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A numerical study of the impact of variance boundary conditions for the Heston model

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Abstract. The well-posedness analysis of a parabolic partial differential equation (PDE), such as the Heston PDE, requires the proper definition of an initial condition and boundary conditions. In contrast to the asset boundary conditions, the variance boundary conditions cannot be directly derived. In the literature different approaches to the variance boundary conditions are discussed, for example they consider the challenge of singularities when the variance approaches zero. This work focuses on the sensitivity of numerical approximations of the solution with respect to the variance boundary conditions.

Keywords: Computational finance, Heston model, boundary conditions, Put option

1 Introduction

This paper discusses different variance boundary conditions for the Heston model and compares them to the closed form solution proposed by Heston. We emphasize that the question of the correct boundary condition is usually avoided by using large spatial domains, highly non-uniform grids, as well as windowing.

However, as we aim to derive a gradient descent algorithm in the long term, which requires multiple solves of the PDE, these strategies are computationally too costly and hence not practical for us. This motivates the following numerical study of the influence of the boundary conditions.

In Section 2, we introduce the Heston model and the various variance boundary conditions, before we describe our approach to solving the Heston PDE in Section 3. In Section 4, we focus on discussing the effect of the different boundary conditions with respect to ν for different test cases. We conclude and give an outlook to future work in Section 5.

2 The Heston model

The Heston model was developed by Steven L. Heston in 1993 and describes the dynamics of the underlying asset through a two-dimensional stochastic differential equation (SDE) involving a stochastic process for the underlying asset S and a Cox-Ingersoll process for the variance ν, the square of the volatility of the asset [2]. The Heston PDE with a risk-neutral measure of the fair price of a vanilla put option $V(S,\nu,t)$ is given on $S>0$, $\nu>0$ by

$$V_t + \frac{S^2}{2}\nu V_{SS} + \frac{\sigma^2}{2}\nu V_{\nu\nu} + \rho\sigma\nu S V_{S\nu} + rSV_S + \kappa(\mu-\nu)V_\nu - rV = 0, \tag{1}$$

where V is the fair price, r is the risk-free interest rate and κ is the mean reversion rate, μ is the long-term mean and σ is the volatility of the variance. The terminal condition is given by the payoff function of the put option

$$\phi(S) = \max(K-S, 0) \tag{2}$$

and Heston proposed the following boundary conditions (BCs)

$$S = 0: \quad V = K\exp\bigl(-r(T-t)\bigr), \tag{3}$$
$$S \to \infty: \quad V = 0, \tag{4}$$
$$\nu = 0: \quad V_t + rSV_S + \kappa\mu V_\nu - rV = 0, \tag{5}$$
$$\nu \to \infty: \quad V = K\exp\bigl(-r(T-t)\bigr). \tag{6}$$

To obtain an initial condition, we apply a time reversal $\tau = T-t$. For the variance process to be positive, the *Feller condition* $2\kappa\mu \geq \sigma^2$ must be satisfied. At $S=0$ and $\nu=0$, the diffusion terms vanish and we use Fichera theory to determine the necessity of BCs [1, 4]. Therefore we rewrite the Heston PDE

$$V_\tau = \frac{1}{2}S^2\nu V_{SS} + \frac{1}{2}\sigma^2\nu V_{\nu\nu} + \rho\sigma\nu S V_{S\nu} - rSV_S - \kappa(\mu-\nu)V_\nu - rV \tag{7}$$

in divergence form

$$V_t - \nabla\cdot A\nabla V + \vec{b}\nabla V - rV = 0 \tag{8}$$

with

$$\vec{b} = -\begin{pmatrix} rS - \nu S - \frac{1}{2}\rho\sigma S \\ \kappa(\mu-\nu) - \frac{1}{2}\sigma^2 - \frac{1}{2}\rho\sigma\nu \end{pmatrix}, \quad A = \frac{1}{2}\nu\begin{pmatrix} S^2 & \rho\sigma S \\ \rho\sigma S & \sigma^2 \end{pmatrix}. \tag{9}$$

At the boundary $S=0$ the *Fichera condition* is given by

$$\lim_{S\to 0^+} \vec{b}\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \left(rS - \nu S - \frac{1}{2}\rho\sigma S\right) = 0.$$

This corresponds to an (unconditional) outflow boundary and thus no boundary condition is required from an analytical perspective. Considering the boundary $\nu=0$, the *Fichera condition* is given by

$$f(\nu) = \lim_{\nu\to 0^+} \vec{b}\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \lim_{\nu\to 0^+}\left(\kappa(\mu-\nu) - \frac{1}{2}\sigma^2 - \frac{1}{2}\rho\sigma\nu\right) = \kappa\mu - \frac{1}{2}\sigma^2. \tag{10}$$

Hence, we have the following cases at $\nu=0$:

- outflow boundary: if $f(\nu)\geq 0$ we must not supply any BCs at $\nu=0$.
- inflow boundary: if $f(\nu)<0$ we have to supply BCs at $\nu=0$.

We obtain an outflow boundary if and only if the Feller condition is satisfied, which is assumed below. Nevertheless, for the implementation we need to specify a numerical closure condition for $S = 0$ and $\nu = 0$. In order to avoid degenerating coefficients, we consider the log-transformed normalized Heston PDE using the transformation $x = \log(S/K)$. This leads to

$$V_\tau = \frac{\nu}{2}V_{xx} + \frac{1}{2}\sigma^2\nu V_{\nu\nu} + (r - \frac{\nu}{2})V_x + \kappa(\mu-\nu)V_\nu + \sigma\nu\rho V_{x\nu} - rV. \tag{11}$$

Note that due to the transformation, there is no singularity at $x \to -\infty$, hence we have to supply a boundary condition anyway.

Analogous to the Heston boundary conditions, the analytical boundary conditions for the transformed Heston PDE are given by

$$V(-\infty,\nu,\tau) = \exp(-r\tau), \quad V(\infty,\nu,\tau) = 0, \quad V(x,\infty,\tau) = \exp(-r\tau). \tag{12}$$

Note that the BC at $x \to \infty$ must be interpolated w.r.t. the variance BCs.

3 Discretization

As we focus on the discussion of BCs, simple and well-known spatial and temporal discretization methods are used to present the approach. We perform a domain truncation to obtain a rectangular grid instead of a semi-unbounded domain. We consider uniform meshes in each direction and obtain $x_i = x_{\min} + i\Delta_x$ for $i = 0,\dots,N_x$ with $\Delta_x = (x_{\max} - x_{\min})/N_x$ and $\nu_j = j\Delta_\nu$ for $j = 0,\dots,N_\nu$ with $\Delta_\nu = \frac{\nu_{\max}}{N_\nu}$, as well as $\tau_k = k\cdot\Delta_\tau$ for $k = 0,\dots,N_\tau$ with time step $\Delta_\tau = \frac{T}{N_\tau}$.

For the spatial discretization we use finite differences and for the temporal discretization the well-known alternating direction implicit (ADI) method, together this yields the second order convergent Hundsdorfer-Verwer scheme [3]. As part of the discretization process, we again need to discuss the BCs. For the closure condition at $\nu = \nu_{\min}$ we consider two different cases. The Dirichlet BC

$$V(x,\nu_{\min},\tau) = \phi(x)\exp(-r\tau) \tag{α}$$

and, since we have a pure outflow boundary here, an extrapolation via a ghost layer at $\nu = \nu_{\min} - \Delta_\nu$ leading to

$$V(x,\nu_{\min}-\Delta_\nu,\tau) = V(x,\nu_{\min},\tau). \tag{β}$$

For the boundary conditions at $\nu = \nu_{\max}$, we consider four different cases. The first condition was proposed by Heston himself

$$V(x,\nu_{\max},\tau) = \exp(-r\tau). \tag{a}$$

Using this BC, a jump between $V(x_{\max},\nu,\tau)$ and $V(x_{\max},\nu_{\max},\tau)$ accrue. Therefore, one can use linear interpolation, considered in the case (a) or an exponential fit at $x_{\max}$ [5] with parameter $c > 0$ given by

$$a = 1 + b, \quad \text{and} \quad b = \frac{\big(\exp(\nu-\nu_{\max})\big)^c}{1-\big(\exp(\nu-\nu_{\max})\big)^c}, \tag{13}$$

which leads to the condition

$$V(x,\nu_{\max},\tau) = \exp(-r\tau), \quad V(x_{\max},\nu,\tau) = a(\exp(\nu-\nu_{\max}))^c - b. \qquad \text{(b)}$$

Note that a high value for c corresponds to a slope at which the option price is zero for most variance values, so the fit to Heston's BC is better. In the following, we use the rather large value $c = 20$. Another approach is to incorporate a dependence on x in the Heston condition [5] for example

$$V(x,\nu_{\max},\tau) = \exp(-r\tau)\Big(1 - \frac{\exp(x) - \exp(x_{\min})}{2\big(\exp(x_{\max}) - \exp(x_{\min})\big)}\Big) \qquad \text{(c)}$$

combined with a linear interpolation in $x_{\max}$.

The final approximation we consider at $\nu = \nu_{\max}$, was proposed by Kùtik and Mikula [4] and uses artificial homogeneous Neumann BCs. The choice is motivated by the independence from the variance of Heston's original BC, given a sufficiently large $\nu_{\max} = \mathcal{O}(1)$. Thus, we perform an extrapolation over the ghost layer at $\nu = \nu_{\max} + \Delta_\nu$ to obtain

$$V(x,\nu_{\max},\tau) = V(x,\nu_{\max}+\Delta_\nu,\tau). \qquad \text{(d)}$$

Table 1 summarizes the different boundary cases. The initial condition has to be adjusted w.r.t. the boundary conditions via interpolation.

Boundary cases	$\nu_{\min}$	$\nu_{\max}$	$x_{\min}$	$x_{\max}$	Boundary cases	$\nu_{\min}$	$\nu_{\max}$	$x_{\min}$	$x_{\max}$
B1	α	a	a	$\exp(-r\tau_k)$	B5	β	a	a	$\exp(-r\tau_k)$
B2	α	b	b	$\exp(-r\tau_k)$	B6	β	b	b	$\exp(-r\tau_k)$
B3	α	c	c	$\exp(-r\tau_k)$	B7	β	c	c	$\exp(-r\tau_k)$
B4	α	d	d	$\exp(-r\tau_k)$	B8	β	d	d	$\exp(-r\tau_k)$

Table 1: Different test cases for the boundary conditions for the Heston model.

4 Numerical Results

We compare the numerical results with Heston's closed-form solution [2] by calculating the mean squared error (MSE) over the entire domain including the boundary itself. For the simulation, we consider two different parameter sets denoted by P1 and P2, see Table 2 and five different grids resulting from Table 3. Note that P1 is taken from [4] with strike K set to 1.

Parameter case	$x_{\min}$	$x_{\max}$	$\nu_{\min}$	$\nu_{\max}$	θ	T	K	r	σ	μ	κ	ρ
P1 [4]	-7	3	0.01	1	0.75	0.05	1	0.1	0.5	0.07	5	-0.5
P2 [5]	-7	3	0.01	1	0.75	1	1	0.05	0.3	0.2	2	-0.5

Table 2: Parameter sets.

Discretization set	D1	D2	D3	D4	D5
N_x	20	40	80	160	320
N_ν	10	20	40	80	160
N_τ	1	4	16	64	256

Table 3: Different discretization grids.

The numerical results show that all considered boundary cases converge with an order of two, see Figure 1. The plot shows that the influence of the different parameter sets is small. Cases B4 and B8 give the best results, i.e., using the extrapolation at $\nu_{\max}$ approximates the solution better than the Dirichlet BCs.

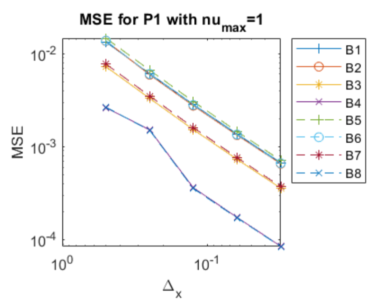

Fig. 1: Visualisation for the MSE presented in Table 4.

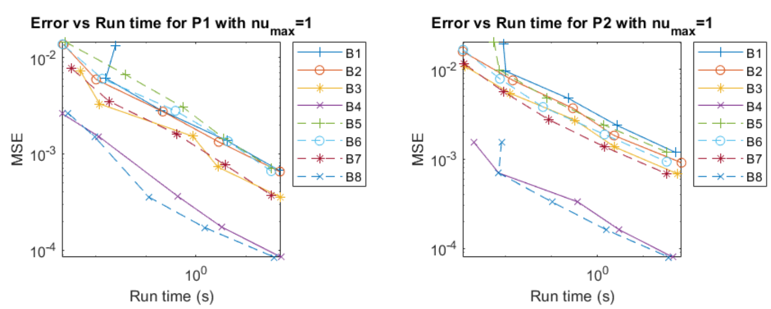

Fig. 2: MSE vs run time (s) for the numerical results in Table 2.

Using the extrapolation for $\nu_{\min}$ as well further improves the MSE, see Table 4. The increased computational cost can be neglected, as Figure 2 shows.

If one is restricted to using Dirichlet-type BCs, the best choice for $\nu_{\max}$ is the BC proposed by Heston in combination with an exponential fit of $x_{\max}$.

P1	D1	D2	D3	D4	D5	P2	D1	D2	D3	D4	D5
B1	13.305	6.092	2.839	1.367	0.671	B1	19.364	9.582	4.768	2.380	1.189
B2	13.666	5.924	2.751	1.331	0.655	B2	15.972	7.586	3.698	1.827	0.909
B3	7.224	3.300	1.523	0.731	0.358	B3	10.885	5.439	2.720	1.361	0.681
B4	**2.618**	**1.514**	**0.362**	**0.174**	**0.085**	**B4**	**1.532**	**0.698**	**0.331**	**0.162**	**0.080**
B5	14.630	6.635	3.051	1.456	0.711	B5	20.544	9.928	4.862	2.404	1.195
B6	13.746	6.029	2.803	1.349	0.662	B6	16.695	7.882	3.817	1.878	0.932
B7	7.720	3.502	1.604	0.763	0.372	B7	11.577	5.637	2.773	1.374	0.684
B8	**2.650**	**1.500**	**0.355**	**0.172**	**0.085**	**B8**	**1.531**	**0.697**	**0.331**	**0.162**	**0.080**

Table 4: MSE scaled by 10^3 between the semi-analytical solution from Heston and the approximation using the different boundary conditions (Table 1) for the different grids responding to Table 3 and the two parameter cases from Table 2.

5 Conclusion and Outlook

To conclude we note that the extrapolation BC for the variance has the smallest error in terms of the run time, it is therefore feasible to use this condition. Especially, if the solution depends on the whole domain and not on a single value in it. If one is limited to Dirichlet-type BCs due to the numerical method, e.g., sparse grids, one should use B3.

An artificial boundary condition for the variance would be advantageous if one is restricted to small computational domains. Hence finding the right boundary conditions for the variance in the Heston model, especially when the Feller condition is not satisfied, is subject to future research.

References

1. Z. Bučková, M. Ehrhardt, M. Günther. Fichera Theory and Its Application in Finance. In: Russo, G., Capasso, V., Nicosia, G., Romano, V. (eds) Progress in Industrial Mathematics at ECMI 2014. ECMI 2014. Mathematics in Industry, vol 22. Springer, Cham, 2016
2. S.L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, Rev. Fin. Stud. 6(2) (1993), 327-343
3. W. Hundsdorfer. Accuracy and Stability of Splitting with Stabilizing Corrections, Appl. Numer. Math., 42(1-3) (2002), 213-233
4. P. Kútik, K. Mikula. Diamond-cell finite volume scheme for the Heston model, DCDS-S, 8(5) (2015), 913-931
5. S. Maurus. A multi-dimensional PDE solver for option pricing based on the Heston model and sparse grids, Master Thesis, TU Munich, 2012