpha}{\mu_h^\alpha} + \Big(N_{\alpha,h}(0) - \frac{\Lambda_h^\alpha}{\mu_h^\alpha}\Big)E_\alpha\big(-(\mu_h t)^\alpha\big), \tag{3.2}$$

and thus

$$\limsup_{t\to\infty} N_{\alpha,h}(t) \leq \frac{\Lambda_h^\alpha}{\mu_h^\alpha}.$$

We proceed similarly to derive the equation of $N_{\alpha,v}(t)$,

$$N_{\alpha,v}(t) = \frac{\Lambda_v^\alpha}{\mu_v^\alpha} + \Big(N_{\alpha,v}(0) - \frac{\Lambda_v^\alpha}{\mu_v^\alpha}\Big)E_\alpha\big(-(\mu_v t)^\alpha\big), \tag{3.3}$$

and conclude that

$$\limsup_{t\to\infty} N_{\alpha,v}(t) \leq \frac{\Lambda_v^\alpha}{\mu_v^\alpha}.$$

As a result, the functions S_h, E_h, I_h, R_h, S_v, E_v and I_v are all non-negative. □

3.2. Stability Analysis. System (2.8) always has a *disease-free equilibrium* (DFE) at:

$$E_{DF} = (N^*_{\alpha,h}, 0, 0, 0, N^*_{\alpha,v}, 0, 0),$$

where

$$N^*_{\alpha,h} = \frac{\Lambda_h^\alpha}{\mu_h^\alpha} \text{ and } N^*_{\alpha,v} = \frac{\Lambda_v^\alpha}{\mu_v^\alpha}.$$

The infection components considered in (2.8) model consist of E_h, I_h, E_v and I_v. By using the next generation approach [**14**], the *basic reproduction number* of the model (2.8) is defined as $R_0^\alpha = \rho(F_\alpha V_\alpha^{-1})$, where F_α represents the new infection matrix and V_α represents the transition matrix. The values for F_α and V_α are provided below:

$$F_\alpha = \begin{pmatrix} 0 & 0 & 0 & \frac{B^\alpha \beta_{vh} N^*_{\alpha,h}}{N^*_{\alpha,v}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{B^\alpha \beta_{hv} N^*_{\alpha,v}}{N^*_{\alpha,h}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_\alpha = \begin{pmatrix} (\nu_h^\alpha + \mu_h^\alpha) & 0 & 0 & 0 \\ -\nu_h^\alpha & (\eta_h^\alpha + \mu_h^\alpha) & 0 & 0 \\ 0 & 0 & (\mu_v^\alpha + \nu_v^\alpha) & 0 \\ 0 & 0 & -\nu_v^\alpha & \mu_v^\alpha \end{pmatrix}.$$

Thus,

$$R_0^\alpha = \sqrt{\frac{\nu_h^\alpha \nu_v^\alpha B^\alpha \beta_{vh} B^\alpha \beta_{hv}}{\mu_v^\alpha (\nu_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)}}.$$

3.3. Local and global stability of DFE. The following theorem discuss the local stability of DFE.

THEOREM 3.3. *The disease-free equilibrium of the proposed fractional-order model is locally asymptotically stable if $R_0^\alpha < 1$ and is unstable if $R_0^\alpha > 1$.*

PROOF. The Jacobian matrix of system (2.8) at DFE is given by,

$$J(E_{DF}) = \begin{pmatrix} -\mu_h^\alpha & 0 & 0 & 0 & 0 & 0 & -\frac{B^\alpha \beta_{vh} N^*_{\alpha,h}}{N^*_{\alpha,v}} \\ 0 & -(\nu_h^\alpha + \mu_h^\alpha) & 0 & 0 & 0 & 0 & \frac{B^\alpha \beta_{vh} N^*_{\alpha,h}}{N^*_{\alpha,v}} \\ 0 & \nu_h^\alpha & -(\eta_h^\alpha + \mu_h^\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_h^\alpha & -\mu_h^\alpha & 0 & 0 & 0 \\ 0 & 0 & -B^\alpha \beta_{hv} \frac{N^*_{\alpha,v}}{N^*_{\alpha,h}} & 0 & -\mu_v^\alpha & 0 & 0 \\ 0 & 0 & B^\alpha \beta_{hv} \frac{N^*_{\alpha,v}}{N^*_{\alpha,h}} & 0 & 0 & -(\mu_v^\alpha + \nu_v^\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu_v^\alpha & -\mu_v^\alpha \end{pmatrix}$$

the characteristic polynomial is then given by,

$$p(X) = (X + \mu_v^\alpha)(X + \mu_h^\alpha)^2(X^4 + a_3X^3 + a_2X^2 + a_1X + a_0)$$

then $X_1 = X_2 = -\mu_v^\alpha$ and $X_3 = -\mu_v^\alpha$ are three eigenvalues. The remaining eigenvalues correspond to the roots of the following polynomial

$$q(X) = a_0X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$$

where

$$a_0 = 1,$$
$$a_1 = \eta_h^\alpha + 2\mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha + \nu_h^\alpha,$$
$$a_2 = (\eta_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha) + (\mu_v^\alpha + \nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha) + \mu_v^\alpha(\nu_h^\alpha + \mu_h^\alpha),$$
$$a_3 = (2\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha) + \mu_v^\alpha(\nu_h^\alpha + 2\mu_h^\alpha + \eta_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha),$$

and

$$a_4 = \mu_v^\alpha(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)\big(1 - (R_0^\alpha)^2\big).$$

The polynomial $q(X)$ has the following Hurwitz matrix :

$$H = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix}$$

So, by the Routh-Hurwitz criterion the roots of $q(X)$ have negative real parts if and only if all principal minors are strictly positive, that is,

$$H_1 = a_1 > 0,$$

$$\begin{aligned} H_2 &= a_1a_2 - a_3 \\ &= (\eta_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)\ (\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha) + \mu_v^\alpha(\mu_h^\alpha + \nu_h^\alpha + \mu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha) \\ &\quad + (\eta_h^\alpha + 2\mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha + \nu_h^\alpha)(\mu_v^\alpha + \nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha) \\ &> 0, \end{aligned}$$

$$\begin{aligned} H_3 &= a_1a_2a_3 - a_1^2a_4 - a_3^2 \\ &= \mu_v^\alpha(\eta_h^\alpha + 2\mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha + \nu_h^\alpha)^2(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)(R_0^\alpha)^2 \\ &\quad + \mu_v^\alpha(\eta_h^\alpha + 2\mu_h^\alpha + 2\mu_v^\alpha + \nu_h^\alpha)(\mu_v^\alpha + \nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha) \\ &\quad + \mu_v^\alpha(\eta_h^\alpha + 2\mu_h^\alpha + \nu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha) \\ &\quad + \mu_v^\alpha(2\mu_v^\alpha + \nu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha)(\mu_v^\alpha + \nu_v^\alpha) \\ &\quad + (\eta_h^\alpha + 2\mu_h^\alpha + \nu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha) \\ &\quad + (2\mu_v^\alpha + \nu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\mu_v^\alpha + \nu_v^\alpha) \\ &\quad + \nu_v^\alpha\nu_v^\alpha\mu_v^\alpha(\eta_h^\alpha + \mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha + \eta_h^\alpha) \\ &\quad + \nu_v^\alpha\mu_v^\alpha\mu_h^\alpha(\eta_h^\alpha + \mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha)(\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha) \\ &\quad + \mu_v^\alpha(\mu_v^\alpha + \nu_v^\alpha)(\mu_h^\alpha + 2\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha) \\ &\quad + \mu_v^\alpha(\mu_v^\alpha + \nu_v^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha + \nu_h^\alpha) \\ &\quad + \mu_v^\alpha\mu_v^\alpha(\mu_v^\alpha + \nu_v^\alpha)(\nu_h^\alpha + 2\mu_h^\alpha + \eta_h^\alpha)(2\mu_v^\alpha + \nu_v^\alpha)(\nu_h^\alpha + \mu_h^\alpha) \\ &\quad + 2\mu_v^\alpha\mu_v^\alpha\mu_h^\alpha(\nu_h^\alpha + \mu_h^\alpha)(\eta_h^\alpha + \mu_h^\alpha)(\mu_h^\alpha + \mu_v^\alpha + \nu_h^\alpha) \end{aligned}$$

$$
\begin{aligned}
&+\mu_v^\alpha\nu_v^\alpha(\nu_h^\alpha+\mu_h^\alpha)\mu_h^\alpha(\mu_v^\alpha+\nu_v^\alpha)\mu_h^\alpha+\mu_v^\alpha\nu_v^\alpha(\eta_h^\alpha+\nu_h^\alpha+\mu_h^\alpha)\mu_h^\alpha\nu_v^\alpha\nu_h^\alpha\\
&+\mu_v^\alpha\mu_v^\alpha\nu_v^\alpha(\nu_h^\alpha+\mu_h^\alpha+\eta_h^\alpha)(\eta_h^\alpha+2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha)\\
&+\mu_v^\alpha\nu_v^\alpha(\nu_h^\alpha+\mu_h^\alpha)(\eta_h^\alpha+\mu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)(\eta_h^\alpha+\mu_h^\alpha)\\
&+(\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha+2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha)(2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)\\
&+\mu_v^\alpha\nu_v^\alpha\nu_h^\alpha(2\mu_h^\alpha+\nu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)(\mu_v^\alpha+\nu_v^\alpha)+\mu_v^\alpha\mu_v^\alpha(\eta_h^\alpha+\mu_h^\alpha)\mu_h^\alpha\nu_h^\alpha(\mu_v^\alpha+\nu_v^\alpha)\\
&+\mu_v^\alpha\mu_v^\alpha(\eta_h^\alpha+\mu_h^\alpha+2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha)(2\mu_h^\alpha+\eta_h^\alpha)(\mu_v^\alpha+\nu_v^\alpha)\\
&+\mu_v^\alpha\mu_v^\alpha(2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha)\nu_h^\alpha(\mu_v^\alpha+\nu_v^\alpha)\\
&+\mu_v^\alpha\mu_v^\alpha(\eta_h^\alpha+\mu_h^\alpha+2\mu_v^\alpha+\nu_v^\alpha)(\eta_h^\alpha+\mu_h^\alpha+\mu_v^\alpha+\nu_v^\alpha)(\nu_h^\alpha+2\mu_h^\alpha+\eta_h^\alpha)(\mu_v^\alpha+\nu_v^\alpha)\\
&+\mu_v^\alpha\nu_v^\alpha(\mu_h^\alpha+2\mu_v^\alpha+\nu_v^\alpha+\nu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)(\eta_h^\alpha+\mu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)\\
&+\mu_v^\alpha\mu_v^\alpha(\nu_h^\alpha+2\mu_h^\alpha+\eta_h^\alpha)(\mu_h^\alpha+\nu_h^\alpha)(\mu_v^\alpha+\nu_v^\alpha)(\mu_v^\alpha+\nu_v^\alpha)\\
&+2\mu_v^\alpha\mu_v^\alpha\nu_h^\alpha(\mu_h^\alpha+2\mu_v^\alpha+\nu_h^\alpha)(\nu_h^\alpha+\mu_h^\alpha)(\eta_h^\alpha+\mu_h^\alpha)\\
&+\mu_v^\alpha\nu_v^\alpha\nu_v^\alpha\nu_h^\alpha\eta_h^\alpha(\eta_h^\alpha+\nu_h^\alpha+\mu_h^\alpha)>0,
\end{aligned}
$$

and

$$H_4=a_4H_3>0.$$

□

In order to prove the global stability of the equilibrium points, we need to recall the following result:

LEMMA 3.4 (See [**10**]). *Let $X(t)\in\mathbb{R}$ be a continuous and differentiable function. Then, for any time instant $t\geq 0$*

$$
{}^{C}D^{\alpha}\Big[X^*\,g\Big(\frac{X(t)}{X^*}\Big)\Big]\leq\Big(1-\frac{X^*}{X(t)}\Big)\,{}^{C}D^{\alpha}X(t),\quad X^*\in\mathbb{R},\,\forall\alpha\in(0,1),\tag{3.4}
$$

where $g(x)=x-1-\ln x$.

Note that for $\alpha=1$, the inequality in (3.4) becomes equality. Now, taking into account the Lyapunov direct method, we provide the global stability of the equilibria in the following theorem.

THEOREM 3.5. *If $(R_0^\alpha)^2<\frac{N_v(0)N_h(0)}{N^*_{\alpha,h}N^*_{\alpha,v}}<1$, then the DFE is globally asymptotically stable.*

PROOF. We consider the following Lyapunov function

$$
\begin{aligned}
V(t)=W_1S_h^*\,g\Big(\frac{S_h(t)}{S_h^*}\Big)+W_2E_h(t)+W_3I_h(t)\\
+W_4(t)S_v^*\,g\Big(\frac{S_v(t)}{S_v^*}\Big)+W_5(t)E_v(t)+W_6(t)I_v(t),
\end{aligned}
$$

where

$$W_1=W_2=\frac{\nu_h^\alpha}{\phi_1},W_3=1,W_4(t)=W_5(t)=\frac{\nu_h^\alpha\nu_v^\alpha B_1^\alpha(t)S_h^*}{\phi_1\phi_3\mu_v^\alpha},\text{and }W_6(t)=\frac{\nu_h^\alpha B_1^\alpha(t)S_h^*}{\phi_1\mu_v^\alpha},$$

with

$$\phi_1=\nu_h^\alpha+\mu_h^\alpha,\phi_2=\eta_h^\alpha+\mu_h^\alpha,\phi_3=\mu_v^\alpha+\nu_v^\alpha,B_1^\alpha(t)=\frac{B^\alpha\beta_{vh}}{N_v(t)}\text{ and }B_2^\alpha(t)=\frac{B^\alpha\beta_{hv}}{N_h(t)}.$$

Now using Lemma 3.4, the derivative of V in the Caputo sense with respect to t is given by:

$$\begin{aligned}
{}^{C}D^{\alpha}V(t) \leq\ & W_1\frac{(S_h(t)-S_h^*)}{S_h(t)}\ {}^{C}D^{\alpha}S_h(t) + W_2\ {}^{C}D^{\alpha}E_h(t) + W_3\ {}^{C}D^{\alpha}I_h(t)\\
&+ W_4(t)\frac{(S_v(t)-S_v^*)}{S_v(t)}\ {}^{C}D^{\alpha}S_v(t) - W_4(t)\frac{S_v^*}{N_v(t)}\,g\Big(\frac{S_v(t)}{S_v^*}\Big)\ {}^{C}D^{\alpha}N_v(t)\\
&+ W_5(t)\ {}^{C}D^{\alpha}E_v(t) - W_5(t)\frac{E_v(t)}{N_v(t)}\ {}^{C}D^{\alpha}N_v(t)\\
&+ W_6(t)\ {}^{C}D^{\alpha}I_v(t) - W_6(t)\frac{I_v(t)}{N_v(t)}\ {}^{C}D^{\alpha}N_v(t)
\end{aligned}$$

and thus

$$\begin{aligned}
{}^{C}D^{\alpha}V(t) \leq\ & W_1\frac{(S_h(t)-S_h^*)}{S_h(t)}\big(\Lambda_h^{\alpha} - (B_1^{\alpha}(t)I_v(t)+\mu_h^{\alpha})S_h(t)\big)\\
&+ W_2\big(B_1^{\alpha}(t)I_v(t)S_h(t) - \phi_1 E_h(t)\big) + W_3\big(\nu_h^{\alpha}E_h(t) - \phi_2 I_h(t)\big)\\
&+ W_4(t)\frac{(S_v(t)-S_v^*)}{S_v(t)}\big(\Lambda_v^{\alpha} - (B_2^{\alpha}(t)I_h(t)+\mu_v^{\alpha})S_v(t)\big)\\
&+ W_5(t)\big(B_2^{\alpha}(t)I_h(t)S_v(t) - \phi_3 E_v(t)\big) + W_6(t)\big(\nu_v^{\alpha}E_v(t) - \mu_v^{\alpha}I_v(t)\big)\\
&- \frac{1}{N_v(t)}\bigg(W_4(t)S_v^*\,g\Big(\frac{S_v(t)}{S_v^*}\Big) + W_5(t)E_v(t) + W_6(t)I_v(t)\bigg)\big(\Lambda_v^{\alpha} - \mu_v^{\alpha}N_v(t)\big)
\end{aligned}$$

which implies

$$\begin{aligned}
{}^{C}D^{\alpha}V(t) \leq\ & -\mu_h^{\alpha}W_1\frac{(S_h(t)-S_h^*)^2}{S_h(t)} - \mu_v^{\alpha}W_4(t)\frac{(S_v(t)-S_v^*)^2}{S_v(t)}\\
&- W_1B_1^{\alpha}(t)(S_h(t)-S_h^*)I_v(t) + W_2B_1^{\alpha}(t)I_v(t)S_h(t) - \phi_1W_2E_h(t)\\
&+ W_3\nu_h^{\alpha}E_h(t) - \phi_2W_3I_h(t) - W_4(t)B_2^{\alpha}(t)(S_v(t)-S_v^*)I_h(t)\\
&+ W_5(t)B_2^{\alpha}(t)I_h(t)S_v(t) - \phi_3W_5(t)E_v(t) + W_6(t)\nu_v^{\alpha}E_v(t) - \mu_v^{\alpha}W_6(t)I_v(t)\\
&- \frac{\mu_v^{\alpha}}{N_v(t)}\bigg(W_4(t)S_v^*\Big(\frac{S_v(t)}{S_v^*}\Big) + W_5(t)E_v(t) + W_6(t)I_v(t)\bigg)(N_{\alpha,v}^* - N_v(t))
\end{aligned}$$

and have

$$\begin{aligned}
{}^{C}D^{\alpha}V(t) \leq\ & -\mu_h^{\alpha}W_1\frac{(S_h(t)-S_h^*)^2}{S_h(t)} - \mu_v^{\alpha}W_4(t)\frac{(S_v(t)-S_v^*)^2}{S_v(t)}\\
&+ B_1^{\alpha}(t)(W_2-W_1)I_v(t)S_h(t) + (W_3\nu_h^{\alpha} - \phi_1W_2)E_h(t)\\
&+ \big(W_4(t)B_2^{\alpha}(t)S_v^* - \phi_2W_3\big)I_h(t) + B_2^{\alpha}(t)\big(W_5(t) - W_4(t)\big)I_h(t)S_v(t)\\
&+ \big(W_6(t)\nu_v^{\alpha} - \phi_3W_5(t)\big)E_v(t) + \big(W_1B_1^{\alpha}(t)S_h^* - \mu_v^{\alpha}W_6(t)\big)I_v(t)\\
&- \frac{\mu_v^{\alpha}}{N_v(t)}\bigg(W_4(t)S_v^*\,g\Big(\frac{S_v(t)}{S_v^*}\Big) + W_5(t)E_v(t) + W_6(t)I_v(t)\bigg)(N_{\alpha,v}^* - N_v(t))
\end{aligned}$$

thus

$$
\begin{aligned}
{}^{C}D^{\alpha}V(t) \leq & -\mu_h^{\alpha} W_1 \frac{(S_h(t)-S_h^*)^2}{S_h(t)} - \mu_v^{\alpha} W_4(t) \frac{(S_v(t)-S_v^*)^2}{S_v(t)} \\
& + \phi_2 \Big((R_0^{\alpha})^2 \frac{S_v^* S_h^*}{N_v(t) N_h(t)} - 1 \Big) I_h(t) \\
& - \frac{\mu_v^{\alpha}}{N_v(t)} \Big(W_4(t) S_v^* \, g\Big(\frac{S_v(t)}{S_v^*}\Big) + W_5(t) E_v(t) + W_6(t) I_v(t) \Big) (N_{\alpha,v}^* - N_v(t)).
\end{aligned}
$$

This implies that if $(R_0^{\alpha})^2 < \frac{N_v(0)N_h(0)}{N_{\alpha,h}^* N_{\alpha,v}^*}$ then ${}^{C}D^{\alpha}V(t) < 0$ for all $(S_h, E_h, I_h, R_h, S_v, E_v, I_v) \neq E_{DF}$ and ${}^{C}D^{\alpha}V(t) = 0$ for $(S_h, E_h, I_h, R_h, S_v, E_v, I_v) = E_{DF}$. Therefore, by LaSalle's invariance principle, the DFE is globally asymptotically stable.

□

4. The Nonstandard Finite Difference Method

In this section we explain the technique of nonstandard finite difference schemes (NSFDs). A NSFD scheme is constructed to satisfy the positivity condition and the conservation laws. Consequently, the solutions are bounded, i.e. stable. Also, only the fixed-points of the ODE systems (2.2), (2.6) appear in the NSFD scheme. The specific full details are not given; we refer to the book of Mickens [**34**] for the discretization strategy.

4.1. Nonstandard Finite Difference Schemes. NSFD methods for the numerical integration of differential equations had their origin in a paper by Mickens published in 1989 [**34**]. In this section, an NSFD scheme is constructed to satisfy the essential positivity condition and the conservation law for $\Lambda_h = \mu_h = 0$, $\Lambda_v = \mu_v = 0$ and $\Lambda_m = \mu_m = 0$ which leads as a byproduct to the stability of the scheme. We will also check that the equilibrium points of the ODE model also appear in the proposed NSFD scheme.

Let us recall that schemes such as those based on Runge-Kutta methods can yield wrong negative solutions (see [**32, 19**]) can produce 'false' or 'spurious' fixed-points, which are not fixed points of the original ODE system, cf. [**35**].

Finally, we will determine in Section 4.4 the so-called denominator function $\phi(h)$, such that we obtain the correct long-time behaviour. We refer to [**6, 42**], where we established an NSFD scheme for a similar compartment model as here.

We remind the reader that a numerical scheme for a system of first-order differential equations is called *NSFD scheme* if at least one of the following conditions [**34**] is satisfied:

- The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives appearing in the differential equations.
- Discrete representations for derivatives must, in general, have nontrivial denominator functions.
  - The first-order derivatives in the system are approximated by the generalized forward difference method (forward Euler method)

$$\frac{\mathrm{d}u}{\mathrm{d}t}\Big|_{t=t_n} \approx \frac{u^{n+1}-u^n}{\phi(h)},$$

with the numerical approximation $u^n \approx u(t_n)$, $n = 0, 1, 2 \ldots$ on the uniform grid $t_n = n\,h$ with the step size $h = \Delta t$.

  - Here, $\phi \equiv \phi(h) > 0$ is the so-called *denominator function* such that $\phi(h) = h + \mathcal{O}(h^2)$. This function $\phi(h)$ is chosen so that the discrete solution has the same asymptotic behaviour as the analytical solution, see Section 4.4.
- The nonlinear terms are approximated by non-local discrete representations, for instance by a suitable function of several points of a mesh, like $u^2(t_n) \approx u^n\, u^{n+1}$ or $u^3(t_n) \approx (u^n)^2\, u^{n+1}$.
- Special conditions that hold for either the ODE and/or its solutions should also apply to the difference equation model and/or its solution, e.g. positivity of the solution, convexity of the solution (in finance), equilibrium points of the ODE system, including their local asymptotic stability properties.

In NSFD schemes, derivatives must be modeled by discrete analogues that take the form, cf. [**34**]

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}\Big|_{t=t_n} \to \frac{u^{n+1} - \psi(h)u^n}{\phi(h)}\,, \tag{4.1}$$

where $t_n = n\,h$, u^n is the approximation of $u(t_n)$, and $\psi(h) = 1+\mathcal{O}(h)$. The purpose of this more general time discretization (4.1) in NSFD schemes, is to properly model the asymptotic long-time behaviour of the solution.

4.2. NSFD scheme for the human-mosquito model. Next, we propose the following NSFD discretization for solving the ODE system (2.2)

$$\begin{aligned}
\frac{S_h^{n+1} - S_h^n}{\phi_h(h)} &= \Lambda_h - \Big(B\beta_{vh}\frac{I_v^n}{N_v^n} + \mu_h\Big)S_h^{n+1},\\
\frac{E_h^{n+1} - E_h^n}{\phi_h(h)} &= B\beta_{vh}\frac{I_v^n}{N_v^n}S_h^{n+1} - (\nu_h + \mu_h)E_h^{n+1},\\
\frac{I_h^{n+1} - I_h^n}{\phi_h(h)} &= \nu_h E_h^{n+1} - (\eta_h + \mu_h)I_h^{n+1},\\
\frac{R_h^{n+1} - R_h^n}{\phi_h(h)} &= \eta_h I_h^{n+1} - \mu_h R_h^{n+1},\\
\frac{S_v^{n+1} - S_v^n}{\phi_v(h)} &= \Lambda_v - \Big(B\beta_{hv}\frac{I_h^n}{N_h^n} + \mu_v\Big)S_v^{n+1},\\
\frac{E_v^{n+1} - E_v^n}{\phi_v(h)} &= B\beta_{hv}\frac{I_h^n}{N_h^n}S_v^{n+1} - (\mu_v + \nu_v)E_v^{n+1},\\
\frac{I_v^{n+1} - I_v^n}{\phi_v(h)} &= -\mu_v I_v^{n+1} + \nu_v E_v^{n+1},
\end{aligned} \tag{4.2}$$

with the denominator functions for each subsystem

$$\phi_h(h) = \frac{e^{\mu_h h} - 1}{\mu_h} \text{ and } \phi_v(h) = \frac{e^{\mu_v h} - 1}{\mu_v}. \tag{4.3}$$

The exact solutions of N_h and N_v are given by

$$N_h(t) = \frac{\Lambda_h}{\mu_h} + \Big(N_h(0) - \frac{\Lambda_h}{\mu_h}\Big)e^{-\mu_h t}, \tag{4.4}$$

and

$$N_v(t) = \frac{\Lambda_v}{\mu_v} + \Big(N_v(0) - \frac{\Lambda_v}{\mu_v}\Big)e^{-\mu_v t}. \tag{4.5}$$

Thus

$$N_h^{n+1}(t) = N_h(t_{n+1}) \text{ and } N_v^{n+1}(t) = N_v(t_{n+1}).$$

Let us briefly comment on the discretizations of the nonlinear (here: quadratic) terms. For example, in the first line (4.2) we have discretized the nonlinear contact term $\beta_{vh} I_v(t) S_h(t)$ in (2.2) by $\beta_{vh} I_v^n S_h^{n+1}$ rather than, say, $I_v^n S_h^n$ or $I_v^{n+1} S_h^{n+1}$. The rule is that exactly one factor of the variable appearing in the time derivative (here S_h) must be taken at the new time level $n+1$. This is needed to obtain a positivity preserving scheme, see (4.6). In order not to destroy the explicit sequential evaluation, all other variables are taken from the previous time level, unless they are already known from a previous step, like $I_h^{n+1} S_v^{n+1}$ in the sixth line. If possible, discrete conservation properties (here: total population of humans, vectors) must also be taken into account.

Observe that although the initial scheme (4.2) can be considered implicit, the variables at the $(n+1)$-th discrete-time level can be explicitly calculated in terms of the previously known variable values as given in the sequence of the equations above, i.e. we can rewrite it as an explicit form

$$\begin{aligned}
S_h^{n+1} &= \frac{S_h^n + \phi_h(h), \Lambda_h}{1 + \phi_h(h)\Big(B\beta_{vh}\frac{I_v^n}{N_v^n} + \mu_h\Big)}, \\
E_h^{n+1} &= \frac{E_h^n + \phi_h(h) B\beta_{vh}\frac{I_v^n}{N_v^n} S_h^{n+1}}{1 + \phi_h(h)(\nu_h + \mu_h)} \\
I_h^{n+1} &= \frac{I_h^n + \phi_h(h)\nu_h E_h^{n+1}}{1 + \phi_h(h)(\eta_h + \mu_h)}, \\
R_h^{n+1} &= \frac{R_h^n + \phi_h(h)\eta_h I_h^{n+1}}{1 + \phi_h(h)\mu_h}, \\
S_v^{n+1} &= \frac{S_v^n + \phi_v(h)\Lambda_v}{1 + \phi_v(h)\Big(B\beta_{hv}\frac{I_h^n}{N_h^n} + \mu_v\Big)}, \\
E_v^{n+1} &= \frac{E_v^n + \phi_v(h) B\beta_{hv}\frac{I_h^n}{N_h^n} S_v^{n+1}}{1 + \phi_v(h)(\mu_v + \nu_v)}, \\
I_v^{n+1} &= \frac{I_v^n + \phi_v(h)\nu_v E_v^{n+1}}{1 + \phi_v(h)\mu_v}.
\end{aligned} \tag{4.6}$$

The calculation must be done in exactly this order. All parameters appearing in these type of epidemic models are always non-negative. This is the convention used in fields related to the spread of diseases. From the explicit representation (4.6) it is easy to deduce that this scheme preserves the positivity, given some natural conditions on the parameters.

4.3. NSFD scheme for the human-mosquito-monkey model. Correspondingly, the NSFD discretization for solving the ODE system (2.6) reads

$$
\begin{aligned}
\frac{S_h^{n+1}-S_h^n}{\phi_h(h)} &= \Lambda_h - \Big(B\beta_{vh}\frac{I_v^n}{N_v^n}+\mu_h\Big)S_h^{n+1},\\
\frac{E_h^{n+1}-E_h^n}{\phi_h(h)} &= B\beta_{vh}\frac{I_v^n}{N_v^n}S_h^{n+1}-(\nu_h+\mu_h)E_h^{n+1},\\
\frac{I_h^{n+1}-I_h^n}{\phi_h(h)} &= \nu_h E_h^{n+1}-(\eta_h+\mu_h)I_h^{n+1},\\
\frac{R_h^{n+1}-R_h^n}{\phi_h(h)} &= \eta_h I_h^{n+1}-\mu_h R_h^{n+1},\\
\frac{S_v^{n+1}-S_v^n}{\phi_v(h)} &= \Lambda_v - \Big(B\beta_{hv}\frac{I_h^n}{N_h^n}+B\beta_{mv}\frac{I_m^n}{N_m^n}+\mu_v\Big)S_v^{n+1},\\
\frac{E_v^{n+1}-E_v^n}{\phi_v(h)} &= \Big(B\beta_{hv}\frac{I_h^n}{N_h^n}+B\beta_{mv}\frac{I_m^n}{N_m^n}\Big)S_v^{n+1}-(\mu_v+\nu_v)E_v^{n+1},\\
\frac{I_v^{n+1}-I_v^n}{\phi(h)} &= \nu_v E_v^{n+1}-\mu_v I_v^{n+1},\\
\frac{S_m^{n+1}-S_m^n}{\phi_m(h)} &= \Lambda_m - \Big(B\beta_{vm}\frac{I_v^n}{N_v^n}+\mu_m\Big)S_m^{n+1},\\
\frac{E_m^{n+1}-E_m^n}{\phi_m(h)} &= B\beta_{vm}\frac{I_v^n}{N_v^n}S_m^{n+1}-(\nu_m+\mu_m)E_m^{n+1},\\
\frac{I_m^{n+1}-I_m^n}{\phi_m(h)} &= \nu_m E_m^{n+1}-(\eta_m+\mu_m)I_m^{n+1},\\
\frac{R_m^{n+1}-R_m^n}{\phi_m(h)} &= \eta_m I_m^{n+1}-\mu_m R_m^{n+1}.
\end{aligned}
\tag{4.7}
$$

Accordingly, we rewrite the scheme (4.7) in an explicit sequential formulation

$$
\begin{aligned}
S_h^{n+1} &= \frac{S_h^n+\phi_h(h)\,\Lambda_h}{1+\phi_h(h)\,\big(B\beta_{vh}\frac{I_v^n}{N_v^n}+\mu_h\big)},\\
E_h^{n+1} &= \frac{E_h^n+\phi_h(h)\,B\beta_{vh}B\beta_{vh}\frac{I_v^n}{N_v^n}S_h^{n+1}}{1+\phi_h(h)\,(\nu_h+\mu_h)},\\
I_h^{n+1} &= \frac{I_h^n+\phi_h(h)\,\nu_h E_h^{n+1}}{1+\phi_h(h)\,(\eta_h+\mu_h)},\\
R_h^{n+1} &= \frac{R_h^n+\phi_h(h)\,\eta_h I_h^{n+1}}{1+\phi_h(h)\,\mu_h},\\
S_v^{n+1} &= \frac{S_v^n+\phi_v(h)\,\Lambda_v}{1+\phi_v(h)\,\big(B\beta_{hv}\frac{I_h^n}{N_h^n}+B\beta_{mv}\frac{I_m^n}{N_m^n}+\mu_v\big)},\\
E_v^{n+1} &= \frac{E_v^n+\phi_v(h)\,\big(B\beta_{hv}\frac{I_h^n}{N_h^n}+B\beta_{mv}\frac{I_m^n}{N_m^n}\big)S_v^{n+1}}{1+\phi_v(h)\,(\nu_v+\mu_v)},\\
I_v^{n+1} &= \frac{I_v^n+\phi_v(h)\,\nu_v E_v^{n+1}}{1+\phi_v(h)\,\mu_v},
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
S_m^{n+1} &= \frac{S_m^n + \phi_m(h)\,\Lambda_m}{1+\phi_m(h)\left(B\beta_{vm}\frac{I_v^n}{N_v^n} + \mu_m\right)},\\
E_m^{n+1} &= \frac{E_m^n + \phi_m(h)\,B\beta_{vm}\frac{I_v^n}{N_v^n}\,S_m^{n+1}}{1+\phi_m(h)\,(\nu_m+\mu_m)},\\
I_m^{n+1} &= \frac{I_m^n + \phi_m(h)\,\nu_m E_m^{n+1}}{1+\phi_m(h)\,(\eta_m+\mu_m)},\\
R_m^{n+1} &= \frac{R_m^n + \phi_m(h)\,\eta_m I_m^{n+1}}{1+\phi_m(h)\,\mu_m}.
\end{aligned}
$$

4.4. The denominator function. Finally, it only remains to correctly determine the denominator function $\phi(h)$. To do so, we reconsider the combined total population $N = N_h, N_v$ or N_m of the ODE systems (2.2) and (2.6)), now without neglecting the birthrates and the natural mortality. Here, we introduce accordingly the combined values $\Lambda = \Lambda_h, \Lambda_v$ or Λ_m, $\mu = \mu_h, \mu_v$ or μ_m for the system (2.2) and the extended system (2.6). At a first glance, it looks inappropriate to add the populations of humans, mosquitos and monkeys, but this has purely mathematical reasons: it is used for the asymptotic behaviour that later leads to the denominator function $\phi(h)$, which must be the same for all components of the ODE system.

Adding the equations of (2.2) or (2.6), we easily obtain the following differential equation describing the dynamics of the combined total population N

$$
\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Lambda - \mu\,N(t)\,. \tag{4.9}
$$

It is solved by

$$
N(t) = \frac{\Lambda}{\mu} + \Big(N(0) - \frac{\Lambda}{\mu}\Big)\,e^{-\mu t} = N(0) + \Big(N(0) - \frac{\Lambda}{\mu}\Big)\,(e^{-\mu t} - 1), \tag{4.10}
$$

with $N(0) = N_h(0) + N_v(0) + N_m(0)$. From (4.10) we immediately deduce that we have in the long term $\lim_{t\to\infty} N(t) = \Lambda/\mu$. Let us briefly note that this link between the transient dynamics and their 'natural' limiting systems can be used to reduce the dimension of this model, cf. [**9**].

Next, adding the equations in the discrete NSFD model (4.2) yields

$$
\frac{N^{n+1} - N^n}{\phi(h)} = \Lambda - \mu\,N^{n+1}, \tag{4.11}
$$

i.e.

$$
\begin{aligned}
N^{n+1} &= \frac{N^n + \phi(h)\Lambda}{1+\phi(h)\,\mu} = N^n - \Big(N^n - \frac{\Lambda}{\mu}\Big)\frac{\phi(h)\,\mu}{1+\phi(h)\,\mu}\\
&= N^n + \Big(N^n - \frac{\Lambda}{\mu}\Big)\Big(\frac{1}{1+\phi(h)\,\mu} - 1\Big).
\end{aligned} \tag{4.12}
$$

The denominator function can be derived by comparing Equation (4.11) with the discrete version of Equation (4.10), that is

$$
N^{n+1} = N^n + \Big(N^n - \frac{\Lambda}{\mu}\Big)\,(e^{-\mu h} - 1), \quad h = \Delta t, \tag{4.13}
$$

such that the (positive) denominator function is defined by

$$
\frac{1}{1+\phi(h)\,\mu} = e^{-\mu h}, \tag{4.14}
$$

i.e.

$$\phi(h) = \frac{e^{\mu h} - 1}{\mu} = \frac{1 + \mu h + \frac{1}{2}\mu^2 h^2 + \cdots - 1}{\mu} = h + \frac{\mu h^2}{2} + \cdots = h + \mathcal{O}(h^2). \tag{4.15}$$

Note that the conservation property requires all the denominator functions $\phi(h)$ for the compartments to be the same. Otherwise, it would be impossible to obtain a discrete analogue like (4.11) which is also needed for stability reasons.

REMARK 4.1. An even more accurate way to compute the denominator function would take into account the transition rate Υ_i at which the i^{th} compartment is entered by individuals for all model compartments $\mathcal{K}_i$, $i = 1, 2, \ldots$ (e.g. β_{vh}, ν_h, η_h, ν_v,...), cf. [**16**]. In this case the parameter μ occurring in the denominator function in Equation (4.15) would be replaced by a parameter $1/T^*$. T^* could be determined as the minimum of the inverse transition parameters:

$$T^* = \min_{i=1,2,\ldots} \Big\{ \frac{1}{\Upsilon_i} \Big\}.$$

4.5. A NSFD scheme for a time-fractional model. Again, let us consider a uniform temporal grid $t_0 = 0 < t_1 < \cdots < t_{N_T} = T$, $t_n = nT/N_T$, where $N_T \in \mathbb{N}$. Next, we present a numerical approximation of the Caputo derivative using the NSFD method. We have

$${}^C D^\alpha X(t)\big|_{t=t_{n+1}} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau} (t_{n+1} - \tau)^{-\alpha} \, d\tau$$

We discretize the term $\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau}$ on the interval $[t_j, t_{j+1}]$ as

$$\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau} = \frac{X^{j+1} - X^j}{\phi_\alpha(h)},$$

where $X^j = X(t_j)$ and $\phi_\alpha(h)$ from (4.15).

$${}^C D^\alpha X(t)\big|_{t=t_{n+1}} \approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \Delta_{\alpha,n}^j \frac{X^{j+1} - X^j}{\phi_\alpha(h)},$$

where

$$\Delta_{\alpha,n}^j = \big((t_{n+1} - t_j)^{1-\alpha} - (t_{n+1} - t_{j+1})^{1-\alpha} \big).$$

Each equation in (2.8) can be written as

$${}^C D^\alpha X(t) = F\big(X(t)\big),$$

at the point $t = t_{n+1}$, we have

$$\frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \Delta_{\alpha,n}^j \frac{X^{j+1} - X^j}{\phi_\alpha(h)} - F(X^{n+1}) = 0 \quad n = 1, \ldots, N_T - 1. \tag{4.16}$$

Now, we apply the scheme (4.16) to the system (2.8), we obtain

$$S_h^{n+1} = \frac{h^{1-\alpha} S_h^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (S_h^{j+1} - S_h^j) + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\Lambda_h^\alpha}{\Big(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\Big(B^\alpha \beta_{vh} \frac{I_v^n}{N_{\alpha,v}^n} + \mu_h^\alpha \Big) \Big)},$$

$$E_h^{n+1} = \frac{h^{1-\alpha} E_h^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (E_h^{j+1} - E_h^j) + \Gamma(2-\alpha)\phi_{\alpha,h}(h) B^\alpha \beta_{vh} \frac{I_v^n}{N_{\alpha,v}^n} S_h^{n+1}}{\big(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,h}(h)(\nu_h^\alpha + \mu_h^\alpha) \big)},$$

$$
\begin{aligned}
I_h^{n+1} &= \frac{h^{1-\alpha} I_h^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (I_h^{j+1} - I_h^j) + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\nu_h^\alpha E_h^{n+1}}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,h}(h)(\eta_h^\alpha + \mu_h^\alpha)\right)}, \\
R_h^{n+1} &= \frac{h^{1-\alpha} R_h^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (R_h^{j+1} - R_h^j) + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\eta_h^\alpha I_h^{n+1}}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\mu_h^\alpha\right)}, \\
N_{\alpha,h}^{n+1} &= \frac{h^{1-\alpha} N_{\alpha,h}^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (N_{\alpha,h}^{j+1} - N_{\alpha,h}^j) + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\Lambda_h^\alpha}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\mu_h^\alpha\right)} \\
S_v^{n+1} &= \frac{h^{1-\alpha} S_v^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (S_v^{j+1} - S_v^j) + \Gamma(2-\alpha)\phi_{\alpha,v}(h)\Lambda_v^\alpha}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,v}(h)(B^\alpha \beta_{hv} \frac{I_h^n}{N_{\alpha,h}^n} + \mu_v^\alpha)\right)}, \\
E_v^{n+1} &= \frac{h^{1-\alpha} E_v^n - \sum_{j=0}^{n} \Delta_{\alpha,n}^j (E_v^{j+1} - E_v^j) + \Gamma(2-\alpha)\phi_{\alpha,v}(h) B^\alpha \beta_{hv} \frac{I_h^n}{N_{\alpha,h}^n} S_v^{n+1}}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,v}(h)(\nu_v^\alpha + \mu_v^\alpha)\right)}, \\
I_v^{n+1} &= \frac{h^{1-\alpha} I_v^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (I_v^{j+1} - I_v^j) + \Gamma(2-\alpha)\phi_{\alpha,v}(h)\nu_v^\alpha E_v^{n+1}}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,v}(h)\mu_v^\alpha\right)}, \\
N_{\alpha,v}^{n+1} &= \frac{h^{1-\alpha} N_{\alpha,v}^n - \sum_{j=0}^{n-1} \Delta_{\alpha,n}^j (N_{\alpha,v}^{j+1} - N_{\alpha,v}^j) + \Gamma(2-\alpha)\phi_{\alpha,v}(h)\Lambda_v^\alpha}{\left(h^{1-\alpha} + \Gamma(2-\alpha)\phi_{\alpha,v}(h)\mu_v^\alpha\right)}.
\end{aligned} \tag{4.17}
$$

Setting $n = 0$, equations of $N_{\alpha,h}^{n+1}$ and $N_{\alpha,v}^{n+1}$ in (4.17) give

$$
N_{\alpha,h}^1 \approx \frac{h^{1-\alpha} N_{\alpha,h}^0}{h^{1-\alpha} + \phi_{\alpha,h}(h)\Gamma(2-\alpha)\mu_h^\alpha} + \frac{\phi_{\alpha,h}(h)\Gamma(2-\alpha)\Lambda_h^\alpha}{h^{1-\alpha} + \phi_{\alpha,h}(h)\Gamma(2-\alpha)\mu_h^\alpha} \tag{4.18}
$$

and

$$
N_{\alpha,v}^1 \approx \frac{h^{1-\alpha} N_{\alpha,v}^0}{h^{1-\alpha} + \phi_{\alpha,v}(h)\Gamma(2-\alpha)\mu_v^\alpha} + \frac{\phi_{\alpha,v}(h)\Gamma(2-\alpha)\Lambda_v^\alpha}{h^{1-\alpha} + \phi_{\alpha,v}(h)\Gamma(2-\alpha)\mu_v^\alpha}. \tag{4.19}
$$

The exact solution of the equations (3.2) and (3.3) can be rewritten as

$$
N_{\alpha,h}(t) = N_{\alpha,h}(0) E_\alpha\big(-(\mu_h t)^\alpha\big) + \frac{\Lambda_h^\alpha}{\mu_h^\alpha}\Big(1 - E_\alpha\big(-(\mu_h t)^\alpha\big)\Big) \tag{4.20}
$$

and

$$
N_{\alpha,v}(t) = N_{\alpha,v}(0) E_\alpha\big(-(\mu_v t)^\alpha\big) + \frac{\Lambda_v^\alpha}{\mu_v^\alpha}\Big(1 - E_\alpha\big(-(\mu_v t)^\alpha\big)\Big). \tag{4.21}
$$

The denominator function $\phi_{\alpha,h}(h)$ ($\phi_{\alpha,v}(h)$ respectively) can be derived by comparing the exact version (4.20) ((4.21) respectively) with the discrete version (4.18) ((4.19) respectively), that is

$$
\phi_{\alpha,h}(h) = \frac{h^{1-\alpha}\Big(1 - E_\alpha\big(-(\mu_h h)^\alpha\big)\Big)}{E_\alpha\big(-(\mu_h h)^\alpha\big)\Gamma(2-\alpha)\mu_h^\alpha} \text{ and } \phi_{\alpha,v}(h) = \frac{h^{1-\alpha}\Big(1 - E_\alpha\big(-(\mu_v h)^\alpha\big)\Big)}{E_\alpha\big(-(\mu_v h)^\alpha\big)\Gamma(2-\alpha)\mu_v^\alpha}.
$$

It is not difficult to show that $\phi_{\alpha,h}(h)$ and $\phi_{\alpha,v}(h)$ reduce to the classical $\phi_h(h)$ and $\phi_v(h)$ in (4.3) when $\alpha = 1$.

5. Numerical Results

In this section, we present the numerical solution of the systems (2.2) and (2.6) using the NSFD schemes (4.6) and (4.8). Then, we compare it with the solution computed by the `ode45` solver of Matlab.

5.1. The human-mosquito Model. We denote by Y the matrix of order $N_T \times 7$ that contains the approximated solution determined by the `ode45` solver which is given by

$$Y = \begin{pmatrix} S_h(t_1) & E_h(t_1) & I_h(t_1) & R_h(t_1) & S_v(t_1) & E_v(t_1) & I_v(t_1) \\ S_h(t_2) & E_h(t_2) & I_h(t_2) & R_h(t_2) & S_v(t_2) & E_v(t_2) & I_v(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_h(t_{N_T}) & E_h(t_{N_T}) & I_h(t_{N_T}) & R_h(t_{N_T}) & S_v(t_{N_T}) & E_v(t_{N_T}) & I_v(t_{N_T}) \end{pmatrix}.$$

The parameters used to simulate the model are listed in the Table 1. The initial conditions are always set to

$$S_h(0) = 9e4, \quad E_h(0) = 0, \quad I_h(0) = 1e4, \quad R_h(0) = 0,$$
$$S_v(0) = 1.188e5, \quad E_v(0) = 0, \quad I_v(0) = 1.2e3.$$

TABLE 1. Fixed and operational parameters for disease-free and disease-endemic equilibrium.

	DFE	EE
Λ_h	4.6e2	4.6e2
μ_h	6e-04	6e-04
B	0.1523	0.1932
β_{hv}	0.0805	0.773
β_{vh}	0.0741	0.7823
ν_h	0.0833	0.0833
η_h	0.2	0.2
Λ_v	3.2e4	3.2e4
μ_v	0.0333	0.0333
ν_v	0.1	0.1
T (days)	22×365	22×365

The following Figures 3–8 represent the trajectories in the three dimensional space of the human and the vector populations, respectively. They show that the NSFD method remains stable and approaches the disease-free equilibrium (DFE) or endemic equilibrium (EE) points.

The Figures 9 and 10 show that the approximate solutions obtained by the NSFD method and `ode45` method are very closed to each other. However, the solution Y obtained by the `ode45` solver becomes negative for some values of t.

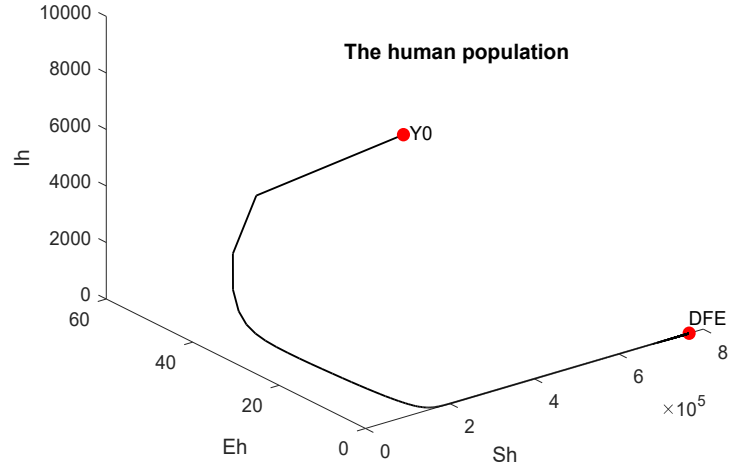

FIGURE 3. The convergence of the discrete system (4.6) to the DFE

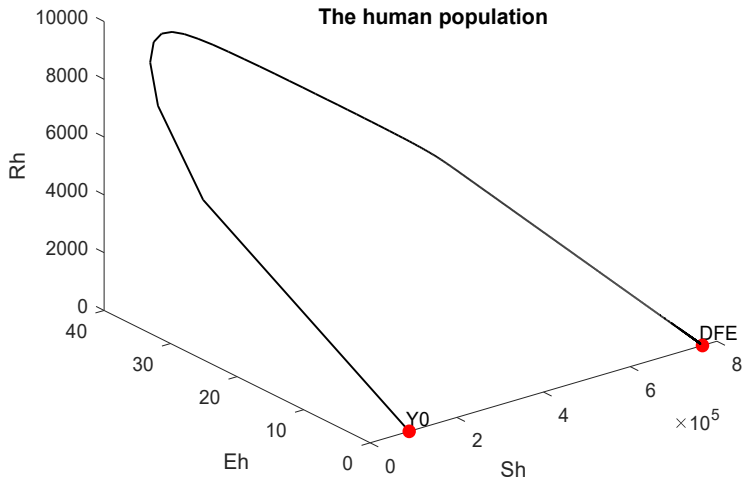

FIGURE 4. The convergence of the discrete system (4.6) to the DFE

This does not figure clearly in the curves because the smallest negative value of Y is $-4.02e-07$.

The Table 2 presents the percentage of negative values in the matrix Y simulating the human-mosquito model (2.2) with the `ode45` solver using the parameters for the disease-free point in the Table 1. The results given in Table 2 show that the NSFD preserves the positivity for all step sizes in $[0, T]$, which is a desirable modeling property. On the other side, the `ode45` method yields solutions that becomes negative for some value of t.

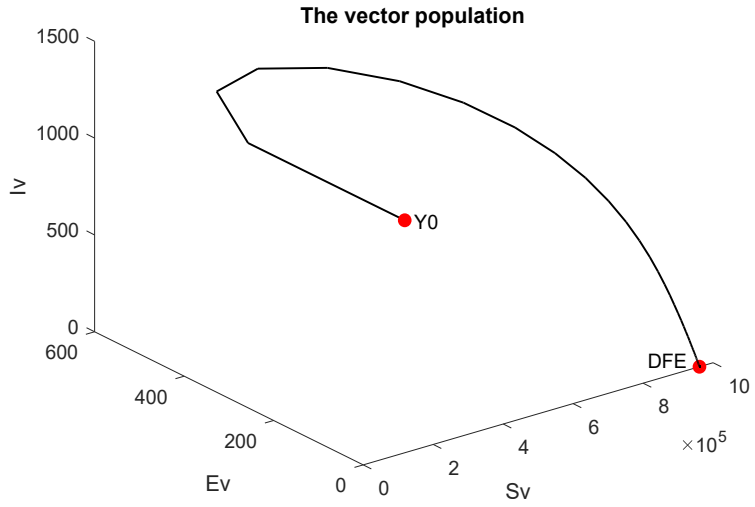

FIGURE 5. The convergence of the discrete system (4.6) to the DFE

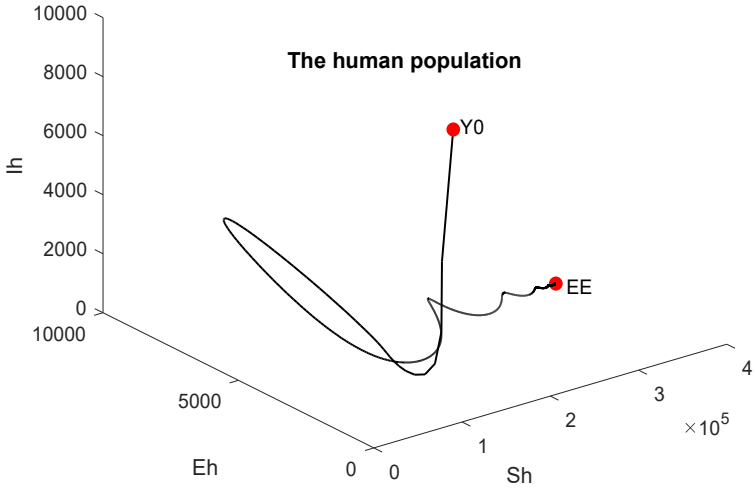

FIGURE 6. The convergence of the discrete system (4.6) to the EE

TABLE 2. Percentage of negative paths for the standard `ode45` solver .

	$N_T = 100$	$N_T = 200$	$N_T = 400$	$N_T = 800$	$N_T = 1000$	$N_T = 1200$	$N_T = 2000$
ode45	17.57%	17.57%	17.5%	17.59%	17.59%	17.54%	17.6%
min(Y)	-3.67e-07	-1.13e-07	-4.02e-07	-4.02e-07	-4.02e-07	-4.02e-07	-4.02e-07

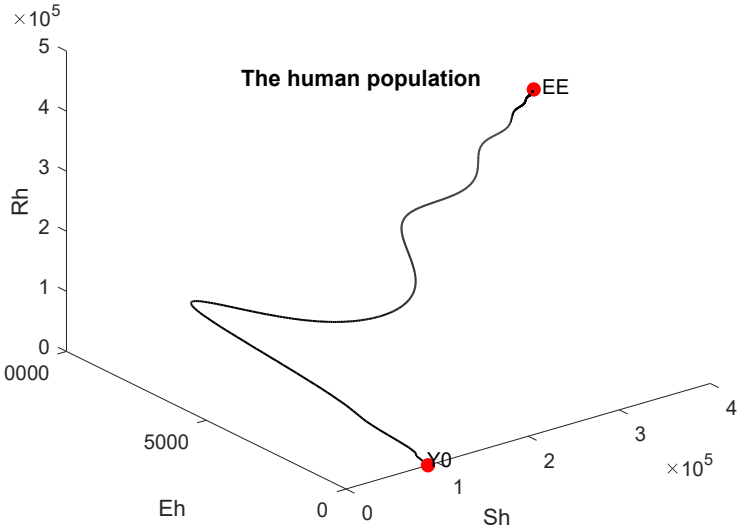

FIGURE 7. The convergence of the discrete system (4.6) to the EE

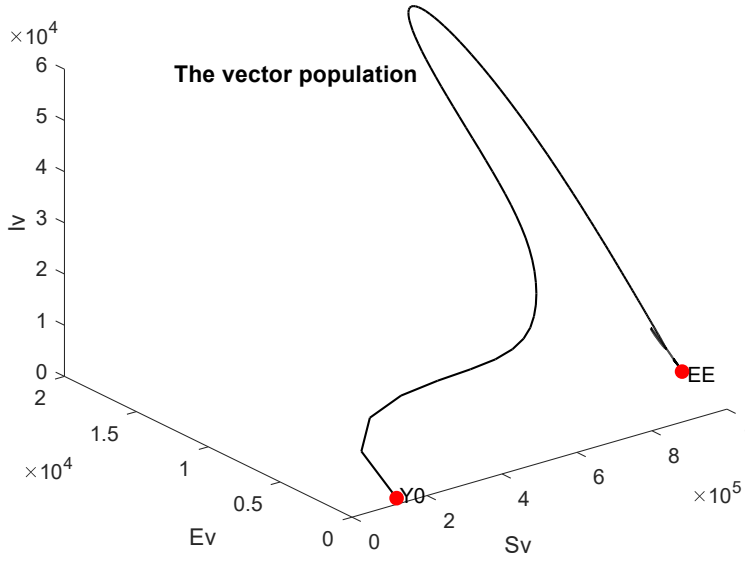

FIGURE 8. The convergence of the discrete system (4.6) to the EE

5.2. The human-mosquito-monkey model. Now we simulate the system for the data given in Tables 2 and 2. The initial conditions are always set to

$$S_m(0) = 6.4e4, \quad E_m(0) = 0, \quad I_m(0) = 1.6e4, \quad R_m(0) = 0.$$

Figures 11–14 show that the numerical solution approximates very well the solution of the continuous system by preserving positivity and converging towards the equilibrium points DFE or EE. Table 4 gives the percentage of negative values for

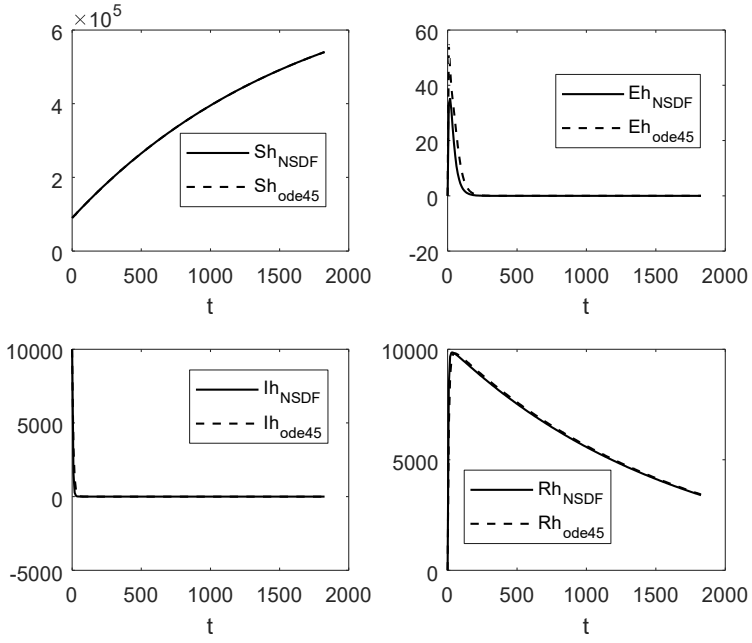

FIGURE 9. The NSFD and `ode45` method numerical simulations of human sub-populations $S_h(t)$, $E_h(t)$, $I_h(t)$ and $R_h(t)$ for model (2.2) with $N_T = 200$ and $t \in [0, 1825]$.

the NSFD method and the `ode45` solver. It can easily be seen that NSFD preserves the positivity of the continuous system where the `ode45` solver failed in some cases.

TABLE 3. Fixed and operational parameters for disease-free and disease-endemic equilibria (Monkey population).

	DFE	EE
Λ_m	$1e3$	$1e3$
μ_m	$3.87e-4$	$3.87e-4$
β_{mv}	0.0805	0.773
β_{vm}	0.0741	0.7823
ν_m	0.035	0.035
η_m	0.2	0.2

5.3. The time-fractional model. In this section, we provide some numerical simulations of the discrete model (4.17) with different values of fractional order α. To proceed with the simulation, we use the parameter values in Table 1 and the initial conditions in (5.1). The numerical simulation results for the NSFD fractional

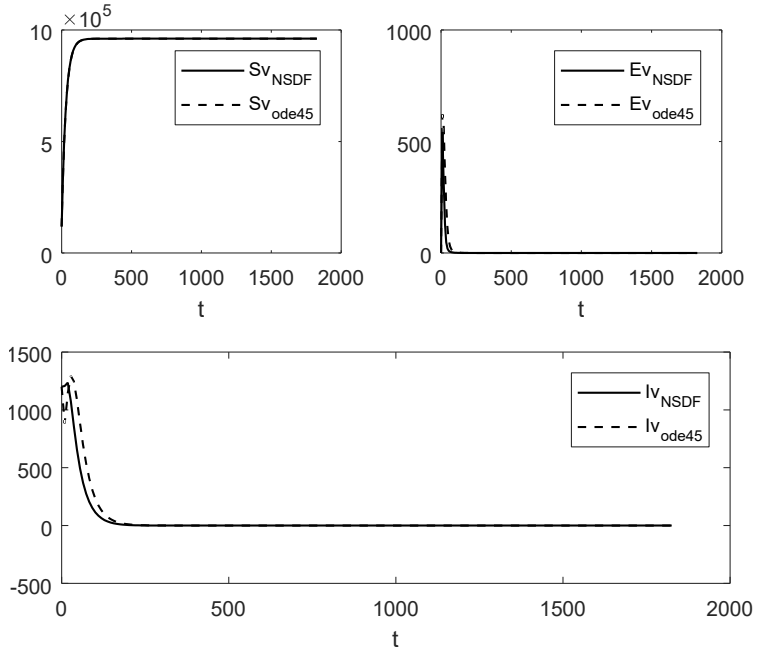

FIGURE 10. The NSFD and `ode45` method numerical simulations of vector sub-populations $S_v(t)$, $E_v(t)$ and $I_v(t)$ for model (2.2) with $N_T = 200$ and $t \in [0, 1825]$.

TABLE 4. Percentage of negative paths for the standard `ode45` solver.

	$N_T = 100$	$N_T = 200$	$N_T = 400$	$N_T = 800$	$N_T = 1000$	$N_T = 1200$	$N_T = 2000$
ode45	14.73%	14%	14.1%	14.16%	14.14%	14.24%	14.17%
$\min(Y)$	-1.18-06	-9.05-07	-1.14e-06	-1.14e-06	-1.18e-06	-1.2e-06	-1.2e-06

order obtained for different values of α are displayed in Figures 15–20. These figures show two different scenarios:

Case 1 DFE. : The dynamical behavior of system for different values of α is shown in Figures 15–17 for $R_0^\alpha < 1$ which implies that it converges to the DFE. It is noticeable that due to the memory property of the Caputo fractional derivatives, the evolution of the system becomes slower each time the α decreases. Therefore, the system decays to the equilibrium like $t^{-\alpha}$, as previously established in [**31**].

Case 2 EE.: For $R_0^\alpha > 1$, Figures 18–20 show the impact of changing the Caputo fractional order α on Zika dynamics. The observed behavior from these figures demonstrates that the EE is shifted towards $EE, EE_{\alpha_1}, EE_{\alpha_2}$ and EE_{α_3} when α is decreasing.

The numerical results above show the memory effect for the fractional dynamical system which does not occur in the ODE system as already proved by [**3, 4**]. And show also that the new approach is very effective, preserves the positivity of

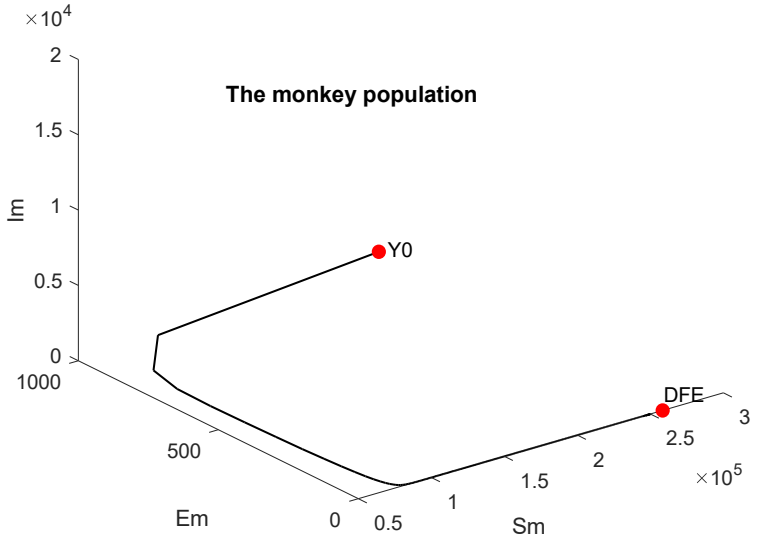

FIGURE 11. The convergence of the discrete system (4.8) to the DFE

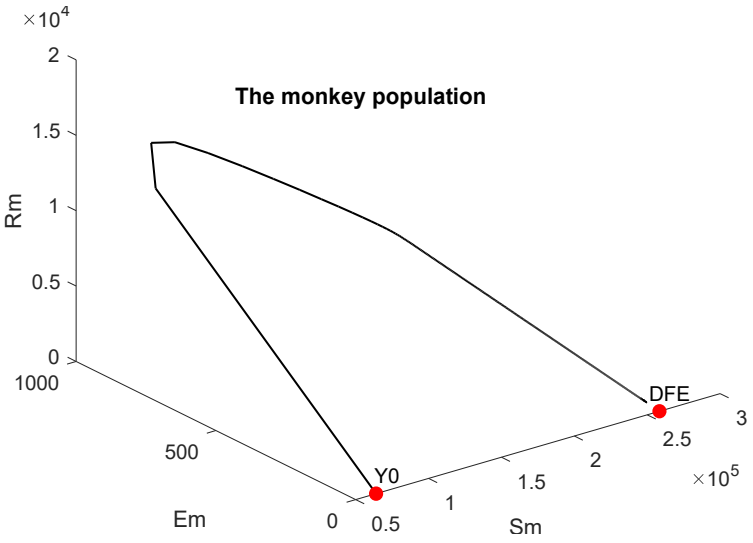

FIGURE 12. The convergence of the discrete system (4.8) to the DFE

the system, applies simpler and can be used as an alternate method for solving fractional differential problems.

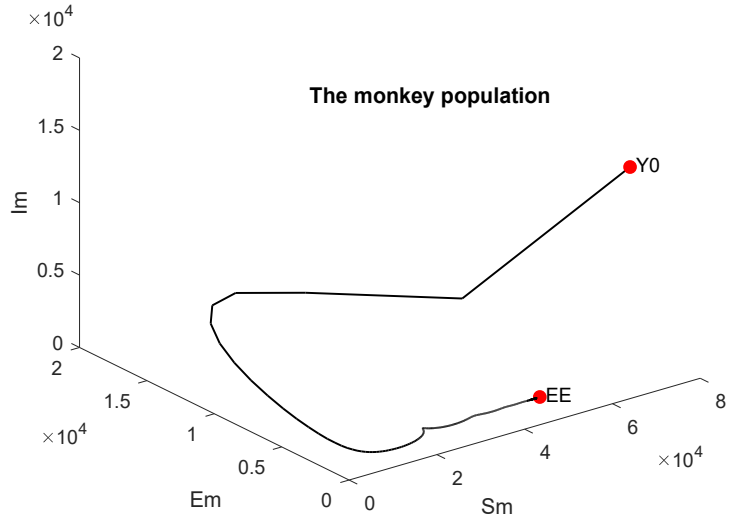

FIGURE 13. The convergence of the discrete system (4.8) to the EE

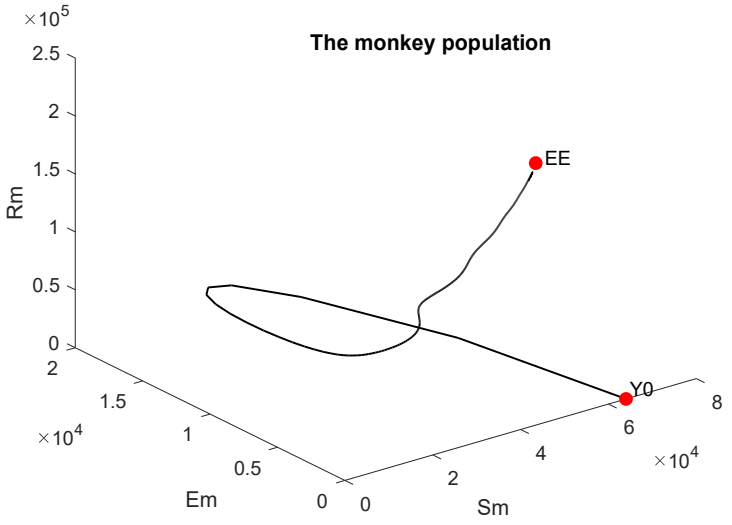

FIGURE 14. The convergence of the discrete system (4.8) to the EE

6. Conclusion and Outlook

In this work we have presented a novel nonstandard finite difference (NSFD) method for calculating numerical solutions to a SEIR model for the spread of the Zika virus. In the absence of the exact solution and in order to prove the efficiency of

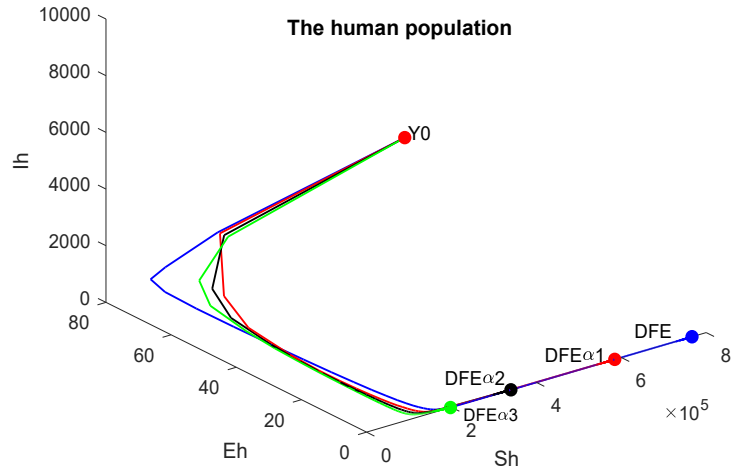

FIGURE 15. Impact of α on the DFE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

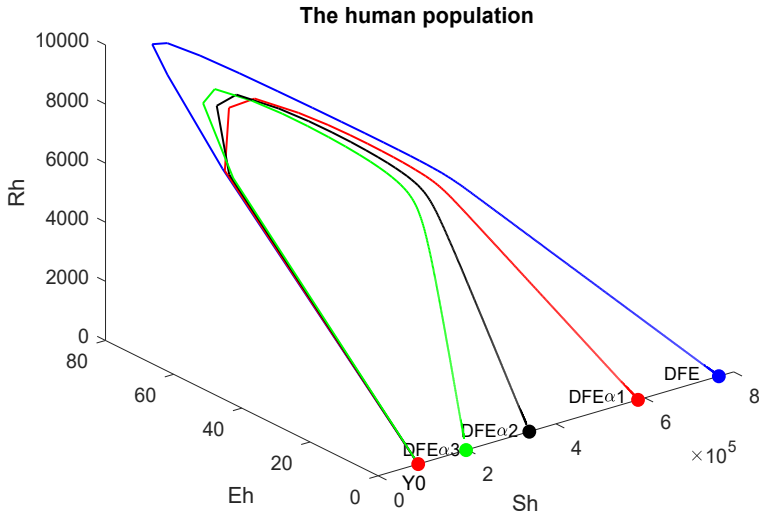

FIGURE 16. Impact of α on the DFE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

the method, the approximate solution is compared with the `ode45` solver solution. The numerical simulations show that the discrete system converges to the same equilibrium points as that of the continuous system. They also prove that the positivity is preserved in case of the NSFD scheme and may be violated using a standard ODE solver.

It is worth recalling that we have used Caputo-type fractional derivatives to describe the temporal dynamics of epidemiological models. The most important reason for using a system of ODEs/PDEs of time-fractional order equations is to

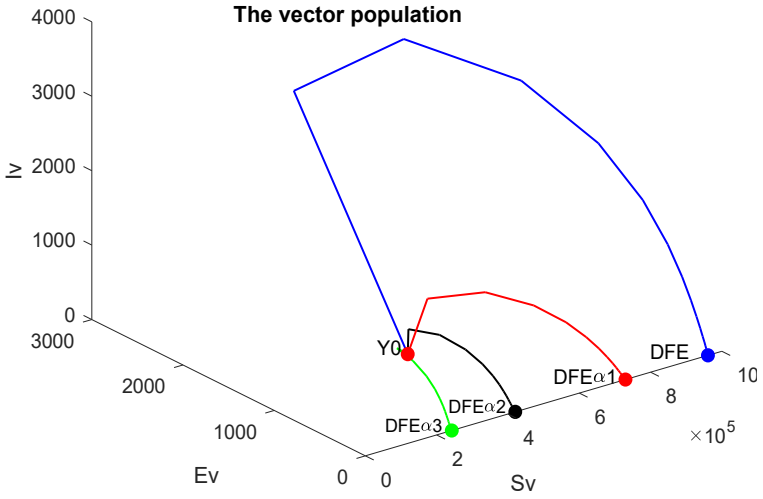

FIGURE 17. Impact of α on the DFE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

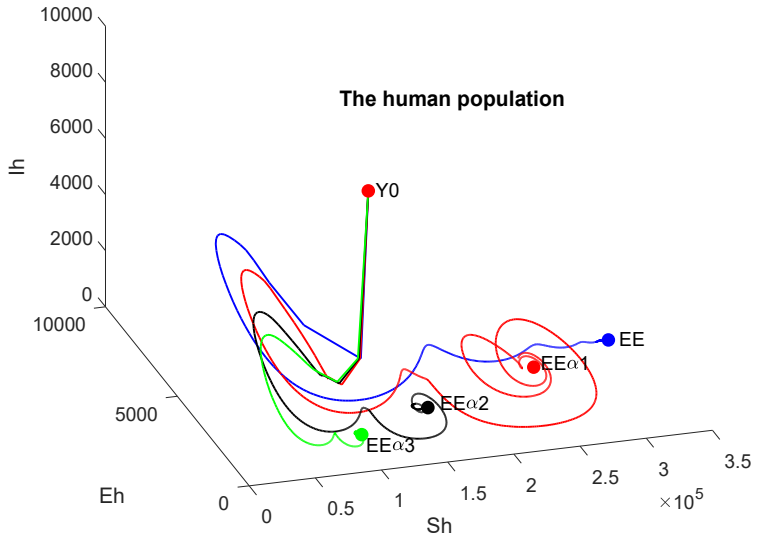

FIGURE 18. Impact of α on the EE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

account for memory effects. These types of effects exist e.g. in many realistic systems like in endemic models to describe the waning effects of the vaccination or a biphasic decline behavior of infections or diseases.

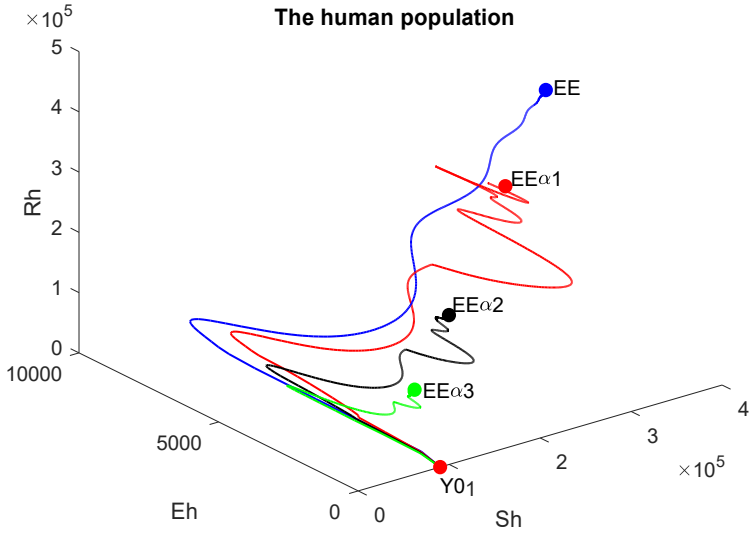

FIGURE 19. Impact of α on the EE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

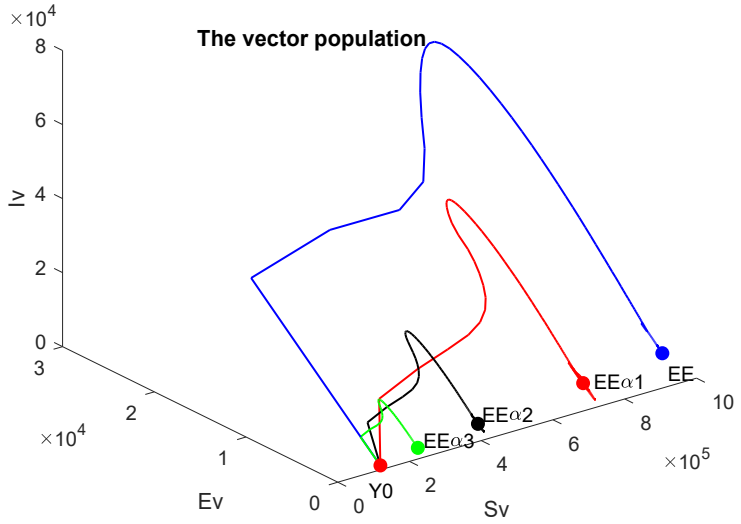

FIGURE 20. Impact of α on the EE with $\alpha_1 = 0.98$, $\alpha_2 = 0.94$ and $\alpha_3 = 0.9$.

Appendix

A.1. The human-mosquito model. The system (2.8) has a unique endemic equilibrium point that exists whenever $R_0^\alpha > 1$ and it is given by

$$
\begin{aligned}
S_h^* &= \frac{\Lambda_h^\alpha N_{\alpha,v}^*}{B^\alpha \beta_{vh} I_v^* + \mu_h^\alpha N_{\alpha,v}^*},\\
E_h^* &= \frac{B^\alpha \beta_{vh} \Lambda_h^\alpha I_v^*}{(\nu_h^\alpha + \mu_h^\alpha)(B^\alpha \beta_{vh} I_v^* + \mu_h^\alpha N_{\alpha,v}^*)},\\
R_h^* &= \frac{\eta_h^\alpha}{\mu_h^\alpha} I_h^*,\\
S_v^* &= \frac{\Lambda_v^\alpha N_{\alpha,h}^*}{B^\alpha \beta_{hv} I_h^* + \mu_v^\alpha N_{\alpha,h}^*},\\
E_v^* &= \frac{B^\alpha \beta_{hv} \Lambda_v^\alpha I_h^*}{(\mu_v^\alpha + \nu_v^\alpha)(B^\alpha \beta_{hv} I_h^* + \mu_v^\alpha N_{h,\alpha}^*)},\\
I_v^* &= \frac{\nu_v^\alpha E_v^*}{\mu_v^\alpha},\\
I_h^* &= \frac{\Lambda_h^\alpha \mu_v^\alpha (\mu_v^\alpha + \nu_v^\alpha)\big((R_0^\alpha)^2 - 1\big)}{B^\alpha \beta_{hv}\big(\mu_h^\alpha(\mu_v^\alpha + \nu_v^\alpha) + \nu_v^\alpha B^\alpha \beta_{vh}\big)}.
\end{aligned}
$$

A.2. The human-mosquito-monkey model. The system (2.6) has two equilibrium points, the disease-free equilibrium $DFE = (\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_v}{\mu_v}, 0, 0, \frac{\Lambda_m}{\mu_m}, 0, 0, 0)^\top$ and the endemic equilibrium $EE = (S_h^{**}, E_h^{**}, I_h^{**}, R_h^{**}, S_v^{**}, E_v^{**}, I_v^{**}, S_m^{**}, E_m^{**}, I_m^{**}, R_m^{**})^\top$, where

$$
\begin{aligned}
S_h^{**} &= \frac{\Lambda_h N_v^*}{B\beta_{vh} I_v^{**} + \mu_h N_v^*},\\
E_h^{**} &= \frac{B\beta_{vh}\Lambda_h I_v^{**}}{(\nu_h + \mu_h)(B\beta_{vh} I_v^{**} + \mu_h N_v^*)},\\
I_h^{**} &= \frac{\nu_h B\beta_{vh}\Lambda_h I_v^{**}}{(\eta_h + \mu_h)(\nu_h + \mu_h)(B\beta_{vh} I_v^{**} + \mu_h N_v^*)},\\
R_h^{**} &= \frac{\eta_h I_h^{**}}{\mu_h},\\
S_v^{**} &= N_v^* - \frac{(\mu_v + \nu_v) I_v^{**}}{\nu_v},\\
E_v^{**} &= \frac{\mu_v}{\nu_v} I_v^{**},\\
S_m^{**} &= N_m^* - \frac{(\nu_m + \mu_m) E_m^{**}}{\mu_m},\\
E_m^{**} &= \frac{B\beta_{vm} I_v^{**} \mu_m N_m^*}{(\nu_m + \mu_m)(\mu_m N_v^* + B\beta_{vm} I_v^{**})},\\
I_m^{**} &= \frac{\nu_m E_m^{**}}{(\eta_m + \mu_m)},\\
R_m^{**} &= \frac{\eta_m}{\mu_m} I_m^{**},
\end{aligned}
$$

I_v^{**} is implicitly given as the zero of the following rational fraction expression

$$P(I_v^{**}) = \frac{\mu_h\nu_h B\beta_{hv}B\beta_{vh}\big(\nu_v N_v^* - (\mu_v+\nu_v)I_v^{**}\big)}{(\eta_h+\mu_h)(\nu_h+\mu_h)(B\beta_{vh}I_v^{**}+\mu_h N_v^*)} + \frac{\mu_m\nu_m B\beta_{mv}B\beta_{vm}\big(\nu_v N_v^* - (\mu_v+\nu_v)I_v^{**}\big)}{(\eta_m+\mu_m)(\nu_m+\mu_m)(\mu_m N_v^*+B\beta_{vm}I_v^{**})} - \mu_v(\mu_v+\nu_v),$$

which is determined numerically. The basic reproduction number of (2.6) is

$$R_0 = \sqrt{R_0^{hv}+R_0^{mv}},$$

where

$$R_0^{hv} = \frac{\nu_v\nu_h B^2\beta_{vh}\beta_{hv}}{\mu_v(\mu_v+\nu_v)(\mu_h+\eta_h)(\mu_h+\nu_h)},$$

and

$$R_0^{mv} = \frac{\nu_v\nu_m B^2\beta_{mv}\beta_{vm}}{\mu_v(\mu_v+\nu_v)(\mu_m+\nu_m)(\mu_m+\eta_m)}.$$

A.3. The time-fractional model. The proof of the theorem requires the following lemma :

LEMMA A.1. *If $X^0, X^1, \dots, X^n \geq 0$ then*

$$h^{1-\alpha}X^n - \sum_{j=0}^{n-1}\Delta_{\alpha,n}^j\big(X^{j+1}-X^j\big) \geq 0.$$

PROOF. For $n \in \mathbb{N}^*$, we have

$$h^{1-\alpha}X^n - \sum_{j=0}^{n-1}\Delta_{\alpha,n}^j\big(X^{j+1}-X^j\big) = \big(h^{1-\alpha}-\Delta_{\alpha,n}^{n-1}\big)X^n + \Delta_{\alpha,n}^0X^0 + \sum_{j=1}^{n-1}\big(\Delta_{\alpha,n}^j-\Delta_{\alpha,n}^{j-1}\big)X^j.$$

and

$$h^{1-\alpha} - \Delta_{\alpha,n}^{n-1} = \big(2-2^{1-\alpha}\big)h^{1-\alpha} \geq 0.$$

Thus

$$h^{1-\alpha}X^n - \sum_{j=0}^{n-1}\Delta_{\alpha,n}^j\big(X^{j+1}-X^j\big) \geq 0$$

□

THEOREM A.2 (Positivity of solution). *Let the initial data $S_h^0, E_h^0, I_h^0, R_h^0, S_v^0, E_v^0$, and $I_v^0 \geq 0$, then all the components $S_h^{n+1}, E_h^{n+1}, I_h^{n+1}, R_h^{n+1}, S_v^{n+1}, E_v^{n+1}$, and $I_v^{n+1} \geq 0$ in the system (4.17) are satisfied for all $n \in \mathbb{N}$.*

PROOF. We have for $n = 0$

$$S_h^1 = \frac{h^{1-\alpha}S_h^0 + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\Lambda_h^\alpha}{\left(h^{1-\alpha}+\Gamma(2-\alpha)\phi_{\alpha,h}(h)\Big(B^\alpha\beta_{vh}\frac{I_v^0}{N_{\alpha,v}^0}+\mu_h^\alpha\Big)\right)} \geq 0,$$

$$E_h^1 = \frac{h^{1-\alpha}E_h^0 + \Gamma(2-\alpha)\phi_{\alpha,h}(h)B^\alpha\beta_{vh}\frac{I_v^0}{N_{\alpha,v}^0}S_h^1}{\big(h^{1-\alpha}+\Gamma(2-\alpha)\phi_{\alpha,h}(h)(\nu_h^\alpha+\mu_h^\alpha)\big)} \geq 0,$$

$$I_h^1 = \frac{h^{1-\alpha}I_h^0 + \Gamma(2-\alpha)\phi_{\alpha,h}(h)\nu_h^\alpha E_h^1}{\big(h^{1-\alpha}+\Gamma(2-\alpha)\phi_{\alpha,h}(h)(\eta_h^\alpha+\mu_h^\alpha)\big)} \geq 0,$$

$$R_h^1 = \frac{h^{1-\alpha}Rh^0 + \phi_{\alpha,h}(h)\Gamma(2-\alpha)\eta_h^\alpha I_h^1}{\big(h^{1-\alpha}+\phi_{\alpha,h}(h)\Gamma(2-\alpha)\mu_h^\alpha\big)} \geq 0,$$

$$S_v^1 = \frac{h^{1-\alpha} S_v^0 + \phi_{\alpha,v}(h)\Gamma(2-\alpha)\Lambda_v^\alpha}{\left(h^{1-\alpha} + \phi_{\alpha,v}(h)\Gamma(2-\alpha)(B^\alpha \beta_{hv} \frac{I_h^0}{N_{\alpha,h}^0} + \mu_v^\alpha)\right)} \geq 0,$$

$$E_v^1 = \frac{h^{1-\alpha} E_v^0 + \phi_{\alpha,v}(h)\Gamma(2-\alpha) B^\alpha \beta_{hv} \frac{I_h^0}{N_{\alpha,h}^0} S_v^1}{\left(h^{1-\alpha} + \phi_{\alpha,v}(h)\Gamma(2-\alpha)(\nu_v^\alpha + \mu_v^\alpha)\right)} \geq 0,$$

$$I_v^1 = \frac{h^{1-\alpha} I_v^0 + \phi_{\alpha,v}(h)\Gamma(2-\alpha)\nu_v^\alpha E_v^1}{\left(h^{1-\alpha} + \phi_{\alpha,v}(h)\Gamma(2-\alpha)\mu_v^\alpha\right)} \geq 0.$$

We suppose that for $1, 2, \ldots, n$, S_h^n, E_h^n, I_h^n, R_h^n, S_v^n, E_v^n and $I_v^n \geq 0$. The hypothesis of induction and Lemma A.1 allow for the statement for $n+1$, i.e.

$$S_h^{n+1}, E_h^{n+1}, I_h^{n+1}, R_h^{n+1}, S_v^{n+1}, E_v^{n+1}, \text{ and } I_v^{n+1} \geq 0.$$

□

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Department of Mathematics and Computer Science, Abdelhamid Ibn Badis University, Algeria
Email address: maghnia.hamoumaamar@gmail.com

Chair of Applied and Computational Mathematics, University of Wuppertal, Germany
Email address: ehrhardt@uni-wuppertal.de

Department of Mathematics and Computer Science, Abdelhamid Ibn Badis University, Algeria
Email address: louiza.tabharit@univ-mosta.dz