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A combined method for stability analysis of linear time invariant and nonlinear continuous-time control systems based on the Hermite-Fujiwara matrix and Cholesky decomposition

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Abstract- In this paper, we have developed an integrative method for checking the stability of linear time-invariant (LTI) systems as well as nonlinear continuous-time ones. In our method, we first apply the iterative Faddeev-Leverrier algorithm to obtain the characteristic polynomial of the LTI system. Subsequently, the associated Hermite-Fujiwara matrix will be evaluated by a particularly efficient technique for the calculation of the Bézoutian matrices. The positive-definiteness of the Hermite-Fujiwara form, as the stability criterion, is next tested by performing the Cholesky decomposition. Our method is extended to nonlinear continuous-time systems with the help of the Jacobian matrix concept. The proposed method is demonstrated to approximately be 3.7085 times faster than the classical Routh-Hurwitz algorithm, at least for matrices with dimensions less than 40, according to a performed CPU time analysis. For the sake of illustration, four numerical examples are given, including dynamical models for a real-world hydrolysis reactor and a well-mixed bioreactor.

Keywords: Hermite-Fujiwara matrix; Bézoutian matrix; Cholesky decomposition; stability analysis; control system.

1. Introduction

Process control is a complementary and vital component of process design for chemical engineering plants. Since the advent of microprocessors in the 1980s, many enhanced control schemes are being developed every day with multiple goals, including tracking references, rejecting disturbances, ignoring measurement noises, and compensating for time delays [1-3].

It is the primary task of any control design to attain and secure stability for the target dynamical system. In other words, a controlled system must respond boundedly to any limited input. This concept is called bounded-input bounded-output (BIBO) stability. Consequently, it is essential to check the stability of any theoretically proposed control structure in advance of manufacturing and deployment [4,5].

It has been proved that a linear time-invariant system of the state-space form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \end{cases} \tag{1}$$

is BIBO stable if and only if all the zeroes of the characteristic polynomial associated with the state matrix $\mathbf{A}$ have strictly negative real parts, and thus lie in the open left side of the complex plane. Unfortunately, it is not possible to exactly obtain all zeroes of the characteristic polynomial in a systematic way, particularly for orders greater than four, as proved by Abel's impossibility theorem [6]. Therefore, more attempts have been made to localize, instead of finding, the zeroes of characteristic polynomials in the left half plane (LHP) of the complex plane [7-9].

Hurwitz's method is a well-known criterion that gives the necessary and sufficient conditions for a linear system to be BIBO stable [10]. If the characteristic polynomial of our system is

$$p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n, \tag{2}$$

then its associated n-by-n Hurwitz matrix is defined as

$$
\mathbf{H}(p)=\begin{bmatrix}
a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\
0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\
\vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\
\vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\
\vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\
\vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\
\vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n
\end{bmatrix}. \tag{3}
$$

According to the Hurwitz method, a system with characteristic polynomial (2) with $a_0 > 0$ is stable if and only if and all the principal (starting from upper left) minors of $\mathbf{H}(p)$ are strictly positive. That is,

$$
\begin{aligned}
m_1 &= \det\left(\left[a_1\right]\right) = a_1 > 0, \\
m_2 &= \det\left(\begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}\right) > 0, \\
m_3 &= \det\left(\begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}\right) > 0, \\
&\vdots \\
m_n &= \det\left(\mathbf{H}(p)\right) > 0.
\end{aligned} \tag{4}
$$

When $a_0 > 0$, the system is stable if and only if the principal minors (4) form an alternating sign pattern starting with a negative number. The principal minors (4) are also referred to as Hurwitz determinants. A recursive algorithm was also proposed by Edward J. Routh and is nowadays known as the Routh test, which is similar to the Hurwitz criterion but is relatively more efficient from the computational perspective [11].

The Hurwitz criterion suffers from two major limitations, 1) the characteristic polynomial of the system under study must have real coefficients only, and 2) it is not applicable to a control system whose open-loop transfer function incorporates transcendental terms like dead times. Furthermore, the computation of the *n* Hurwitz determinants or the tabulation of the Routh array in the Routh test can be demanding, particularly for higher-order systems. We emphasize that higher-order systems are often encountered in chemical engineering industries; for instance, a distillation column in the iso-butane–butene alkylation plant has been modeled as a system with $n = 191$ [12].

Recently, a novel stability criterion has been proposed, which relies on the matrix sign function for eigenvalue separation of the system's state matrix [13]. The involved matrix sign function calculations were carried out by means of the Adomian decomposition method [14-17].

In this paper, we propose an integrative framework that enables the stability assessment of control systems (LTI and nonlinear continuous-time). Briefly put, a particular method for calculating the Bézout matrix of two polynomials is presented, which facilitates computing the Hermite-Fujiwara matrix associated with our control system. Furthermore, the assessment of positive definiteness of the obtained Hermite-Fujiwara matrix, which corresponds to the BIBO stability condition, is proposed based on the Cholesky decomposition. Unlike the Hurwitz method, our approach is determinant-free. A number of illustrative examples are given in the penultimate section to demonstrate the application of our method.

2. Mathematical Preliminaries

Definition 2.1. Let $p(z)$ and $q(z)$ be complex polynomials of degree *n* and *m*, respectively, with $m \le n$. The Bézoutian matrix $\mathbf{B}(p,q) = (b_{ij})_{i,j=1,2,\ldots,n}$ associated with the two polynomials *p* and *q* is an *n*-by-*n* square matrix whose entries are obtained from the following identity [18]:

$$\frac{p(x)q(y) - p(y)q(x)}{x - y} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\, x^{i-1} y^{j-1}. \qquad (5)$$

Definition 2.2. Let $\hat{p}(z) = p(-z)$ and $p(z)$ be the same as in Definition 2.1. The Hermite-Fujiwara matrix $\mathbf{H}_F(p) = (h_{ij})_{i,j=1,2,\ldots,n}$ associated with polynomial *p* is obtained by [19]

$$\mathbf{H}_F(p) = \mathbf{B}(p, \hat{p})\mathbf{D}, \qquad (6)$$

where $\mathbf{D}$ is an *n*-by-*n* diagonal matrix defined as follows:

$$\mathbf{D} = diag\left(1,-1,1,\ldots,(-1)^{n-1}\right). \tag{7}$$

Theorem 2.1. (Faddeev-Leverrier) Let $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ be the normalized characteristic polynomial of matrix **A**, i.e., $a_0 = 1$. The following recursive relation gives the coefficients a_i:

$$\begin{cases} a_0 = 1, \quad \mathbf{C}_1 = \mathbf{A} \\ a_i = -\dfrac{trace(\mathbf{C}_i)}{i}, \quad 1 \le i \le n \\ \mathbf{C}_{i+1} = \mathbf{A}(\mathbf{C}_i + a_i \mathbf{I}), \quad 1 \le i \le n \end{cases}, \tag{8}$$

where *trace*() is the sum of the elements on the main diagonal of its input matrix and **I** denotes the identity matrix [20,21].

Remark 2.1. The Bézoutian matrices are symmetric, i.e., they remain unchanged when transposed.

Remark 2.2. The Hermite-Fujiwara matrices are Hermitian, i.e., they are equal to their own conjugate transpose.

Definition 2.5. An *n*-by-*n* Hermitian complex matrix **A** is called positive-definite if and only if $\mathbf{x}^*\mathbf{A}\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$, where $\mathbf{x}^*$ denotes the conjugate transpose of $\mathbf{x}$.

Theorem 2.2. Consider the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. The matrix **A** is positive-definite if and only if it can be Cholesky decomposed [22].

Corollary 2.1. Consider the following algorithm (Cholesky decomposition) for $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$:

$$
\begin{aligned}
& \mathit{for}\ \ j = 1\ \ \mathit{to}\ \ n \\
& \qquad \mathit{for}\ \ i = 1\ \ \mathit{to}\ \ j-1 \\
& \qquad\qquad r_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}}{r_{ii}} \\
& \qquad \mathit{next} \\
& r_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} r_{kj}^2} \\
& \mathit{next}
\end{aligned}
$$

The matrix **A** is positive-definite if and only if the components $\left(r_{ii}\right)_{i=1,2,\ldots,n}$ are all real and strictly positive.

Remark 2.1. If in the previous algorithm r_{ii} for some $i<n$ becomes negative or non-real, we terminate the algorithm and conclude that the matrix **A** is not positive-definite. This saves time as we do not have to run the remaining steps of the algorithm.

3. The Proposed Method

Assume that the closed-loop transfer function of a dynamical system in the Laplace domain is given. It is readily known that if we factor out the coefficient corresponding to the highest power of the unknown variable in the denominator of the aforementioned transfer function, we will obtain the characteristic polynomial of the system. Obviously, it is a monic polynomial. On the other hand, if the mathematical model for the system is available in the state-space representation, we can apply the Faddeev-Leverrier algorithm, see Theorem 2.1, to the system state matrix for determining the characteristic polynomial. For a consistent notation, let us take $p(x)$ as the characteristic polynomial of our system:

$$p\left(x\right) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n . \tag{9}$$

Also, as noted in Definition 2.2, we take

$$\hat{p}(x) = p(-x) = (-1)^n x^n + (-1)^{n-1} a_1 x^{n-1} + (-1)^{n-2} a_2 x^{n-2} + \cdots - a_{n-1} x + a_n . \tag{10}$$

In what follows, we are interested in the calculation of the Bézoutian matrix of p and $\hat{p}$. However, it is a challenging task to compute the entries of the Bézoutian matrix directly by Eq. (5). Therefore, inspired by Barnett [23], we propose a more computationally effective procedure.

Let the triangular square matrix $\mathbf{T}$ be defined as

$$\mathbf{T} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0 \\ a_{n-3} & a_{n-4} & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{11}$$

and consider the transpose of the Frobenius companion matrix of p as

$$\mathbf{C}^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}. \tag{12}$$

Now, the Hermite-Fujiwara matrix of p can be calculated by

$$\mathbf{H}_F(p) = \mathbf{T}\,\hat{p}(\mathbf{C}^T)\,\mathbf{D}, \tag{13}$$

where $\mathbf{D}$ is defined by Eq. (7).

It can be proved, see [18], that our system is BIBO stable if and only if the matrix $\mathbf{H}_F(p)$ is positive-definite. By Theorem 2.2 and Corollary 2.1, in order to assess the stability of the system, we only need to perform a Cholesky decomposition on $\mathbf{H}_F(p)$, and check the diagonal entries of the obtained upper triangular matrix. If they are all real and strictly positive, our system is stable. Otherwise, the system is deemed unstable.

For nonlinear continuous-time systems, the stability of the equilibrium point can similarly be analyzed. For this purpose, first, we evaluate the Jacobian matrix of the system at the intended equilibrium point and then apply the Faddeev-Leverrier algorithm to the Jacobian matrix for calculating the coefficients of the relevant characteristic polynomial. The remaining steps are identical to those for LTI systems.

4. Illustrative Examples

Example 1.

The hydrolysis of propylene oxide to propylene glycol, $C_3H_6O \rightarrow C_3H_8O_2$, can be controlled in a continuously stirred tank reactor (CSTR). The dynamical model of such reaction system, at temperature 343.1 K, is as follows [24]:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -0.0664 & 0 & -0.0001 & 0 \\ 0.0365 & -0.0299 & 0.0001 & 0 \\ 54.9420 & 0 & 0.1329 & 0.0138 \\ 0 & 0 & 0.0144 & -0.3297 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.0188 & 0 \\ -0.0188 & 0 \\ -18.3005 & 0 \\ 0 & -1.1978 \end{bmatrix} \mathbf{u}, \\ y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}. \end{cases} \tag{14}$$

Without any difficulty, we can obtain the characteristic polynomial of the system (14) by the Faddeev-Leverrier algorithm, i.e. Theorem 2.1, as

$$p(x) = x^4 + 0.2931x^3 - 0.0176x^2 - 0.0019x - 0.3322\times10^{-4}. \tag{15}$$

Clearly, we can write

$$\hat{p}(x) = p(-x) = x^4 - 0.2931x^3 - 0.0176x^2 + 0.0019x - 0.3322\times10^{-4}. \tag{16}$$

Next, in view of Eq. (11), we propose the matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{bmatrix} -0.0019 & -0.0176 & 0.2931 & 1 \\ -0.0176 & 0.2931 & 1 & 0 \\ 0.2931 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Also, the transpose of the Frobenius companion matrix for this system is

$$\mathbf{C}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3322\times10^{-4} & 0.0019 & 0.0176 & -0.2931 \end{bmatrix}. \tag{18}$$

Hence, the Hermite-Fujiwara matrix associated with p is computed as

$$\mathbf{H}_F\left(p\right) = \begin{bmatrix} 1.26236\times10^{-7} & -1.35525\times10^{-12} & -1.94735\times10^{-5} & -3.46944\times10^{-18} \\ 0 & 0.000086353 & 0 & -0.0038 \\ -1.9473\times10^{-5} & 0 & -0.00651712 & 0 \\ 0 & -0.0038 & 0 & 0.5862 \end{bmatrix}. \tag{19}$$

Now, we perform a Cholesky decomposition on $\mathbf{H}_F\left(p\right)$ following the algorithm explained in Corollary 2.1. The obtained diagonal elements r_{ii} are computed as follows:

$$\begin{aligned} r_{11} &= 0.0003552971, \\ r_{22} &= 0.009292662, \\ r_{33} &= 0.0975765i, \\ r_{44} &= 0.647287. \end{aligned} \tag{20}$$

Since there is one non-real component, r_{33}, the matrix $\mathbf{H}_F\left(p\right)$ is not positive-definite; hence, the system (14) is unstable. This conclusion is corroborated by knowing that at least one of the eigenvalues of the state matrix of system (14) has a positive real part. These eigenvalues are −0.0299, 0.1004, −0.0335, and −0.3301.

Example 2.

To better assess the performance of our method, we have devised a hypothetical stable system of higher-order. The characteristic polynomial of the system is as follows:

$$\begin{aligned} p\left(x\right) &= x^{14} + 10.5x^{13} + 50.05x^{12} + 143.325x^{11} + 274.9747x^{10} + 373.12275x^{9} \\ &+368.411615x^{8} + 268.1453775x^{7} + 144.09322928x^{6} + 56.66336676x^{5} \\ &+15.972160568x^{4} + 3.109892604x^{3} + 0.392156797824x^{2} + 0.028346564736x \\ &+0.000871782912. \end{aligned} \tag{21}$$

It is clear that,

$$
\begin{aligned}
\hat{p}(x) = p(-x) = {} & x^{14} - 10.5x^{13} + 50.05x^{12} - 143.325x^{11} + 274.9747x^{10} - 373.12275x^{9} \\
& + 368.411615x^{8} - 268.1453775x^{7} + 144.09322928x^{6} - 56.66336676x^{5} \\
& + 15.972160568x^{4} - 3.109892604x^{3} + 0.392156797824x^{2} - 0.028346564736x \\
& + 0.000871782912.
\end{aligned}
\tag{22}
$$

Subsequently, by Eqs. (11) and (12), we can obtain the matrices $\mathbf{T}$ and $\mathbf{C}^{T}$ as follows. Note that we have rounded the entries of $\mathbf{C}^{T}$ to five decimal places for a better presentation. Next, the Hermite-Fujiwara matrix is calculated according to Eq. (13).

$$
\mathbf{T}=\begin{bmatrix}
0.028346564736 & 0.392156797824 & 3.109892604 & 15.972160568 & 56.66336676 & 144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 \\
0.392156797824 & 3.109892604 & 15.972160568 & 56.66336676 & 144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 \\
3.109892604 & 15.972160568 & 56.66336676 & 144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 \\
15.972160568 & 56.66336676 & 144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 \\
56.66336676 & 144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 \\
144.09322928 & 268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 \\
268.1453775 & 368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
368.411615 & 373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
373.12275 & 274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
274.9747 & 143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
143.325 & 50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
50.05 & 10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{23}
$$

$$\mathbf{C}^{T}=\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
-8.71782\times 10^{-4} & -0.02834 & -0.39215 & -3.10989 & -15.97216 & -56.66336 & -144.09322 & -268.145377 & -368.41161 & -373.1227 & -274.9747 & -143.325 & -50.05 & -10.5
\end{bmatrix}. \qquad (24)$$

$$
\mathbf{H}_F\left(p\right)=\begin{bmatrix}
4.95e\text{-}5 & \text{-}3.14e\text{-}6 & 0.00547 & \text{-}3.43e\text{-}4 & 0.101 & \text{-}0.00596 & 0.482 & \text{-}0.0272 & 0.686 & \text{-}0.0355 & 0.274 & \text{-}0.0112 & 0.0216 & \text{-}4.85e\text{-}4 \\
1.66e\text{-}8 & 0.0168 & 7.07e\text{-}6 & 0.807 & 2.88e\text{-}4 & 7.7 & 0.00239 & 20.2 & 0.00586 & 15.3 & 0.00387 & 2.82 & 5.32e\text{-}4 & 0.0566 \\
0.00542 & \text{-}5.9e\text{-}8 & 1.63 & \text{-}6.63e\text{-}6 & 36.7 & \text{-}1.17e\text{-}4 & 190.0 & \text{-}5.03e\text{-}4 & 277.0 & \text{-}7.03e\text{-}4 & 110.0 & \text{-}2.4e\text{-}4 & 8.18 & \text{-}9.92e\text{-}6 \\
1.47e\text{-}10 & 0.807 & 5.4e\text{-}8 & 62.6 & 2.8e\text{-}6 & 706.0 & 1.51e\text{-}5 & 2011.0 & 3.55e\text{-}5 & 1600.0 & 2.98e\text{-}5 & 303.0 & 2.41e\text{-}6 & 6.22 \\
0.0988 & \text{-}5.68e\text{-}10 & 36.7 & \text{-}1.32e\text{-}7 & 1100.0 & \text{-}2.03e\text{-}6 & 6555.0 & \text{-}5.96e\text{-}6 & 1.03e4 & \text{-}6.44e\text{-}6 & 4288.0 & \text{-}4.89e\text{-}6 & 329.0 & \text{-}2.44e\text{-}7 \\
2.05e\text{-}12 & 7.7 & 2.33e\text{-}10 & 706.0 & 4.1e\text{-}8 & 9788.0 & 1.79e\text{-}7 & 3.14e4 & \text{-}4.17e\text{-}7 & 2.69e4 & 7.75e\text{-}7 & 5344.0 & \text{-}5.96e\text{-}8 & 113.0 \\
0.468 & 9.09e\text{-}12 & 190.0 & \text{-}1.4e\text{-}9 & 6555.0 & 7.45e\text{-}9 & 4.58e4 & 1.04e\text{-}7 & 8.06e4 & 8.94e\text{-}8 & 3.6e4 & \text{-}5.96e\text{-}8 & 2911.0 & \text{-}1.86e\text{-}9 \\
0.0 & 20.2 & 2.18e\text{-}11 & 2011.0 & \text{-}9.31e\text{-}10 & 3.14e4 & 3.73e\text{-}9 & 1.17e5 & \text{-}5.22e\text{-}8 & 1.12e5 & 0.0 & 2.39e4 & \text{-}1.86e\text{-}9 & 536.0 \\
0.651 & \text{-}2.27e\text{-}13 & 277.0 & \text{-}7.28e\text{-}11 & 1.03e4 & 9.31e\text{-}10 & 8.06e4 & 3.73e\text{-}9 & 1.63e5 & \text{-}7.45e\text{-}9 & 8.17e4 & \text{-}4.19e\text{-}9 & 7200.0 & \text{-}1.31e\text{-}10 \\
0.0 & 15.3 & 9.09e\text{-}13 & 1600.0 & \text{-}2.91e\text{-}11 & 2.69e4 & \text{-}2.33e\text{-}10 & 1.12e5 & \text{-}9.31e\text{-}10 & 1.24e5 & 4.66e\text{-}10 & 3.01e4 & 2.91e\text{-}11 & 746.0 \\
0.25 & \text{-}7.11e\text{-}15 & 110.0 & \text{-}9.09e\text{-}13 & 4288.0 & 0.0 & 3.6e4 & 5.82e\text{-}11 & 8.17e4 & \text{-}1.75e\text{-}10 & 4.87e4 & \text{-}2.91e\text{-}11 & 5033.0 & 0.0 \\
0.0 & 2.82 & 2.84e\text{-}14 & 303.0 & 0.0 & 5344.0 & 0.0 & 2.39e4 & 0.0 & 3.01e4 & \text{-}1.46e\text{-}11 & 9322.0 & \text{-}9.09e\text{-}13 & 287.0 \\
0.0183 & 0.0 & 8.18 & 0.0 & 329.0 & 2.27e\text{-}13 & 2911.0 & 0.0 & 7200.0 & 0.0 & 5033.0 & 0.0 & 764.0 & 0.0 \\
0.0 & 0.0567 & 0.0 & 6.22 & 0.0 & 113.0 & 0.0 & 536.0 & 0.0 & 746.0 & 0.0 & 287.0 & 0.0 & 21.0
\end{bmatrix}. \tag{25}
$$

Then, we perform a Cholesky decomposition on the matrix $\mathbf{H}_F\left(p\right)$ and calculate the diagonal entries r_{ii}, see Corollary 2.1, as

$$\begin{array}{ll} r_{11}=0.0070, & r_{88}=100.3046,\\ r_{22}=0.1297, & r_{99}=111.3961,\\ r_{33}=1.0145, & r_{1010}=97.4212,\\ r_{44}=4.8880, & r_{1111}=66.7087,\\ r_{55}=16.1504, & r_{1212}=35.0311,\\ r_{66}=38.8301, & r_{1313}=13.4308,\\ r_{77}=70.8448, & r_{1414}=3.2830. \end{array} \tag{26}$$

Since all of these quantities are real and positive, we conclude that the system under study is stable. It is doubly confirmed by knowing that the roots of the characteristic equation of our system are −0.1, −0.2, −0.3, −0.4, −0.5, −0.6, −0.7, −0.8, −0.9, −1, −1.1, −1.2, −1.3, and −1.4, which are all located in the left half of the complex plane.

Example 3.

The mathematical model of a continuously stirred biological reactor is given as [25]:

$$\begin{cases} \dfrac{ds_p}{dt}=Ds_{p,f}-Ds_p-Ks_p,\\ \dfrac{dz}{dt}=-Dz+cK\left(s_p-\dfrac{Ds_{p,f}}{D+K}\right),\\ \dfrac{ds}{dt}=D\left(s_f-s\right)+cKs_p-\dfrac{\mu_{\max}s}{K_s+s}\left(z-s+s_f+\dfrac{cs_{p,f}K}{D+K}\right), \end{cases} \tag{27}$$

where the state functions s is the concentration of soluble organic substrate, s_p denotes the concentration of the particulate substrate, and z is a function of the concentration of the microorganism, x, according to this law

$$z = \frac{x}{Y} + s - s_f - \frac{cs_{p,f}K}{D+K}. \tag{28}$$

The numerical values for the model parameters, Eqs. (27) and (28), are listed in Table 1. We are interested in examining the system stability at its steady-state, i.e., equilibrium point, corresponding to $\left(s, s_p, x\right) = \left(1.483, 35892.224, 1551.6922\right)$.

Table 1) Numerical values for the parameters of the bioreactor	
s_f	$0\ mg/l$
D	$0.2544\ d^{-1}$
μ_{max}	$4.2\ d^{-1}$
K_s	$23\ mg/l$
Y	$0.11\ mg/mg$
$s_{p,f}$	$50000\ mg/l$
K	$0.1\ d^{-1}$
c	$1\ mg/mg$

The Jacobian matrix of the system (27) can be determined parametrically as

$$\mathbf{J} = \begin{bmatrix} -D-K & 0 & 0 \\ cK & -D & 0 \\ 0 & -\frac{\mu_{\max} s}{K_s + s} & -D - \mu_{\max}\left(\frac{K_s}{\left(K_s+s\right)^2}\left(z - s + s_f + \frac{cs_{p,f}K}{D+K}\right) - \frac{s}{K_s+s}\right) \end{bmatrix} \tag{29}$$

,

which is numerically evaluated at the aforementioned equilibrium points:

$$\mathbf{J}_{eq} = \begin{bmatrix} -0.3544 & 0 & 0 \\ 0.1 & -0.2544 & 0 \\ 0 & -0.2544050974 & -2273.32093 \end{bmatrix}. \tag{30}$$

To check the stability of the equilibrium point, we first apply the Faddeev-Leverrier algorithm to $\mathbf{J}_{eq}$ to derive the characteristic polynomial as

$$p(x) = x^3 + 2273.9297x^2 + 1384.0879x + 204.9611. \tag{31}$$

Clearly,

$$\hat{p}(x) = p(-x) = -x^3 + 2273.9297x^2 - 1384.0879x + 204.9611. \tag{32}$$

Similar to the previous examples, we evaluate the matrix **T** from Eq. (11) as follows:

$$\mathbf{T} = \begin{bmatrix} 1384.0879 & 2273.9297 & 1 \\ 2273.9297 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{33}$$

Next, the transpose of the Frobenius companion matrix of *p*, see Eq. (12), is derived:

$$\mathbf{C}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -204.9611 & -1384.0879 & -2273.9297 \end{bmatrix}. \tag{34}$$

Afterwards, we can obtain the Hermite-Fujiwara matrix of *p* by Eq. (13) effortlessly.

$$\mathbf{H}_F(p) = \begin{bmatrix} 567368.3569 & 0 & 409.9222 \\ 0 & 6294227.2442 & 0 \\ 409.9222 & 0 & 4547.8594 \end{bmatrix}. \tag{35}$$

Then, we perform a Cholesky decomposition on $\mathbf{H}_F(p)$, see Corollary 2.1, and evaluate the diagonal entries r_{ii}:

$$\begin{aligned} r_{11} &= 753.2385, \\ r_{22} &= 2508.8298,. \\ r_{33} &= 67.4356, \end{aligned} \tag{36}$$

which are all real and positive. This indicates that the equilibrium of the system (27) is stable. Our postulation can be validated by knowing that the eigenvalues of $\mathbf{J}_{eq}$, being −2273.3209, −0.2544, and −0.3544, are all located left to the imaginary axis in the complex plane.

Example 4.

The block diagram of a negative feedback control system with a tuned PID controller is presented in Fig. 1. We would like to assess its closed-loop stability.

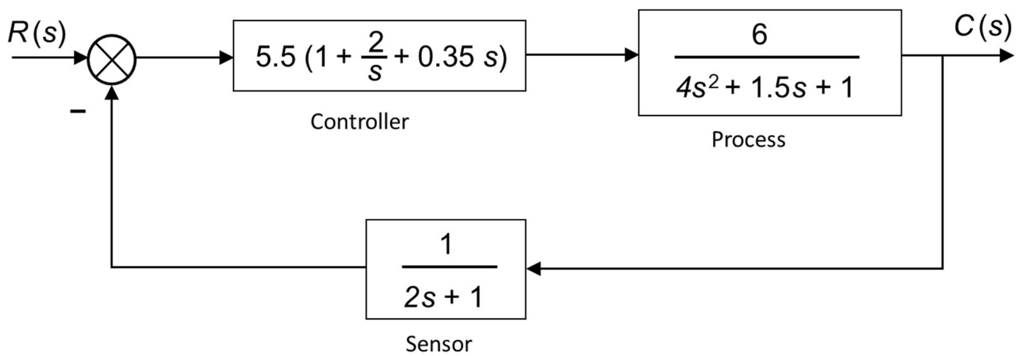

Fig. 1) Block diagram of the PID-controlled system

As the first step, we simplify the block diagram and obtain the closed-loop transfer function of the system as

$$\frac{C}{R} = \frac{90.585s^2 + 179.4925s + 67.1}{8s^3 + 17.065s^2 + 37.05s + 68.1}. \tag{37}$$

We can identify the characteristic equation of the closed-loop system from the denominator in Eq. (37) as

$$8s^3 + 17.065s^2 + 37.05s + 68.1 = 0. \tag{38}$$

However, the polynomial in Eq. (38) is not monic and its Hermite-Fujiwara matrix cannot be calculated by Eq. (13). Hopefully, a scalar multiplication preserves the stability status of polynomials. Therefore, we divide the denominator in Eq. (37) by eight and obtain the normalized characteristic equation:

$$p(x) = x^3 + 2.133125x^2 + 4.63125x + 8.5125, \tag{39}$$

and readily,

$$\hat{p}(x) = p(-x) = -x^3 + 2.133125x^2 - 4.63125x + 8.5125. \tag{40}$$

The corresponding matrices $\mathbf{T}$, $\mathbf{C}^T$, and $\mathbf{H}_F(p)$ are straightforwardly calculated in the sequel.

$$\mathbf{T} = \begin{bmatrix} 4.63125 & 2.133125 & 1 \\ 2.133125 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{41}$$

$$\mathbf{C}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8.5125 & -4.63125 & -2.133125 \end{bmatrix}, \tag{42}$$

$$\mathbf{H}_F(p) = \begin{bmatrix} 78.8470 & -0.0000 & 17.0250 \\ 0 & 2.7331 & 0 \\ 17.0250 & 0 & 4.2663 \end{bmatrix}. \tag{43}$$

Now, we perform a Cholesky decomposition on $\mathbf{H}_F(p)$ to obtain the following diagonal components:

$$\begin{aligned} r_{11} &= 8.8796, \\ r_{22} &= 1.6532, . \\ r_{33} &= 0.7682, \end{aligned} \tag{44}$$

which are all positive and real that is indicative of the stability of our closed-loop control system. We are assured of the truth of this statement as the poles of the closed-loop transfer function are -1.9728 and $-0.0802 \pm 2.0757i$, which are located in the left half of the complex plane.

5. Computational Performance

In this section, we will test the performance of our method from a computational viewpoint. To do so, we have conducted a CPU time analysis to compare the run times of our method with those of the Routh-Hurwitz algorithm. We generated random matrices with sizes ranging from 3 to 40 with the "*rand*" procedure of the MATLAB software package and measured the CPU times with the "*tic toc*" function. Each matrix was considered as the state matrix of an LTI system or the Jacobian of a nonlinear system, and the corresponding characteristic polynomial was computed as described in the previous sections . The results of this comparison are depicted in Fig 2. It is concluded that in average our method is 3.7085 times faster than the Routh-Hurwitz criterion in judging the stability of dynamical systems.

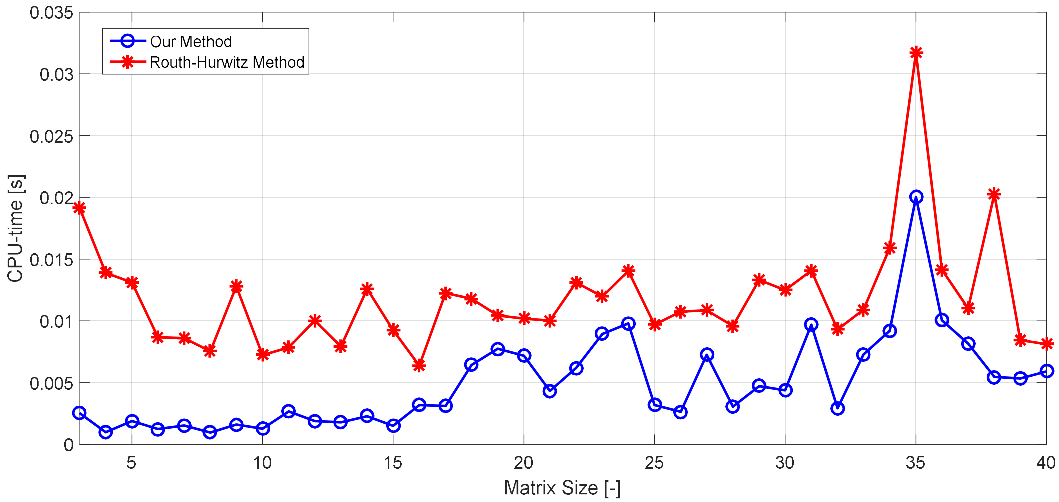

Fig. 2) Comparison of the CPU times of our method and the Routh-Huwritz criterion for stability analysis of randomly generated state matrices (size= 3 to 40)

Conclusion

A stability assessment method for linear time-invariant and nonlinear continuous-time dynamical systems was proposed. For LTI systems described in the state-space form, we first invoked the Faddeev-Leverrier algorithm to obtain the corresponding characteristic polynomial from the system's state matrix. Next, an efficient approach for evaluating the Bézoutian matrix was followed in order to obtain the Hermite-Fujiwara form associated with the system's characteristic equation. The necessary and sufficient condition for stability of the system under study was stated as positive-definiteness of the aforementioned Hermite-Fujiwara matrix. An efficient technique for the calculation of the Bézoutian matrix was adopted in the course of determining the Hermite-Fujiwara matrix. Finally, the positive-definiteness of the Hermite-Fujiwara matrix was verified by the Cholesky decomposition. In the case of nonlinear continuous-time systems, a similar strategy was followed by replacing the Jacobian matrix, evaluated at an equilibrium point, with the system's state matrix. Unlike the classical Hurwitz technique, our method avoids any tedious determinant calculations. Four illustrative examples were given to demonstrate the application of our integrative approach. Based on a CPU-time analysis, it was demonstrated that our combined method is roughly 3.7 times faster than the classical Routh-Hurwitz algorithm in an average sense.

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