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A second-order nonstandard finite difference method for a general Rosenzweig-MacArthur predator-prey model

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Abstract

In this paper, we consider a general Rosenzweig-MacArthur predator-prey model with logistic intrinsic growth of the prey population. We develop the Mickens' method to construct a dynamically consistent second-order nonstandard finite difference (NSFD) scheme for the general Rosenzweig-MacArthur predator-prey model. The second-order NSFD method is based on a novel nonlocal approximation using right-hand side function weights and nonstandard denominator functions.

Through rigorous mathematical analysis, we show that the NSFD method not only preserves two important and prominent dynamical properties of the continuous model, namely positivity and asymptotic stability independent of the values of the step size, but also is convergent of order 2. Therefore, it provides a solution to the contradiction between the dynamic consistency and high-order accuracy of NSFD methods.

The proposed NSFD method improves positive and elementary stable nonstandard numerical schemes constructed in a previous work of Dimitrov and Kojouharov, [Journal of Computational and Applied Mathematics 189 (2006) 98-108]. Moreover, the present approach can be extended to construct second-order NSFD methods for some classes of nonlinear dynamical systems. Finally, the theoretical insights and advantages of the constructed NSFD

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scheme are supported by some illustrative numerical simulations.

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1. Introduction

In the 1980s, Mickens proposed nonstandard finite difference methods (NSFD) to overcome a serious drawback of standard numerical methods, the so-called "numerical instability" [31, 32, 33, 34, 35, 36, 40, 41]. One of the outstanding and prominent advantages of NSFD methods over standard methods is their dynamic consistency, i.e., they can correctly preserve the dynamic properties of differential equations for all values of the step size. NSFD methods have been intensively studied in the last decades and have become powerful and efficient numerical methods for differential equations.

However, most of the existing dynamically consistent NSFD methods are convergent only up to the first order (see, e.g., [8, 15, 16, 19, 20, 21, 22, 28, 39, 45]), which can be considered as an inherent drawback of NSFD methods. For this reason, the problem of improving the accuracy of NSFD methods has attracted the attention of many researchers, and consequently, higher-order NSFD methods have been proposed for some classes of nonlinear dynamical systems using different approaches (see, for example, [7, 10, 18, 23, 24, 27, 29, 30]). More recently, some classes of second-order NSFD methods for general one-dimensional autonomous dynamical systems were presented in [18, 19, 27]. However, higher-order NSFD methods for nonlinear dynamical systems are still an important unsolved problem.

Motivated and inspired by the above reason, in this paper we consider a general Rosenzweig-MacArthur predator-prey model with a logistic intrinsic growth of the prey population (see [6, p. 182]) of the form

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= f_1(x,y) := bx(1-x) - ag(x)xy, \quad x(0) = x_0 \geq 0,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= f_2(x,y) := g(x)xy - dy, \quad y(0) = y_0 \geq 0,
\end{aligned} \tag{1}$$

where

- x and y stand for the prey and predator population sizes, respectively;

- $b > 0$ is the intrinsic growth rate of the prey;
- $a > 0$ represents the capturing rate;
- $d > 0$ is the predator death rate;
- the function $g(x)$ is assumed to satisfy $xg(x)$ is bounded as $x \to \infty$ and

$$g(x) \geq 0, \quad g'(x) \leq 0, \quad [xg(x)]' \geq 0.$$

We refer readers to [6] for more details of the model (1). It is easy to see that the model (1) admits the set $\mathbb{R}_2^+ = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$ as a positive invariant set. On the other hand, the equilibria of the model (1) and their local asymptotic stability (or linear stability) in [12] were given as follows

(i) A trivial equilibrium point $E_0 = (0, 0)$ always exists, and it is also always unstable.

(ii) A boundary equilibrium point $E_1 = (1, 0)$ always exists, and it is locally asymptotically stable if $g(1) < d$ and is unstable if $g(1) > d$.

(iii) A unique positive (interior) equilibrium point $E^* = (x^*, y^*)$ exists if and only if $g(1) > d$, where

$$x^* g(x^*) = d, \qquad y^* = \frac{bx^*(1 - x^*)}{ad}.$$

Furthermore, it is locally asymptotically stable if $b + ay^* g'(x^*) > 0$ and is unstable when $b + ay^* g'(x^*) < 0$.

Dimitrov and Kojouharov [12] constructed *positive and elementary stable nonstandard* (PESN) schemes for the model (1) in the following form

$$\begin{aligned} \frac{x_{k+1} - x_k}{\phi(\Delta t)} &= b\, x_k - b\, x_{k+1}\, x_k - a\, g(x_k)\, x_{k+1}\, y_k, \\ \frac{y_{k+1} - y_k}{\phi(\Delta t)} &= g(x_k)\, x_k\, y_k - d\, y_{k+1}, \end{aligned} \tag{2}$$

where Δt is the step size, $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ as $\Delta t \to 0$ and $0 < \phi(\Delta t) < 1$ for $\Delta t > 0$, x_k and y_k are the approximations of $x(t_k)$ resp. $y(t_k)$ for $k \geq 1$. Then, easily verifiable conditions for the denominator function $\phi(\Delta t)$ are determined so that the NSFD method (2) is PESN (see [12, Section 3]). The NSFD method (2) can be viewed as a special case of a general class of NSFD methods introduced in [8]. Therefore, it is easy to show that it is convergent only of order one (see [8, Theorem 5.2] and [8, Appendix B]).

In this work, we introduce a simple approach, different from the approaches to formulate higher-order NSFD methods used in the above works, to construct a second-order NSFD method that preserves the positivity and asymptotic stability of the model (1). First, a novel nonlocal approximation with weights for the right-hand side functions is used. Then, conditions for the dynamic consistency of the NSFD method are imposed on the weights. Finally, the nonstandard denominator functions in the discretization of the first-order derivatives are renormalized to ensure that the NSFD method is convergent of order two. Thus, a dynamically consistent second-order NSFD method is obtained. The proposed NSFD method improves the PESN methods constructed in [12], and in particular, the present approach can be very useful in constructing higher-order NSFD methods for some classes of nonlinear dynamical systems.

The plan for this work is as follows. In Section 3, some preliminary remarks and auxiliary results are presented. The second-order NSFD method is proposed and analyzed in Section 3. In Section 4 we report a set of illustrative numerical experiments. Some remarks on generalized versions of the constructed NSFD method are given in Section 5. Finally, concluding remarks and some open problems are discussed in the last section.

2. Preliminaries and auxiliary results

Consider an initial value problem for an autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \qquad y(0) = y_0 \in \mathbb{R}^n, \tag{3}$$

where the right hand side $f(y)$ satisfies a Lipschitz condition (with the Lipschitz constant L^C) in order to guarantee a unique solution. For solving (3) we consider a general one-step numerical method that approximates solutions of (3) in the form

$$D_{\Delta t}(y_k) = F_{\Delta t}(f; y_k), \tag{4}$$

where $\Delta t > 0$ is the step size, y_k is the approximation of $y(t_k)$, $t_k = k\Delta t$, $k \geq 0$. $D_{\Delta t}$ and $F_{\Delta t}$ denote the approximations of dy/dt and $f(y)$, respectively.

Since any non-autonomous system can be written as an autonomous system, we can restrict ourselves to $f(y)$ here for simplicity.

The following concept of an NSFD scheme is derived from Mickens' methodology [31, 32, 33, 34, 35, 36].

Definition 1 ([2, 14])**.** *The finite difference scheme* (4) *is called an NSFD scheme if at least one of the following conditions is satisfied:*

- $D_{\Delta t}(y_k) = \dfrac{y_{k+1} - y_k}{\phi(\Delta t)}$, *where* $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ *is a non-negative function and is called a nonstandard denominator function;*
- $F_{\Delta t}(f; y_k) = g(y_k, y_{k+1}, \Delta t)$, *where* $g(y_k, y_{k+1}, \Delta t)$ *is a nonlocal approximation of the right-hand side of the system* (3)*.*

Next, we define the notion of dynamically consistent scheme.

Definition 2 ([2, 3, 36])**.** *Let us consider the differential equation* $dy/dt = f(y)$*. Let a finite difference scheme for this equation be* $y_{k+1} = F(y_k; \Delta t)$*. Let the differential equation and/or its solutions have the property* $\mathcal{P}$*. The discrete model equation is dynamically consistent with the differential equation if it and/or its solutions also have the property* $\mathcal{P}$*.*

In practice, these aforementioned properties $\mathcal{P}$ are diverse, e.g. positivity, equilibria and their stability, boundedness, conservation laws, physical properties, periodicity, etc.

Before ending this section, we present some results on the stability of equilibria of time-continuous and discrete-time dynamical systems.

Definition 3 ([26, 44])**.** *The equilibrium point* $y^* = 0$ *of* (3) *is said to be:*

- *stable, if, for each* $\epsilon > 0$*, there is* $\delta = \delta(\epsilon)$ *such that*
$$\|y(0)\| < \delta \quad \textit{implies that} \quad \|y(t)\| < \epsilon, \quad \forall t \geq 0;$$
- *unstable if it is not stable;*
- *(locally) asymptotically stable if it is stable and* δ *can be chosen such that*
$$\|y(0)\| < \delta \quad \textit{implies that} \quad \lim_{t \to \infty} y(t) = 0.$$

Let us consider general discrete-time dynamical systems defined by first-order difference equations of the form

$$Y_{k+1} = F(Y_k), \qquad Y_0 \in \mathbb{R}^n. \tag{5}$$

The stability concepts for equilibria of discrete-time dynamical systems are defined in the same way as in Definition 3.

First, we give a well-known characterization of locally asymptotically stable equilibria.

Theorem 1 ([17, 44])**.** *Let $F \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Then a fixed point Y^* of the system* (5) *is locally asymptotically stable if the eigenvalues of the Jacobian matrix $dF(Y^*)$ are strictly inside the unit circle. If one of the eigenvalues lies outside the unit circle, the fixed point is unstable.*

The following result is a direct consequence of the Jury conditions or Schur-Cohn criteria (see [1, Theorem 2.13]). It is very useful in the analysis of the asymptotic stability of equilibria of discrete-time two-dimensional dynamical systems.

Theorem 2. *Given the polynomial*

$$p(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

where a_1 and a_2 are real numbers. Then, the solutions λ_1 and λ_2 of the equation $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$ if and only if

$$|a_1| < 1 + a_2 < 2.$$

3. Construction of the second-order NSFD method

We now construct a second-order NSFD scheme for the model (1) and discuss its properties. First, we apply Mickens' method [31, 33, 34, 35] to discretize the differential equation model (1) as follows

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big|_{t=t_k} \approx \frac{x_{k+1} - x_k}{\phi_1(\Delta t, x_k, y_k)}, \qquad \frac{\mathrm{d}y(t)}{\mathrm{d}t}\Big|_{t=t_k} \approx \frac{y_{k+1} - y_k}{\phi_2(\Delta t, x_k, y_k)} \tag{6}$$

and

$$\begin{aligned} &b\,x(t_k)\,(1 - x(t_k)) - a\,g(x(t_k))\,x(t_k)\,y(t_k) \\ &\qquad\qquad \approx b\,x_k - b\,x_{k+1}\,x_k - a\,g(x_k)\,x_{k+1}\,y_k + w_1\,x_k - w_1\,x_{k+1}, \\ &g(x(t_k))\,x(t_k)\,y(t_k) - d\,y(t_k) \approx g(x_k)\,x_k\,y_k - d\,y_{k+1} + w_2\,y_k - w_2\,y_{k+1}, \end{aligned} \tag{7}$$

with the denominator functions $\phi_i(\Delta t, x, y)\colon \mathbb{R}^3_+ \to \mathbb{R}_+$, $i = 1, 2$ satisfying

$$\phi_i(\Delta t, x, y) = \Delta t + \mathcal{O}(\Delta t^2) \text{ as } \Delta t \to 0, \quad \phi_i(\Delta t, x, y) > 0 \text{ for } \Delta t > 0,\ x, y \geq 0.$$

In the nonstandard discretization of the right-hand side (7) $w_1, w_2 \in \mathbb{R}$ play a role as weights in the discretization of the zero function, namely, 0 can be discretized as $0 = z - z \to wz_k - wz_{k+1}$.

In summary, the discretizations (6) and (7) lead to the following NSFD model

$$\begin{aligned}
\frac{x_{k+1} - x_k}{\phi_1(\Delta t, x_k, y_k)} &= b\, x_k - b\, x_{k+1}\, x_k - a\, g(x_k)\, x_{k+1}\, y_k + w_1\, x_k - w_1\, x_{k+1}, \\
\frac{y_{k+1} - y_k}{\phi_2(\Delta t, x_k, y_k)} &= g(x_k)\, x_k\, y_k - d\, y_{k+1} + w_2\, y_k - w_2\, y_{k+1}.
\end{aligned} \tag{8}$$

Remark 1. *A key difference between the NSFD method* (8) *and most NSFD methods constructed in previous works, including the NSFD model* (2)*, is that the nonstandard denominator functions ϕ_i depend not only on Δt but also on the solution (x_k, y_k) and the appearance of the weights w_1 and w_2.*

As will be seen later, the denominator functions will ensure the convergence of order 2 while the weights guarantee the dynamic consistency of the NSFD method (7).

Remark 2. *The NSFD model* (2) *is a special case of* (8) *with $w_1 = w_2 = 0$ and $\phi_1(\Delta t, x_k, y_k) = \phi_2(\Delta t, x_k, y_k) = \phi(\Delta t)$.*

The price one has to pay for the higher order is usually different denominator functions, so that one loses a possibly existing conservation property of the overall system, which can be quite important in some applications and also directly provides the stability of the scheme.

Next, we show that the proposed scheme yields positive solutions for positive initial data.

Theorem 3 (Positivity of the NSFD method)**.** *Let $w_1, w_2 \in \mathbb{R}$ satisfying $w_1, w_2 \geq 0$. Then, the NSFD model* (8) *is dynamically consistent with respect to the positivity of the model* (1) *for all the values of the step size Δt, that is, $x_k, y_k \geq 0$ for all $k \geq 1$ whenever $x_0, y_0 \geq 0$.*

Proof. This theorem is proved by mathematical induction. Namely, the system (8) can be transformed into the explicit form

$$
\begin{aligned}
x_{k+1} &= \frac{x_k + \phi_1(\Delta t, x_k, y_k) b x_k + \phi_1(\Delta t, x_k, y_k) w_1 x_k}{1 + \phi_1(\Delta t, x_k, y_k)(b x_k + a g(x_k) y_k) + \phi_1(\Delta t, x_k, y_k) w_1},\\
y_{k+1} &= \frac{y_k + \phi_2(\Delta t, x_k, y_k) g(x_k) x_k y_k + \phi_2(\Delta t, x_k, y_k) w_2 y_k}{1 + \phi_2(\Delta t, x_k, y_k) d + \phi_2(\Delta t, x_k, y_k) w_2}.
\end{aligned}
\tag{9}
$$

Hence, we deduce that if $x_k \geq 0$ and $y_k \geq 0$ then $x_{k+1} \geq 0$ and $y_{k+1} \geq 0$. This concludes the proof. $\square$

We now analyze the asymptotic stability of the NSFD model (8). Note that the system (9) can be rewritten in the form

$$
\begin{aligned}
x_{k+1} &= x_k + \phi_1(\Delta t, x_k, y_k) \frac{f_1(x_k, y_k)}{1 + \phi_1(\Delta t, x_k, y_k)(b x_k + a g(x_k) y_k) + \phi_1(\Delta t, x_k, y_k) w_1},\\
y_{k+1} &= y_k + \phi_2(\Delta t, x_k, y_k) \frac{f_2(x_k, y_k)}{1 + \phi_2(\Delta t, x_k, y_k) d + \phi_2(\Delta t, x_k, y_k) w_2},
\end{aligned}
\tag{10}
$$

which implies that the sets of equilibria of the NSFD model (8) and the continuous model (1) are identical. On the other hand, if $\mathcal{E}_0 = (x_0^*, y_0^*)$ is an equilibrium point of the model (8), then it follows from (10) that the Jacobian matrix of (8) evaluating at $\mathcal{E}_0$ is given by

$$
J^D(\mathcal{E}_0) = \begin{pmatrix}
1 + \dfrac{\phi_1(\Delta t, x_0^*, y_0^*) J_{11}^C(\mathcal{E}_0)}{1 + \phi_1(\Delta t, x_0^*, y_0^*)(b x_0^* + a g(x_0^*) y_0^*) + \phi_1(\Delta t, x_0^*, y_0^*) w_1} & \dfrac{\phi_1(\Delta t, x_0^*, y_0^*) J_{12}^C(\mathcal{E}_0)}{1 + \phi_1(\Delta t, x_0^*, y_0^*)(b x_0^* + a g(x_0^*) y_0^*) + \phi_1(\Delta t, x_0^*, y_0^*) w_1}\\
\dfrac{\phi_2(\Delta t, x_0^*, y_0^*) J_{21}^C(\mathcal{E}_0)}{1 + \phi_2(\Delta t, x_0^*, y_0^*) d + \phi_2(\Delta t, x_0^*, y_0^*) w_2} & 1 + \dfrac{\phi_2(\Delta t, x_0^*, y_0^*) J_{22}^C(\mathcal{E}_0)}{1 + \phi_2(\Delta t, x_0^*, y_0^*) d + \phi_2(\Delta t, x_0^*, y_0^*) w_2}
\end{pmatrix},
\tag{11}
$$

where $J^C = (J_{ij}^C)$ is the Jacobian of the continuous system (1) at the equilibrium $\mathcal{E}_0$, that is,

$$
J^C(\mathcal{E}_0) = \begin{pmatrix}
b - 2b x_0^* - a y_0^* \big(x_0^* g'(x_0^*) + g(x_0^*)\big) & -a g(x_0^*) y_0^*\\
y_0 \big(x_0^* g'(x_0^*) + g(x_0^*)\big) & g(x_0^*) x_0^* - d
\end{pmatrix}.
$$

In this section, we always assume positive weights: $w_1, w_2 \geq 0$. The stability properties of the equilibrium points of the NSFD model (8) are determined as follows.

Theorem 4 (Stability analysis for the equilibria of the discrete system)**.**

(i) The trivial equilibrium point $E_0 = (0, 0)$ is always unstable.

(ii) If $g(1) > d$, then the boundary equilibrium point $E_1 = (1, 0)$ is unstable.

(iii) If $g(1) < d$, then the boundary equilibrium point $E_1 = (1, 0)$ is locally asymptotically stable.

(iv) Suppose that the equilibrium point E^* exists and $b + ay^* g'(x^*) > 0$. Let w_1 and w_2 be real numbers satisfying

$$\begin{aligned} w_1 &\geq w_1^* := x^* \frac{b + ay^* g'(x^*)}{2}, \\ w_2 &\geq w_2^* := \frac{ag(x^*) y^* (g(x^*) + x^* g'(x^*))}{b + ay^* g'(x^*)}. \end{aligned} \tag{12}$$

Then, E^* is locally asymptotically stable.

(v) If the equilibrium point E^* exists and $b + ay^* g'(x^*) < 0$, then it is unstable.

Proof. **Proof of Part (i):** We deduce from (11) that the Jacobian matrix of the system (8) evaluated at trivial equilibrium point $E_0 = (0, 0)$ reads

$$J^D(E_0) = \begin{pmatrix} 1 + b\phi_1(\Delta t, 0, 0) & 0 \\ 0 & 1 - d\phi_2(\Delta t, 0, 0) \end{pmatrix}.$$

Consequently, $J^D(E_0)$ has two eigenvalues $\lambda_1 = 1 + b\phi_1(\Delta t, 0, 0) > 1$ and $\lambda_2 = 1 - d\phi_2(\Delta t, 0, 0)$. This implies that E_0 is unstable.

Proof of Part (ii): The Jacobian matrix of the system (8) J^D evaluated at E_1 is given by

$$J^D(E_1) = \begin{pmatrix} 1 - \dfrac{b\phi_1(\Delta t, 1, 0)}{1 + \phi_1(\Delta t, 1, 0)b + \phi_1(\Delta t, 1, 0)w_1} & -\dfrac{ag(1)\phi_1(\Delta t, 1, 0)}{1 + \phi_1(\Delta t, 1, 0)b + \phi_1(\Delta t, 1, 0)w_1} \\ 0 & 1 + \dfrac{\phi_2(\Delta t, 1, 0)(g(1) - d)}{1 + \phi_2(\Delta t, 1, 0)d + \phi_2(\Delta t, 1, 0)w_2} \end{pmatrix}.$$

Hence, two eigenvalues of $J^D(E_1)$ are

$$\begin{aligned} \lambda_1 &= 1 - \frac{b\phi_1(\Delta t, 1, 0)}{1 + \phi_1(\Delta t, 1, 0)b + \phi_1(\Delta t, 1, 0)w_1}, \\ \lambda_2 &= 1 + \frac{\phi_2(\Delta t, 1, 0)(g(1) - d)}{1 + \phi_2(\Delta t, 1, 0)d + \phi_2(\Delta t, 1, 0)w_2}. \end{aligned}$$

Hence, if $g(1) > d$ then $\lambda_2 > 1$, which implies that E_1 is unstable.
Proof of Part (iii): It is easy to see that the first eigenvalue of $J^D(E_1)$ always satisfies $\lambda_1 \in (-1, 1)$. On the other hand, it follows from $g(1) \geq 0$ and $g(1) - d < 0$ that $\lambda_2 \in (-1, 1)$. Thus, E_1 is a stable equilibrium point.
Proof of Part (iv): We recall that the unique positive equilibrium point E^* exists if and only if $g(1) > d$. The Jacobian matrix of the system (8) J^D at E^* reads

$$J^D(E^*) = \begin{pmatrix} 1 - \dfrac{\phi_1^*(\Delta t)x^*(b + ay^*g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1} & -\dfrac{\phi_1^*(\Delta t)ag(x^*)x^*}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1} \\ \dfrac{\phi_2^*(\Delta t)y^*(g(x^*) + x^*g'(x^*))}{1 + \phi_2^*(\Delta t)d + \phi_2^*(\Delta t)w_2} & 1 \end{pmatrix}, \quad (13)$$

where $\phi_i^*(\Delta t) = \phi_i(\Delta t, x^*, y^*)$ for $i = 1, 2$. The characteristic polynomial of $J^D(E^*)$ is given by

$$\lambda^2 - \mathrm{Tr}(J^D)\lambda + \det(J^D). \quad (14)$$

Next, by using Theorem 2, we conclude that E^* is an asymptotically stable equilibrium point if

$$\det(J^D) < 1, \quad 1 - \mathrm{Tr}(J^D) + \det(J^D) > 0, \quad 1 + \mathrm{Tr}(J^D) + \det(J^D) > 0. \quad (15)$$

We now show that all the conditions of (15) are satisfied. Indeed, it is easy to see that for the determinant and the trace we have

$$\begin{aligned} \det(J^D) &= 1 - \frac{\phi_1^*(\Delta t)x^*(b + ay^*g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1} \\ &\quad + \frac{\phi_1^*(\Delta t)ag(x^*)x^*}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1}\frac{\phi_2^*(\Delta t)y^*(g(x^*) + x^*g'(x^*))}{1 + \phi_2^*(\Delta t)d + \phi_2^*(\Delta t)w_2} \\ \mathrm{Tr}(J^D) &= 2 - \frac{\phi_1^*(\Delta t)x^*(b + ay^*g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1}. \end{aligned} \quad (16)$$

Hence,

$$\begin{aligned} \det(J^D) - 1 &= -\frac{\phi_1^*(\Delta t)x^*(b + ay^*g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1} \\ &\quad + \frac{\phi_1^*(\Delta t)ag(x^*)x^*}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t)w_1}\frac{\phi_2^*(\Delta t)y^*(g(x^*) + x^*g'(x^*))}{1 + \phi_2^*(\Delta t)d + \phi_2^*(\Delta t)w_2}, \end{aligned} \quad (17)$$

which implies that $\det(J^D) - 1 < 0$ if

$$-w_2 x^*(b + ay^* g'(x^*)) + ax^* g(x^*) y^*(g(x^*) + x^* g'(x^*)) \leq 0. \tag{18}$$

It is clear that the inequality (18) is satisfied if

$$w_2 \geq \frac{ag(x^*)y^*(g(x^*) + x^* g'(x^*))}{b + ay^* g'(x^*)}. \tag{19}$$

Next, by using (16) we obtain

$$\begin{aligned} &1 - \mathrm{Tr}(J^D) + \det(J^D) \\ &= \frac{\phi_1^*(\Delta t) ag(x^*) x^*}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t) w_1} \frac{\phi_2^*(\Delta t) y^*(g(x^*) + x^* g'(x^*))}{1 + \phi_2^*(\Delta t) d + \phi_2^*(\Delta t) w_2} > 0. \end{aligned}$$

Lastly, since

$$\begin{aligned} 1 + \mathrm{Tr}(J^D) + \det(J^D) &= 4 - 2\frac{\phi_1^*(\Delta t) x^*(b + ay^* g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t) w_1} \\ &+ \frac{\phi_1^*(\Delta t) ag(x^*) x^*}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t) w_1} \frac{\phi_2^*(\Delta t) y^*(g(x^*) + x^* g'(x^*))}{1 + \phi_2^*(\Delta t) d + \phi_2^*(\Delta t) w_2}, \end{aligned}$$

we deduce that $1 + \mathrm{Tr}(J^D) + \det(J^D) > 0$ if

$$4 - 2\frac{\phi_1^*(\Delta t) x^*(b + ay^* g'(x^*))}{1 + \phi_1^*(\Delta t)(bx^* + ag(x^*)y^*) + \phi_1^*(\Delta t) w_1} \geq 0. \tag{20}$$

The above inequality (20) will be satisfied if

$$4w_1 - 2x^*(b + ay^* g'(x^*)) \geq 0,$$

or equivalently,

$$w_1 \geq x^* \frac{b + ay^* g'(x^*)}{2}. \tag{21}$$

Combining (19) and (21), we conclude that if (12) holds then (15) is satisfied. Consequently, we obtain the stability of E^*, which is the desired conclusion.

Proof of part (v): We see from (17) that $\det(J^D) - 1 > 0$ if $b + ay^* g'(x^*) > 0$, or equivalently, $\det(J^D) > 1$. This implies that E^* is unstable.

The proof is completed. $\square$

Remark 3. *From Theorem 4, we obtain the conditions for the NSFD model* (8) *to be dynamically consistent with respect to the stability of the model* (1).

We now determine the conditions such that the NSFD scheme (8) is convergent of order 2.

Theorem 5. *Let $\phi_1(\Delta t, x, y)$ and $\phi_2(\Delta t, x, y)$ be functions satisfying the following conditions*

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial \Delta t^2}(0,x,y) &= 2(bx + ag(x)y + w_1) + \frac{\partial f_1(x,y)}{\partial x} + \frac{\partial f_1(x,y)}{\partial y}\frac{f_2(x,y)}{f_1(x,y)}, \\ \frac{\partial^2 \phi_2}{\partial \Delta t^2}(0,x,y) &= 2(d + w_2) + \frac{\partial f_2(x,y)}{\partial x}\frac{f_1(x,y)}{f_2(x,y)} + \frac{\partial f_2(x,y)}{\partial y} \end{aligned} \tag{22}$$

for all $(x, y) \geq \mathbb{R}^2_+$ and $f_i(x, y) \neq 0$ $(i = 1, 2)$, where $(f_1(x, y), f_2(x, y))^\top$ is the right-hand side function of the model (1)*. Then, the truncation error of the NSFD method* (8) *is $\mathcal{O}(\Delta t^3)$, i.e. the scheme is consistent of order 2.*

Proof. First, using the Taylor's expansion for the solution components $x(t)$ and $y(t)$ we obtain

$$\begin{aligned} x(t_{k+1}) &= x(t_k) + \Delta t x'(t_k) + \frac{\Delta t^2}{2}x''(t_k) + \mathcal{O}(\Delta t^3) \\ &= x(t_k) + \Delta t f_1(x(t_k), y(t_k)) + \frac{\Delta t^2}{2}\frac{\partial f_1(x(t_k), y(t_k))}{\partial t} + \mathcal{O}(\Delta t^3), \\ y(t_{k+1}) &= y(t_k) + \Delta t y'(t_k) + \frac{\Delta t^2}{2}y''(t_k) + \mathcal{O}(\Delta t^3) \\ &= y(t_k) + \Delta t f_2(x(t_k), y(t_k)) + \frac{\Delta t^2}{2}\frac{\partial f_2(x(t_k), y(t_k))}{\partial t} + \mathcal{O}(\Delta t^3). \end{aligned} \tag{23}$$

Let us denote by $\big(F_1(\Delta t, x_k, y_k), F_2(\Delta t, x_k, y_k)\big)^\top$ the right-side function of the model (9) (or also (10)). It follows from (10) that

$$F_1(0, x, y) = x, \quad \frac{\partial F_1(0, x, y)}{\partial \Delta t} = f_1(x, y), \tag{24}$$

$$\frac{\partial^2 F_1(0, x, y)}{\partial \Delta t^2} = f_1(x, y)\left[\frac{\partial^2 \phi_1(0, x, y)}{\partial \Delta t^2} - 2(bx + ag(x)y + w_1)\right], \tag{25}$$

$$F_2(0, x, y) = y, \quad \frac{\partial F_2(0, x, y)}{\partial \Delta t} = f_2(x, y), \tag{26}$$

$$\frac{\partial^2 F_2(0, x, y)}{\partial \Delta t^2} = f_2(x, y)\left[\frac{\partial^2 \phi_2(0, x, y)}{\partial \Delta t^2} - 2(d + w_1)\right]. \tag{27}$$

Combining (24) with the Taylor expansion, we have that

$$\begin{aligned}
x_{k+1} &= F_1(\Delta t, x_k, y_k) = F_1(0,x,y) + \Delta t \frac{\partial F_1(0,x,y)}{\partial \Delta t} + \frac{\Delta t^2}{2}\frac{\partial^2 F_1(0,x,y)}{\partial \Delta t^2} + \mathcal{O}(\Delta t^3),\\
&= x_k + \Delta t f_1(x_k,y_k) + \frac{\Delta t^2}{2} f_1(x_k,y_k)\\
&\quad \left[\frac{\partial^2 \phi_1(0,x_k,y_k)}{\partial \Delta t^2} - 2(bx_k + ag(x_k)y_k + w_1)\right] + \mathcal{O}(\Delta t^3),\\
y_{k+1} &= F_2(\Delta t, x_k, y_k) = F_2(0,x,y) + \Delta t \frac{\partial F_2(0,x,y)}{\partial \Delta t} + \frac{\Delta t^2}{2}\frac{\partial^2 F_2(0,x,y)}{\partial \Delta t^2} + \mathcal{O}(\Delta t^3),\\
&= y_k + \Delta t f_2(x_k,y_k) + \frac{\Delta t^2}{2} f_2(x_k,y_k)\left[\frac{\partial^2 \phi_2(0,x_k,y_k)}{\partial \Delta t^2} - 2(d + w_2)\right] + \mathcal{O}(\Delta t^3).
\end{aligned} \tag{28}$$

Hence, we deduce from (23) and (28) that

$$x_{k+1} - x(t_{k+1}) = \mathcal{O}(\Delta t^3), \quad y_{k+1} - y(t_{k+1}) = \mathcal{O}(\Delta t^3)$$

if (22) holds. This is the desired conclusion and the proof is complete. □

The consistency is a local property of a one-step scheme like the NSFD method (8). The following theorem that for one-step schemes the convergence order follows from the consistency order is well-known.

Theorem 6. *Let $(x(t), y(t))^\top$ be the solution of the initial value problem* (1) *with continuous right hand side f. Let a Lipschitz condition hold for the second argument of the incremental function Φ*

$$\left\|\Phi\Big(t, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\Big) - \Phi\Big(t, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\Big)\right\| \le L^D \left\|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\|, \textit{ for all } \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2_+ \tag{29}$$

where the incremental function Φ can be read from the NSFD scheme (10)

$$\Phi\Big(t, \begin{pmatrix} x \\ y \end{pmatrix}\Big) = \begin{pmatrix} \frac{f_1(x,y)}{1+\phi_1(\Delta t,x,y)(bx+ag(x)y)+\phi_1(\Delta t,x,y)w_1} \\ \frac{f_2(x,y)}{1+\phi_2(\Delta t,x,y)d+\phi_2(\Delta t,x,y)w_2}. \end{pmatrix} \tag{30}$$

Then the convergence of the NSFD method (8) *follows from its consistency and the order of convergence equals the consistency order.*

Remark 4. *The system of conditions* (22) *can be expressed as follows*

$$\begin{aligned}
\frac{\partial^2\phi_1}{\partial\Delta t^2}(0,x,y) &= \tau_1(x,y) := ag(x)y + 2w_1 + b - axyg'(x) - \frac{ag(x)(g(x)xy - dy)}{b - bx - ag(x)y},\\
\frac{\partial^2\phi_2}{\partial\Delta t^2}(0,x,y) &= \tau_2(x,y) := d + 2w_2 + xg(x)\\
&\qquad\qquad + \frac{(g(x) + xg'(x))(bx - bx^2 - ag(x)xy)}{xg(x) - d}.
\end{aligned} \tag{31}$$

Therefore, candidate functions ϕ_i can be chosen in the form

$$\phi_i(\Delta t, x, y) = \begin{cases} \dfrac{e^{\tau_i(x,y)\Delta t} - 1}{\tau_i(x,y)} & \text{if} \quad \tau_i(x,y) \neq 0,\\ \Delta t \quad \text{if} \quad \tau_i(x,y) = 0. \end{cases} \tag{32}$$

The functions ϕ_i in (32) *satisfy not only* (31) *but also $\phi_i(\Delta t, x, y) = \Delta t^2 + \mathcal{O}(\Delta t^2)$ as $\Delta t \to 0$ and $\phi_i(\Delta t, x, y) > 0$ for all $\Delta t > 0, x \geq 0, y \geq 0$.*

Summarizing the results in this section, we obtain a second-order NSFD method that is dynamically consistent with the positivity and stability of the continuous model (1).

Remark 5. *Similarly to the arguments used in the proof of [8, Theorem 5.2] (see [8, Appendix B]), we can prove that the NSFD method* (8) *under the conditions of Theorem 5 is convergent of order 2.*

4. Numerical simulations

In this section, we report some illustrative numerical simulations to support the theoretical findings and demonstrate the advantages of the constructed NSFD method (8). To this end, as in [12], we consider the Rosenzweig-MacArthur predator-prey system (1) with a Holling-type II predator functional response of the form $xg(x) = x/(c + x)$. Consequently, the model (1) becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= bx(1-x) - \frac{axy}{c+x},\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{xy}{c+x} - dy.
\end{aligned} \tag{33}$$

As shown in the numerical examples in [12, Section 5], the standard explicit Euler and Runge-Kutta second-order methods, as well as the modified Patankar-Euler scheme, cannot preserve the positivity and stability of the model (33). In the following numerical examples, the NSFD method (8) is directly compared with the methods constructed in [12] and [46] to show its advantages.

Example 1 (The case $g(1) < d$)**.** *Let us consider the system* (33) *with the following set of the parameters (see [12])*

$$a = 2.0, \quad b = 1.0, \quad c = 0.5, \quad d = 6.0, \quad (x(0), y(0)) = (5, 2).$$

Since $g(1) < d$, the unique interior equilibrium point E^* does not exists and the boundary equilibrium point $E_1 = (1, 0)$ is asymptotically stable.

We now apply the second-order NSFD (2ndNSFD) method (8) for solving the system (33) and then, compute *absolute errors* at the time $t = 1$ and estimate the *rates of convergence* by the formulas (see [4])

$$\begin{aligned} \text{error} &= |x(t_N) - x_N| + |y(t_N) - y_N|, \quad N = \frac{1}{\Delta t}, \quad t_N = 1, \\ \text{rate} &:= \log(\Delta t_1/\Delta t_2)\left(\frac{\text{error}(\Delta t_1)}{\text{error}(\Delta t_2)}\right). \end{aligned}$$

Since it is possible to find the exact solution in close form, we admit the numerical solution obtained using a higher-order Runge-Kutta method, namely the classical four-stage Runge-Kutta method with step size $\Delta t = 10^{-5}$, as the reference solution. The errors and convergence rates of the 2ndNSFD method (8) using $w_1 = w_2 = 0$, the PESN method [12, scheme (11)], and the NSFD method in [46] (1stNSFD method) are given in Table 1. In particular, the errors of the models over the time interval $[0, 1]$ with $\Delta t = 0, 1$ are shown in Figure 1.

Table 1: Errors and rates of convergence of the three NSFD methods.

Step size	2ndNSFD *error*	2ndNSFD *rate*	PESN *error*	PESN *rate*	1stNSFD *error*	1stNSFD *rate*
0.2	0.0095		0.0837		0.1174	
10^{-1}	0.0023	2.0453	0.0343	1.2874	0.0474	1.3080
10^{-2}	2.3298e-005	1.9924	0.0028	1.0957	0.0033	1.1579
10^{-3}	2.3397e-007	1.9982	2.6864e-004	1.0102	3.1364e-004	1.0214
10^{-4}	2.3407e-009	1.9998	2.6801e-005	1.0010	3.1205e-005	1.0022
10^{-5}	2.3375e-011	2.0006	2.6795e-006	1.0001	3.1190e-006	1.0002
10^{-6}	1.4996e-013	2.1928	2.6794e-007	1.0000	3.1187e-007	1.0000

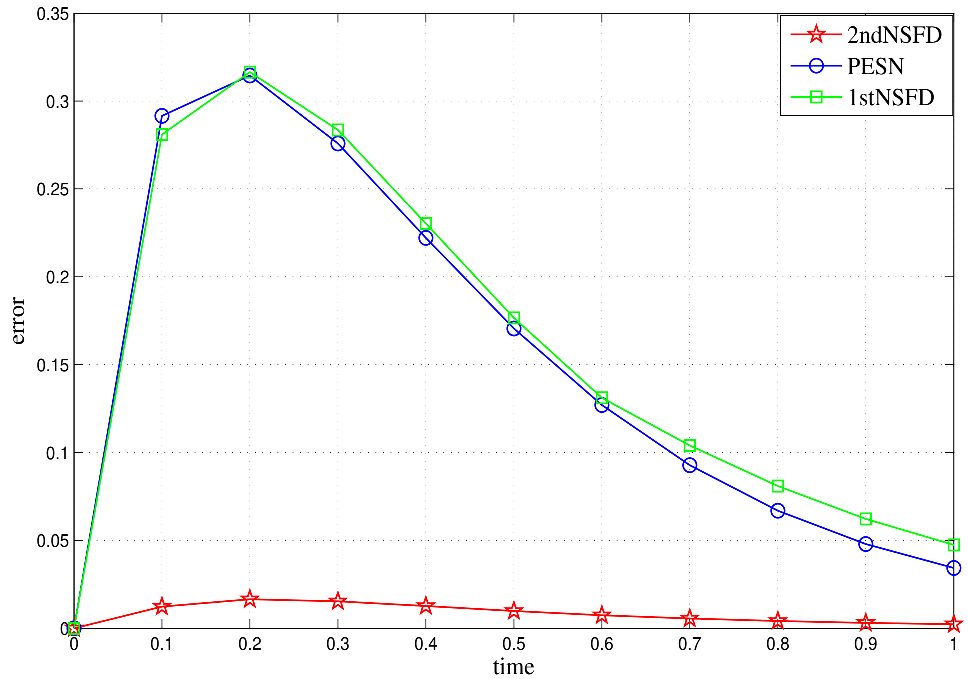

Figure 1: The errors versus time of the NSFD methods with $\Delta t = 0.1$.

From the results in Table 1 and Figure 1, the 2ndNSFD method is convergent with order 2, while the PESN method in [12] and the 1stNSFD method in [46] are convergent only with order 1. This is evidence supporting the theoretical claims presented in Section 3.

Next, the 2ndNSFD method is used to simulate the model (1) over a long time period, namely $t \in [0, 100]$, and show its dynamical consistency. Numerical solutions generated by the 2ndNSFD method using $\Delta t \in \{1.0,\ 0.1,\ 0.001\}$ are sketched in Figures 2-4. In these figures, each blue curve represents a phase plane corresponding to a particular initial value, the green circle indicates the position of the boundary equilibrium point, and the red arrows show the evolution of the predator-prey system. From these figures, it is clear that the NSFD method preserves the dynamics of the predator-prey system regardless of the step sizes chosen. This is in complete agreement with the theoretical results on the dynamic properties of the NSFD method presented in Section 3.

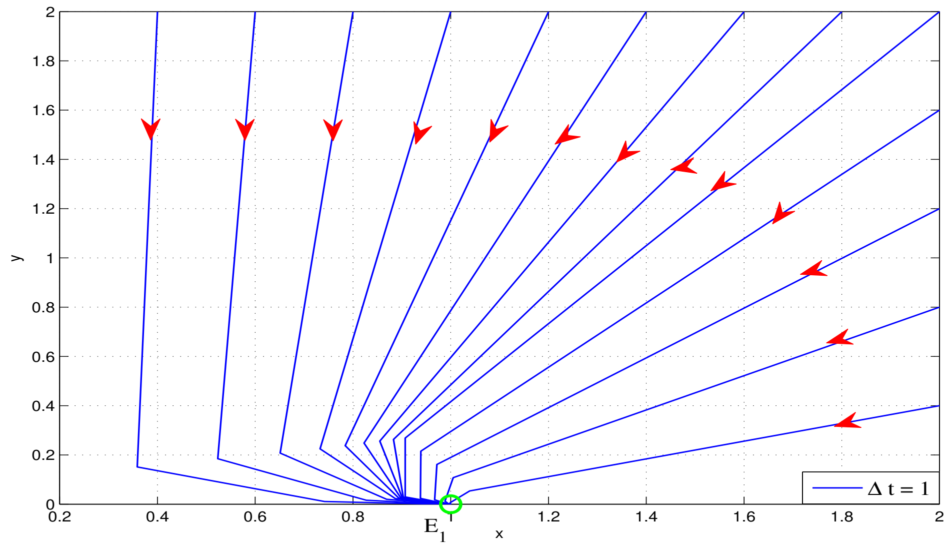

Figure 2: The phase planes generated by the second-order NSFD method using $\Delta t = 1.0$.

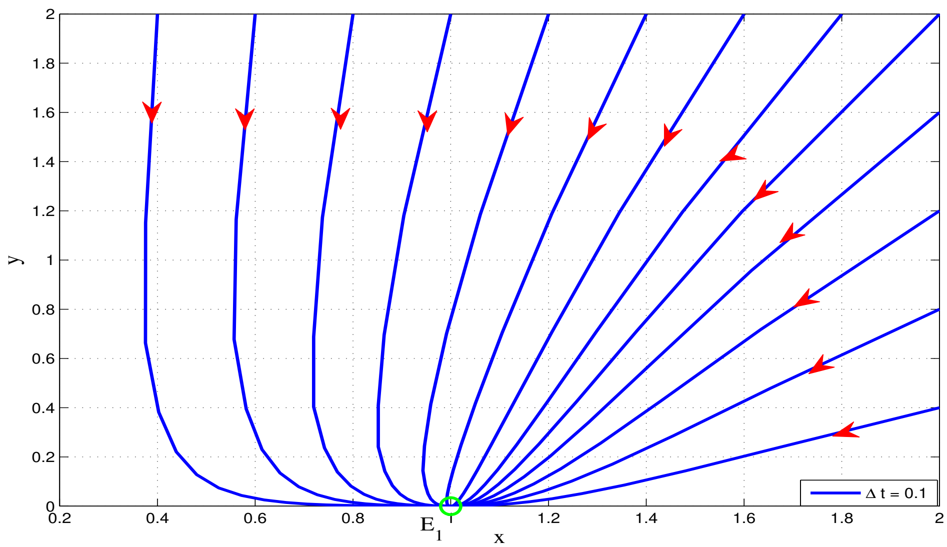

Figure 3: The phase planes generated by the second-order NSFD method using $\Delta t = 0.1$.

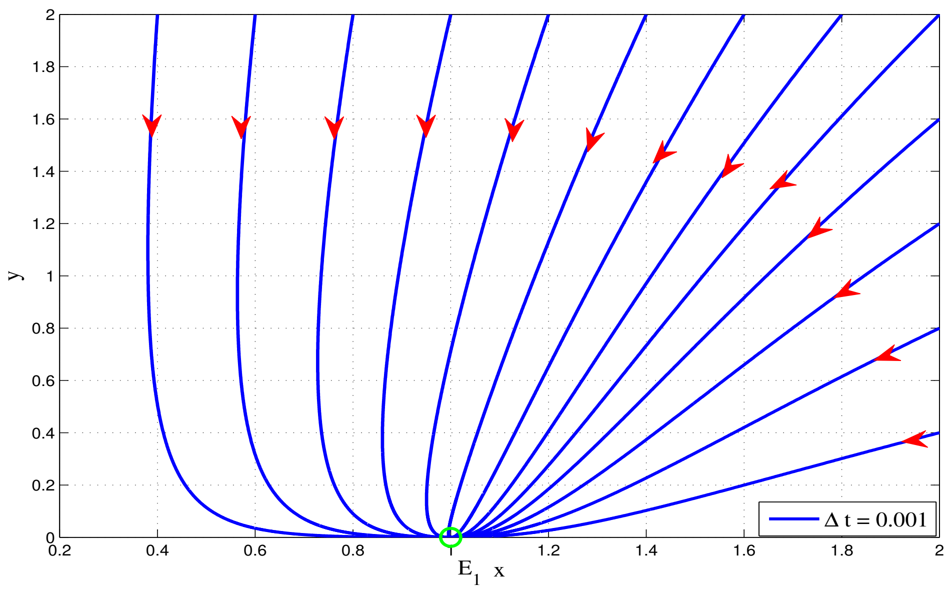

Figure 4: The phase planes generated by the second-order NSFD method using $\Delta t = 0.001$.

Example 2 (The case when $g(1) > d$)**.** *In this example, we consider the system* (33) *with the following set of the parameters (see [12])*

$$a = 2.0, \quad b = 1.0, \quad c = 1.0, \quad d = 0.2.$$

In this case, the model has a unique interior equilibrium point $E^* = (1/4,\ 15/32)$, which is also asymptotically stable. We now apply the second-order NSFD method (8) to solve the model (33) over the interval time $[0, 100]$, where the weights w_1 and w_2 are given by

$$w_1 = 0.1 > w_1^* := 0.05, \qquad w_2 = 1.25 > w_2^* := 1.20.$$

The obtained numerical results are presented in Figures 5-7, respectively. It is clear the NSFD method correctly preserves the dynamics of the continuous model.

It is important to note that in both Examples 1 and 2, the behaviour of the numerical solutions generated by the NSFD method is dependent of the chosen step sizes. So, the dynamic consistency of the constructed NSFD method is supported.

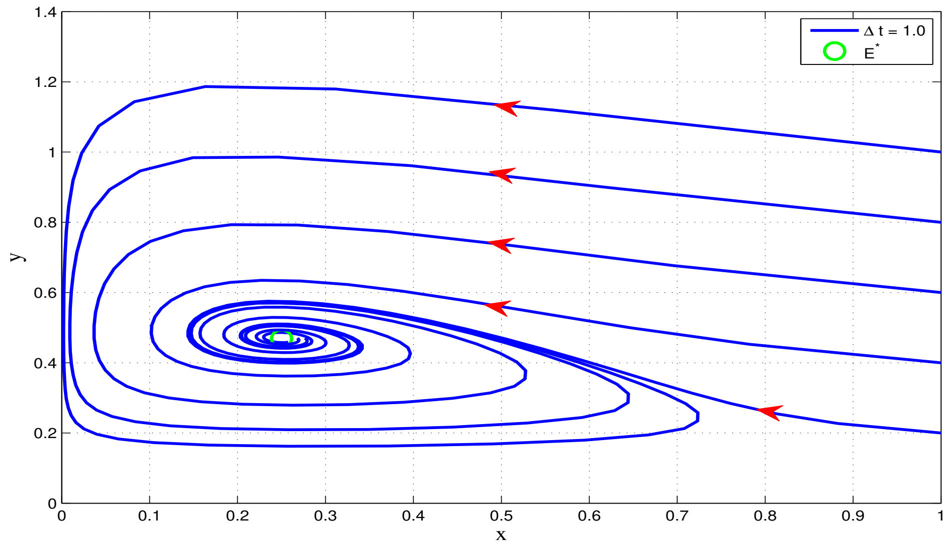

Figure 5: The phase planes in Example 2 provided by the second-order NSFD method using $\Delta t = 1.0$.

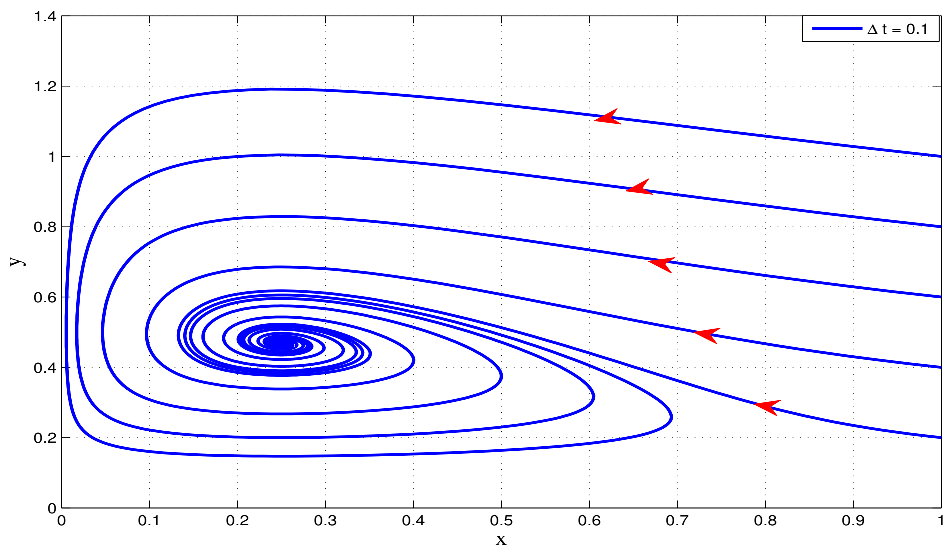

Figure 6: The phase planes in Example 2 provided by the second-order NSFD method using $\Delta t = 0.1$.

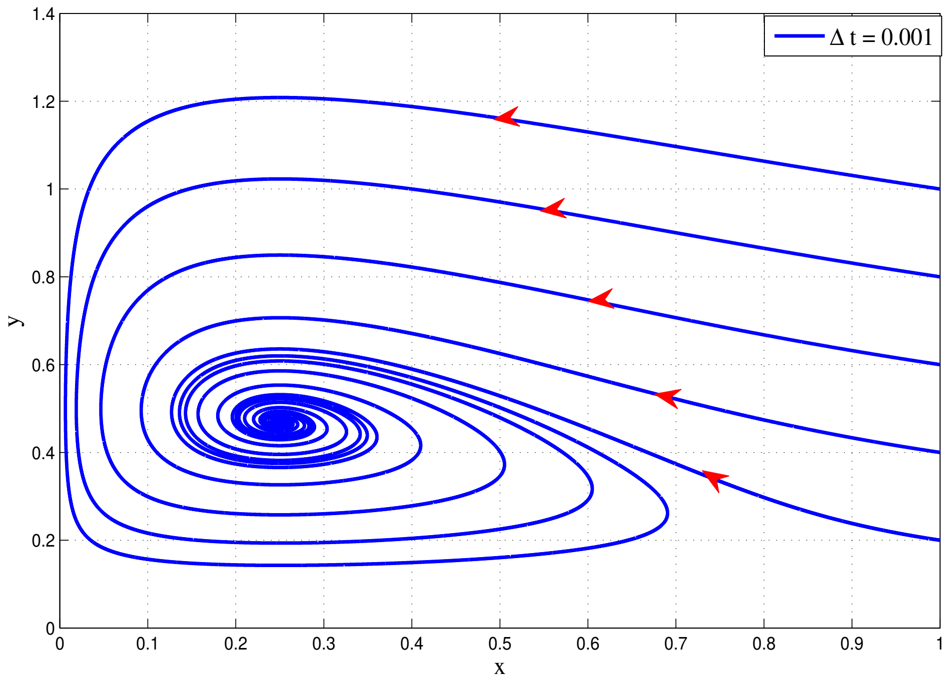

Figure 7: The phase planes in Example 2 provided by the second-order NSFD method using $\Delta t = 0.001$.

5. Remarks on generalized versions of the constructed NSFD method

In this section, we make some remarks on generalized versions of the constructed NSFD method (8).

Dynamically consistent NSFD methods for predator-prey systems modeled by two-dimensional dynamical systems have been intensively studied in recent decades due to the importance of mathematical models describing predator-prey interactions (see, e.g. [5, 11, 12, 13, 38, 42]). In the following paragraphs, we will show that the approach used to construct the second-order NSFD method (8) can be extended for some classes of nonlinear dynamical systems.

To describe how the NSFD method (8) can be extended for predator-prey systems, we first consider a general predator-prey model with a general functional response of the form (see [13])

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= p(x) - af(x,y)y, \quad x(0) \geq 0,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= f(x,y)y - \mu(y), \quad y(0) \geq 0,
\end{aligned} \tag{34}
$$

where

- x and y stand for the prey and predator population sizes, respectively;
- $p(x)$ and $\mu(y)$ describe the intrinsic growth rate of the prey and the mortality rate of the predator, respectively;
- the function $f(x,y)$ is called functional response" and represents the per capita predator "feeding rate" per unit time;
- $a > 0$ is the transformation rate constant, which represents the assimilation efficiency of the predator.

It is important to note that most of the scientific literature on predator-prey models assumes that $\mu(y) = dy$, where $d > 0$ (see [13]). Further details on the model (34) can be found in [13]. It is worth noting that (34) is a generalization of (1). Dimitrov and Kojouharov [13] have proposed and analyzed positive and elementary stable nonstandard (PESN) finite difference methods for the model (34) in the case $\mu(y) = dy$. These NSFD methods are of the form

$$
\begin{aligned}
\frac{x_{k+1} - x_k}{\phi(\Delta t)} &= p(x_k) - ag(x_k, y_k)x_{k+1}y_k,\\
\frac{y_{k+1} - y_k}{\phi(\Delta t)} &= f(x_k, y_k)y_k - dy_{k+1},
\end{aligned} \tag{35}
$$

where $g(x,y) = f(x,y)/x$. However, it is easy to verify that the scheme (35) is only convergent of order one.

Now, applying the approach used in Section 3, we obtain the following new NSFD model for (34)

$$
\begin{aligned}
\frac{x_{k+1} - x_k}{\phi_1(\Delta t, x_k, y_k)} &= p(x_k) - ag(x_k, y_k)x_{k+1}y_k + w_1 x_k - w_1 x_{k+1},\\
\frac{y_{k+1} - y_k}{\phi_2(\Delta t, x_k, y_k)} &= f(x_k, y_k)y_k - dy_{k+1} + w_2 y_k - w_2 y_{k+1}.
\end{aligned} \tag{36}
$$

It is clear that (36) is a generalization of (35).

Using the techniques used in Section 3, it is possible to specify conditions for the denominator functions ϕ_i and w_1 and w_2 such that the NSFD method (36) is dynamically consistent and convergent of order 2, where the conditions for dynamic consistency and convergence of order 2 are set on (w_1, w_2) and ϕ_i, respectively.

More generally, we consider a general class of two-dimensional differential equations involving several models of population dynamics of the form (see [8])

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(f_+(x,y) - f_-(x,y)), \quad x(0) \geq 0, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(g_+(x,y) - g_-(x,y)), \quad y(0) \geq 0, \end{aligned} \tag{37}$$

where f_+, f_- and g_+, g_- are positive for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and of class $\mathcal{C}^1$. Cresson and Pierret [8] proposed the following first-order and dynamically consistent NSFD method for (37)

$$\begin{aligned} \frac{x_{k+1} - x_k}{\phi(\Delta t)} &= x_k f_+(x_k, y_k) - x_{k+1} f_-(x_k, y_k), \\ \frac{y_{k+1} - y_k}{\phi(\Delta t)} &= y_k g_+(x_k, y_k) - y_{k+1} g_-(x_k, y_k). \end{aligned} \tag{38}$$

Using the approach in Section 3, we obtain the following NSFD method for (37)

$$\begin{aligned} \frac{x_{k+1} - x_k}{\phi_1(\Delta t, x_k, y_k)} &= x_k f_+(x_k, y_k) - x_{k+1} f_-(x_k, y_k) + w_1 x_k - w_2 x_{k+1}, \\ \frac{y_{k+1} - y_k}{\phi_2(\Delta t, x_k, y_k)} &= y_k g_+(x_k, y_k) - y_{k+1} g_-(x_k, y_k) + w_2 y_k - w_2 y_{k+1}. \end{aligned} \tag{39}$$

The NSFD method (39) not only generalizes (38), but can also be dynamically consistent and convergent of order 2.

Before ending this section, we consider general autonomous dynamical systems of the following form

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = F_i(y_1(t), y_2(t), \ldots, y_n(t)), \quad y_i(0) = y_{i,0} \geq 0, \quad 1 \leq i \leq n \tag{40}$$

under the hypothesis that the model (40) admits the set $\mathbb{R}^n_+$ as a positively invariant set, i.e., $y_i(t) \geq 0$ for all $t > 0$ if $y_{i,0} \geq 0$ for $i = 1, 2, \ldots, n$. This is equivalent to (see [25, Lemma 1] or [43, Proposition B.7])

$$F_i(y_1, y_2, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) \geq 0, \quad i = 1, 2, \ldots, n, \tag{41}$$

for all $y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \geq 0$. It follows from the condition (41) that the model (40) can be always represented in the form (see [9, 37])

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = f_i(y(t)) - y_i g_i(y(t)), \quad 1 \leq i \leq n, \tag{42}$$

where $y = (y_1, y_2, \ldots, y_n)$ and f_i and g_i are functions satisfy $f_i(y), g_i(y) \geq 0$ for all $y \geq 0$.

By extending the NSFD method (8), we propose the following NSFD method for (42)

$$\frac{y_{i,k+1} - y_{i,k}}{\phi_i(\Delta t, y_k)} = f_i(y_k) - y_{i,k+1} g_i(y_k) + w_i y_{i,k} - w_i y_{i,k+1}, \tag{43}$$

where ϕ_i are denominator functions; w_i are weights and $y_{i,k} \approx y_i(t_k)$. Then, a dynamically consistent NSFD method of second-order for (42) can be obtained by determining suitable conditions for ϕ_i and w_i in (43).

6. Concluding remarks and discussions

As the main conclusion of this work, we have presented a simple approach to construct a dynamically consistent second-order NSFD method for a general Rosenzweig-MacArthur predator-prey model with logistic intrinsic growth of the prey population. The second-order NSFD method was constructed based on a novel nonlocal approximation using right-hand side function weights and nonstandard denominator functions.

We have also shown that the NSFD method not only preserves two important and prominent dynamical properties of the continuous model, namely positivity and asymptotic stability independent of the values of the step size, but also is convergent of order 2. Therefore, it provides a solution to the contradiction between the dynamic consistency and high-order accuracy of NSFD methods.

The proposed NSFD method improves the non-standard numerical methods constructed in [13]. Moreover, the present approach can be extended to construct second-order NSFD methods for some classes of nonlinear dynamical systems encountered in real applications.

In the near future, we will extend the approach and the results obtained to study the construction of dynamically consistent higher-order NSFD methods for differential equations. In particular, generalized versions of the second-order NSFD method (8), discussed in Section 5, will be intensively studied.

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M.T.H.: Conceptualization, Methodology, Software, Formal analysis, Writing-Original draft preparation
M.E.: Methodology, Analysis, Writing - Review & Editing, Supervision.
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References

[1] L. J. S. Allen, An Introduction to Mathematical Biology, Prentice Hall, 2007.

[2] R. Anguelov, J. M.-S. Lubuma, Nonstandard finite difference method by nonlocal approximation, Mathematics and Computers in Simulation 61 (2003) 465-475.

[3] R. Anguelov and J. M.-S. Lubuma, Contributions to the mathematics of the nonstandard finite difference method and applications, Numerical Methods for Partial Differential Equations 17 (2001) 518-543.

[4] U. M. Ascher, L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, Society for Industrial and Applied Mathematics, Philadelphia, 1998.

[5] N. Bairagi, M. Biswas, A predator-prey model with Beddington-DeAngelis functional response: a non-standard finite-difference method, Journal of Difference Equations and Applications 22 (2016) 581-593.

[6] F. Brauer, C. Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology, Springer, New York, 2001.

[7] B. M. Chen-Charpentier, D. T. Dimitrov, H. V. Kojouharov, Combined nonstandard numerical methods for ODEs with polynomial right-hand sides, Mathematics and Computers in Simulation 73 (2006) 105-113.

[8] J. Cresson, F. Pierret, Non standard finite difference scheme preserving dynamical properties, Journal of Computational and Applied Mathematics 303 (2016) 15-30.

[9] J. Cresson, A. Szafrańska, Discrete and continuous fractional persistence problems – the positivity property and applications, Communications in Nonlinear Science and Numerical Simulation 44 (2017) 424-448.

[10] Quang A Dang, Manh Tuan Hoang, Positive and elementary stable explicit nonstandard Runge-Kutta methods for a class of autonomous dynamical systems, International Journal of Computer Mathematics 97 (2020) 2036-2054.

[11] Quang A Dang, Manh Tuan Hoang, Nonstandard finite difference schemes for a general predator-prey system, Journal of Computational Science 36 (2019) 101015.

[12] D. T. Dimitrov, H. V. Kojouharov, Positive and elementary stable nonstandard numerical methods with applications to predator-prey models, Journal of Computational and Applied Mathematics 189 (2006) 98-108.

[13] D. T. Dimitrov, H. V. Kojouharov, Nonstandard finite-difference methods for predator-prey models with general functional response, Mathematics and Computers in Simulation 78 (2008) 1-11.

[14] D. T. Dimitrov, H. V. Kojouharov, Stability-Preserving Finite-Difference Methods for General Multi-Dimensional Autonomous Dynamical Systems, International Journal of Numerical Analysis & Modeling 4 (2007) 280-290.

[15] M. Ehrhardt, Nonstandard Finite Difference Schemes for the Black-Scholes Equation, Chapter 11 in: A. Gumel (ed.), Mathematics of Continuous and Discrete Dynamical Systems, AMS (Contemporary Mathematics), Vol. 618, (2014) 217-228.

[16] M. Ehrhardt, R. E. Mickens, A Nonstandard Finite Difference Scheme for Convection-Diffusion Equations having Constant Coefficients, Applied Mathematics and Computation Vol. 219, Issue 12 (2013) 6591-6604.

[17] S. Elaydi, An Introduction to Difference Equations, Springer New York, 2005.

[18] M. Gupta, J. M. Slezak, F. Alalhareth, S. Roy, H. V. Kojouharov, Second-order Nonstandard Explicit Euler Method, AIP Conference Proceedings 2302 (2020) 110003.

[19] M. T. Hoang, Dynamical analysis of a generalized hepatitis B epidemic model and its dynamically consistent discrete model, Mathematics and Computers in Simulation 205 (2023) 291-314.

[20] M. T. Hoang, Reliable approximations for a hepatitis B virus model by nonstandard numerical schemes, Mathematics and Computers in Simulation 193 (2022) 32-56.

[21] M. T. Hoang, Positivity and boundedness preserving nonstandard finite difference schemes for solving Volterra's population growth model, Mathematics and Computers in Simulation 199 (2022) 359-373.

[22] M. T. Hoang, Dynamically consistent nonstandard finite difference schemes for a virus-patch dynamic model, Journal of Applied Mathematics and Computing 68 (2022) 3397-3423.

[23] M. T. Hoang, A novel second-order nonstandard finite difference method for solving one-dimensional autonomous dynamical systems, Communications in Nonlinear Science and Numerical Simulation 114 (2022) 106654.

[24] M. T. Hoang, A class of second-order and dynamically consistent nonstandard finite difference schemes for nonlinear Volterra's population growth model, Computational and Applied Mathematics 42 (2023) Article number: 85.

[25] Z. Horváth, Positivity of Runge-Kutta and diagonally split Runge-Kutta methods, Applied Numerical Mathematics 28 (1998) 309-326.

[26] H. K. Khalil, Nonlinear systems, Prentice Hall, 2002.

[27] H. V. Kojouharov, S. Roy, M. Gupta, F. Alalhareth, J. M. Slezak, A second-order modified nonstandard theta method for one-dimensional autonomous differential equations, Applied Mathematics Letters 112 (2021) 106775.

[28] M.H. Maamar, M. Ehrhardt, L. Tabharit, A Nonstandard Finite Difference Scheme for a Time-Fractional Model of Zika Virus Transmission, Preprint 22/21, University of Wuppertal, November 2022.

[29] J. Martin-Vaquero, A. Martin del Rey, A. H. Encinas, J. D. Hernandez Guillen, A. Queiruga-Dios, G. Rodriguez Sanchez, Higher-order nonstandard finite difference schemes for a MSEIR model for a malware propagation, Journal of Computational and Applied Mathematics 317 (2017) 146-156.

[30] J. Martin-Vaquero, A. Queiruga-Dios, A. Martin del Rey, A. H. Encinas, J. D. Hernandez Guillen, G. Rodriguez Sanchez, Variable step length algorithms with high-order extrapolated non-standard finite difference schemes for a SEIR model, Journal of Computational and Applied Mathematics 330 (2018) 848-854.

[31] R. E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World Scientific, 1993.

[32] R. E. Mickens, Applications of Nonstandard Finite Difference Schemes, World Scientific, 2000.

[33] R. E. Mickens, Advances in the Applications of Nonstandard Finite Difference Schemes, World Scientific, 2005.

[34] R. E. Mickens, Nonstandard Finite Difference Schemes: Methodology and Applications, World Scientific, 2020.

[35] R. E. Mickens, Nonstandard Finite Difference Schemes for Differential Equations, Journal of Difference Equations and Applications 8 (2002) 823-847.

[36] R. E. Mickens, Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations, Journal of Difference Equations and Applications 11 (2005) 645-653.

[37] R. E. Mickens, T. M. Washington, NSFD discretizations of interacting population models satisfying conservation laws, Computers and Mathematics with Applications 66 (2013) 2307-231.

[38] S. M. Moghadas, M. E. Alexander, B. D. Corbett, A non-standard numerical scheme for a generalized Gause-type predator-prey model, Physica D: Nonlinear Phenomena 188 (2004) 134-151.

[39] G. Morais Rodrigues Costa, M. Lobosco, M. Ehrhardt, R.F. Reis, Mathematical Analysis and a Nonstandard Scheme for a Model of the Immune Response against COVID-19, in: A. Gumel (ed.), Mathematical and Computational Modeling of Phenomena Arising in Population Biology and Nonlinear Oscillations: In honour of the 80th birthday of Ronald E. Mickens, AMS Contemporary Mathematics, 2023.

[40] K. C. Patidar, On the use of nonstandard finite difference methods, Journal of Difference Equations and Applications 11 (2005) 735-758.

[41] K. C. Patidar, Nonstandard finite difference methods: recent trends and further developments, Journal of Difference Equations and Applications 22 (2016) 817-849.

[42] K. C. Patidar, A. Ramanantoanina, A non-standard finite difference scheme for a class of predator-prey systems with non-monotonic functional response, Journal of Difference Equations and Applications 27 (2021) 1310-1328.

[43] H. L. Smith, P. Waltman, The Theory of the Chemostat: Dynamics of Microbial Competition, Cambridge University Press, 1995.

[44] A. Stuart, A. R. Humphries, Dynamical systems and numerical analysis, Cambridge University Press, 1998.

[45] S. Treibert, H. Brunner, M. Ehrhardt, A Nonstandard Finite Difference Scheme for the SVICDR model to predict COVID-19 dynamics, Mathematical Biosciences and Engineering 19(2) (2022) 1213-1238.

[46] D. T. Wood, H. V. Kojouharov, A class of nonstandard numerical methods for autonomous dynamical systems, Applied Mathematics Letters 50 (2015) 78-82.