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On solvability of dissipative partial differential-algebraic equations

Birgit Jacob and Kirsten Morris

***Abstract*— We investigate the solvability of infinite-dimensional differential algebraic equations. Such equations often arise as partial differential-algebraic equations (PDAEs). A decomposition of the state-space that leads to an extension of the Hille-Yosida Theorem on reflexive Banach spaces is described. For dissipative partial differential equations the Lumer-Phillips generation theorem characterizes solvability and also boundedness of the associated semigroup. An extension of the Lumer-Phillips generation theorem to dissipative differential-algebraic equations is given. The results are illustrated by coupled systems and the Dzektser equation.**

I. Introduction

We consider infinite-dimensional differential-algebraic equations (DAEs)

$$\frac{d}{dt}Ex(t) = Ax(t), \quad t \geq 0, \qquad Ex(0) = z_0. \tag{1}$$

Here A and E are linear operators from $\mathcal{X}$ to $\mathcal{Z}$ and $z_0 \in \mathcal{Z}$, where $\mathcal{X}$ and $\mathcal{Z}$ are reflexive Banach spaces. The operator E is bounded from $\mathcal{X}$, but A is densely defined and closed on $\mathcal{X}$. Such equations arise from the coupling of partial differential equations where one sub-system is in equilibrium and also from constraints.

Establishing well-posedness of these equations, particularly when E is not invertible, is non-trivial; see [Rei08], [Tro20], [TT01], [TT96], [FY04], [Sho10], [Yag91], [FY99]. Sufficient conditions in terms of Hille-Yosida type resolvent estimates can be found in [RT05], [Tro20]. In order to show solvability, Trostorff [Tro20] exploit Wong sequences associated with (E, A). In [TT96] the splitting $\mathcal{X} = \ker E \oplus \overline{\operatorname{ran} E^*}$ and $\mathcal{Z} = \ker E^* \oplus \overline{\operatorname{ran} E}$ (where $\oplus$ indicates the direct sum of spaces) and the solvability of (1) is investigated.

In [SF03] a concept called (E, p)-radiality is introduced that also leads to Hille-Yosida type conditions for generation of a semigroup. Under associated conditions, there exists a splitting of $\mathcal{X}$ and $\mathcal{Z}$ into $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$ and $\mathcal{Z} = \mathcal{Z}^0 \oplus \mathcal{Z}^1$. Unlike the approach in [TT96] this splitting is not in general orthogonal. In the case $p = 0$ equations (1) are rewritten as

$$\frac{d}{dt}\begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix}\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad t \geq 0, \tag{2}$$
$$E_1 x^1(0) = (x_0)^1.$$

where $E_1 : \mathcal{X}^1 \to \mathcal{Z}^1$ is bounded and invertible, $A_0 : \mathcal{D}(A) \cap \mathcal{X}^0 \to \mathcal{Z}^0$ is closed and invertible, $A_1 : \mathcal{D}(A) \cap \mathcal{X}^1 \to \mathcal{Z}^1$ is closed, and $A_1 E_1^{-1}$ generates a C_0-semigroup in $\mathcal{Z}^1$. Here $\mathcal{D}(A)$ denotes the domain of the operator A. Since we work with reflexive spaces, we are able to simplify and weaken the required conditions considerably.

The difficulty with this approach, as with the classical Hille-Yosida Theorem, and other resolvent estimates, is that it can be difficult to confirm that the assumptions are satisfied. The Lumer-Phillips Theorem e.g. [CZ95] is a very useful tool in the standard $E = I$ situation for establishing that an operator A generates a C_0-semigroup. Favini and Yagi [FY99, Page 37] show that if $(\lambda E - A)^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ for some $\lambda > 0$ and $\operatorname{Re}\langle Ax, Ex\rangle_{\mathcal{Z}} \leq 0$ for all $x \in \mathcal{D}(A)$, then for every $x_0 \in E(\mathcal{D}(A))$ the DAE (1) has a unique classical solution; that is $x : [0, T] \to \mathcal{D}(A)$ and $Ex \in C^1([0, T]; \mathcal{Z})$, $Ax \in C([0, T]; Z)$, and DAE (1) is satisfied. They further provide results for parabolic DAEs. However, they do not investigate a splitting of the state space as in (2), nor do they show generation of a C_0-semigroup. The main result of this paper is a generalization of the Lumer-Phillips Theorem to dissipative infinite-dimensional DAEs. It is shown that assumptions similar to those of the classical Lumer-Phillips Theorem imply E-radiality and hence generation of a contraction semigroup on a closed subspace of $\mathcal{Z}$.

The framework of (E, r)-radiality is first summarized and adapted to the reflexive Banach space situation in the next section. Some new results are proven. In Section 3 we prove a Lumer-Phillips Theorem for dissipative infinite-dimensional DAEs on Hilbert spaces. Well-posedness of a class of coupled systems is shown in Section 4. Finally in Section 5, the results are applied to the Dzektser equation.

Notation: By $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ we denote the set of all linear and bounded operators from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$. We denote the kernel of an operator A by $\ker A$ and its range by $\operatorname{ran} A$. If an operator A is closable [Kre78, sect. 10.3], $\overline{A}$ denotes its closure.

II. Radiality and semigroup generation

Let $\mathcal{X}$, $\mathcal{Z}$ be reflexive Banach spaces, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{Z}$ densely defined and closed, and $z_0 \in \mathcal{Z}$. The *resolvent set* of the operator pencil (E, A) is denoted

$$\varrho(E, A) := \{s \in \mathbb{C} \mid (sE - A)^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})\}.$$

For $s \in \varrho(E, A)$, we define the right- and left-E resolvents of A (with respect to E) by

$$R^E(s, A) = (sE - A)^{-1}E, \quad L^E(s, A) = E(sE - A)^{-1}.$$

Definition 2.1: The operator A is E*-radial* if

- $s \in \varrho(E, A)$ for all real $s > 0$,

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- there exists $K > 0$ such that for all $n \in \mathbb{N}$ and for all real $s > 0$

$$\| \left(R^E(s,A)\right)^n \|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq \frac{K}{s^n}, \tag{3}$$

$$\| \left(L^E(s,A)\right)^n \|_{\mathcal{L}(\mathcal{Z},\mathcal{Z})} \leq \frac{K}{s^n}, \tag{4}$$

In the case $E = I$ statements (3) and (4) are equivalent; and either statement implies generation of a C_0-semigroup by the Hille-Yosida Theorem. In [SF03] a more general concept, (E,p) radiality, is considered. In that framework, an E-radial operator is $(E,0)$-radial. Here, only $p = 0$ is considered and also the spaces are assumed to be reflexive Banach spaces.

Definition 2.2: The operator A is *weakly E-radial* if $s \in \varrho(E,A)$ for all $s > 0$, and (3)-(4) holds with $n = 1$.
Clearly any E-radial operator is weakly E-radial. The converse holds if $K \leq 1$.

Define for some $\alpha \in \varrho(E,A)$,

$$\mathcal{X}^0 = \ker R^E(\alpha,A), \qquad \mathcal{Z}^0 = \ker L^E(\alpha,A),$$
$$\mathcal{X}^1 = \overline{\operatorname{ran} R^E(\alpha,A)}, \qquad \mathcal{Z}^1 = \overline{\operatorname{ran} L^E(\alpha,A)}.$$

It is easy to show that $\mathcal{X}^0 = \ker E$. Also, $z \in \mathcal{Z}^0$ if and only if $x = (\alpha E - A)^{-1} z \in \ker E$. Rewriting,

$$(\alpha E - A)x = z$$

and since $x \in \ker E$, $z = Ax$ for some $x \in \mathcal{D}(A) \cap \ker E$. Thus,

$$\mathcal{Z}^0 = \{Ax \mid x \in \mathcal{D}(A) \cap \ker E\}.$$

These spaces are independent of the choice of α ([SF03, Lem. 2.1.2, pg. 18]). Also, if A is weakly E-radial, then

$$\lim_{s\to\infty} sR^E(s,A)x = x, \quad \text{for all } x \in \mathcal{X}^1,$$
$$\lim_{s\to\infty} sL^E(s,A)z = z, \quad \text{for all } z \in \mathcal{Z}^1,$$

see [SF03, Lem. 2.2.6]. If A is weakly E-radial, then since $\mathcal{X}$ and $\mathcal{Z}$ are reflexive, [SF03, Theorem 2.5.1] implies

$$\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1 \qquad \text{and} \qquad \mathcal{Z} = \mathcal{Z}^0 \oplus \mathcal{Z}^1.$$

If A is weakly E-radial, then

- $P : \mathcal{X} \to \mathcal{X}$ defined by

$$Px := \lim_{s\to\infty} sR^E(s,A)x$$

  is a projection onto $\mathcal{X}^1$ with $\ker P = \mathcal{X}^0$, $\operatorname{ran} P = \mathcal{X}^1$;
- $Q : \mathcal{Z} \to \mathcal{Z}$ defined by

$$Qz := \lim_{s\to\infty} sL^E(s,A)z$$

  is a projection onto $\mathcal{Z}^1$ with $\ker Q = \mathcal{Z}^0$, $\operatorname{ran} Q = \mathcal{Z}^1$.

The assumption that A is weakly E-radial implies that P and Q are bounded operators. In general, both P and Q are non-orthogonal projections.

Define restrictions of E and A as follows:

$$E_0 := E|_{\mathcal{X}^0}, \quad A_0 := A|_{D(A_0)},\ D(A_0) = \mathcal{X}^0 \cap \mathcal{D}(A),$$

$$E_1 := E|_{\mathcal{X}^1}, \quad A_1 := A|_{D(A_1)},\ D(A_1) = \mathcal{X}^1 \cap \mathcal{D}(A).$$

In [SF03, Lem. 2.2.1, pg. 20] it is shown that $E_0 \in \mathcal{L}(\mathcal{X}^0, \mathcal{Z}^0)$ and $A_0 : \mathcal{D}(A_0) \to \mathcal{Z}^0$. Further, if A is weakly E-radial, then A_0 is boundedly invertible; that is,

$$A_0^{-1} \in \mathcal{L}(\mathcal{Z}^0, \mathcal{X}^0),$$

see [SF03, Lem. 2.2.4, pg. 22]. Also, here E_0 is the zero operator

$$A_0^{-1}E_0 = 0, \quad E_0 A_0^{-1} = 0$$

on $\mathcal{X}^0$ and $\mathcal{Z}^0$, respectively, by [SF03, Lem. 2.2.5, pg. 22].

The following proposition has been proved in [SF03, Cor. 2.5.1,pg 38] with assumptions that can be weakened because we deal with reflexive spaces.

Proposition 2.3: If A is weakly E-radial, then

1) for all $x \in \mathcal{D}(A)$, $Px \in \mathcal{D}(A)$ and $APx = QAx$,
2) for all $x \in \mathcal{X}$, $EPx = QEx$.

Proof: Recall that the operator P is defined by

$$Px = \lim_{s\to\infty} sR^E(s,A)x.$$

For any $x \in \mathcal{D}(A) \subset \mathcal{X}$, by [SF03, Equation (2.1.8), pg. 17]

$$AR^E(s,A)x = L^E(s,A)Ax.$$

Let $x \in \mathcal{D}(A)$. Since $R^E(s,A)x \in \mathcal{D}(A)$, and A is closed, $Px \in \mathcal{D}(A)$. Thus

$$\begin{aligned} APx &= A(\lim_{s\to\infty} sR^E(s,A)x) \\ &= \lim_{s\to\infty} AsR^E(s,A)x \\ &= \lim_{s\to\infty} sL^E(s,A)Ax \\ &= QAx. \end{aligned}$$

This proves Part 1). Part 2) follows easily using the fact that $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. For any $x \in \mathcal{X}$,

$$\begin{aligned} EPx &= E \lim_{s\to\infty} sR^E(s,A)x \\ &= \lim_{s\to\infty} sER^E(s,A)x \\ &= \lim_{s\to\infty} sE(sE-A)^{-1}Ex \\ &= \lim_{s\to\infty} sL^E(s,A)Ex \\ &= QEx. \end{aligned}$$

This concludes the proof. □

Thus, if A is weakly E-radial, then the operators A_0, A_1, E_0 and E_1 are invariant with respect to the projected spaces. More precisely, if A is weakly E-radial, by [SF03, Lem. 2.2.1, pg. 20 and Cor. 2.5.2, pg. 39]

- $E_0 \in \mathcal{L}(\mathcal{X}^0, \mathcal{Z}^0)$,
- $E_1 \in \mathcal{L}(\mathcal{X}^1, \mathcal{Z}^1)$,
- $A_0 : \mathcal{D}(A_0) \subset \mathcal{X}^0 \to \mathcal{Z}^0$ is densely defined, closed, and boundedly invertible,
- $A_1 : \mathcal{D}(A_1) \subset \mathcal{X}^1 \to \mathcal{Z}^1$ is densely defined and closed.

The following proposition was proven in [SF03, Thm. 2.5.3, pg. 40] with stronger assumptions.

Proposition 2.4: If A is weakly E-radial and $\operatorname{ran} E$ is closed in $\mathcal{Z}$, then $E_1 \in \mathcal{L}(\mathcal{X}^1, \mathcal{Z}^1)$ is boundedly invertible.

Proof: The fact that $E_1 \in \mathcal{L}(\mathcal{X}^1, \mathcal{Z}^1)$ follows from [SF03, Cor. 2.5.2, pg. 39]. Since $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, and $\mathcal{X}^0 = \ker E$, it follows that E_1 is injective. By definition of $\mathcal{Z}_1$,

$$\mathcal{Z}^1 = \overline{E(\mathcal{D}(A))} \subset \operatorname{ran} E = \operatorname{ran} E_1.$$

Since $\operatorname{ran} E_1 \subset \mathcal{Z}^1$, E_1 is surjective. Thus E_1 has an inverse defined on all of $\mathcal{Z}^1$ and so by the Closed Graph Theorem this inverse is bounded. □

If A is weakly E-radial and $\operatorname{ran} E$ is closed, then the system can be decomposed into simpler subsystems using the non-orthogonal projections P and Q. Define

$$\widetilde{P} = \begin{bmatrix} I - P \\ P \end{bmatrix} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^0 \times \mathcal{X}^1),$$
$$\widetilde{Q} = \begin{bmatrix} I - Q \\ Q \end{bmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^0 \times \mathcal{Z}^1),$$

and also

$$\widetilde{P}^{-1} = \begin{bmatrix} I & I \end{bmatrix} \in \mathcal{L}(\mathcal{X}^0 \times \mathcal{X}^1, \mathcal{X}),$$
$$\widetilde{Q}^{-1} = \begin{bmatrix} I & I \end{bmatrix} \in \mathcal{L}(\mathcal{Z}^0 \times \mathcal{Z}^1, \mathcal{Z}),$$

where I above indicates the natural injection on the various spaces; the different spaces are not explicitly indicated. Let $z \in \mathcal{D}(A)$ and $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} := \widetilde{P} z$. System (1) can be written

$$E\widetilde{P}^{-1} \begin{bmatrix} \dot{z}_0 \\ \dot{z}_1 \end{bmatrix} = A\widetilde{P}^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}.$$

Premultiplying by $\widetilde{Q}$,

$$\widetilde{Q}E\widetilde{P}^{-1} \begin{bmatrix} \dot{z}_0 \\ \dot{z}_1 \end{bmatrix} = \widetilde{Q}A\widetilde{P}^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}.$$

Now,

$$\widetilde{Q}E\widetilde{P}^{-1} = \begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix}, \quad \widetilde{Q}A\widetilde{P}^{-1} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}.$$

Since $E_0 = 0$, (1) is equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{z}_0 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & E_1^{-1}A_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}.$$

Our main result in this section is as follows.

Theorem 2.5: If $A - \alpha E$ is E-radial and $\operatorname{ran} E$ is closed, then the operator $E_1^{-1}A_1$ with domain $\mathcal{D}(A) \cap \mathcal{X}^1$ generates a C_0-semigroup $(S(t))_{t \geq 0}$ on $\mathcal{X}^1$ with bound $Ke^{\alpha t}$. The component on $\mathcal{X}^0$ is identically zero.

Proof: First define $\widetilde{A} = (A - \alpha E)$. By our assumption the operator $E_1^{-1}\widetilde{A}_1$ with domain $\mathcal{D}(A) \cap \mathcal{X}^1$ is well-defined, closed and densely defined. The definition of E-radiality further implies $(0, \infty) \in \varrho(E_1^{-1}\widetilde{A}_1)$ and there exists $K > 0$ such that for all $n \in \mathbb{N}$ and for all real $s > 0$

$$\|(sI - E_1^{-1}\widetilde{A}_1)^n\|_{\mathcal{L}(\mathcal{X}^1, \mathcal{X}^1)} \leq \frac{K}{s^n}.$$

The Hille-Yosida Theorem implies $E_1^{-1}\widetilde{A}_1$ generates a bounded C_0-semigroup on $\mathcal{Z}^1$ with bound K. Since $\widetilde{A}_1 = (A - \alpha E)_1 = A_1 - \alpha E_1$, the theorem follows. □

Alternatively, the same projections can be applied to

$$\frac{d}{dt}Ez = Az.$$

This yields

$$\begin{aligned} & \frac{d}{dt}Ez = Az \\ \iff \quad & \frac{d}{dt}\widetilde{Q}E\widetilde{P}^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \widetilde{Q}A\widetilde{P}^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \\ \iff \quad & \frac{d}{dt} \begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \\ \iff \quad & \frac{d}{dt} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_1E_1^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \end{aligned}$$

where $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0^{-1}x_0 \\ E_1^{-1}x_1 \end{bmatrix}$.

Corollary 2.6: If $A - \alpha I$ is E-radial and $\operatorname{ran} E$ is closed, then the operator $A_1E_1^{-1}$ with domain $E_1(\mathcal{D}(A) \cap \mathcal{X}^1)$ generates a C_0 semigroup on $\mathcal{Z}^1$ with bound $Ke^{\alpha t}$.

Thus well-posedness of the Cauchy problem $E\frac{d}{dt}x = Ax$ reduces to well-posedness of $\dot{x}_1 = E_1^{-1}A_1x_1$ and well-posedness of $\frac{d}{dt}(Ex) = Ax$ is reduced to well-posedness of $\dot{z}_1 = A_1E_1^{-1}z_1$.

III. Dissipative pencils

It is not in general easy to verify that an operator A is E-radial. Therefore, in this section we aim to generalize the Lumer-Phillips theorem to dissipative differential-algebraic equations. We start with the following definition. The spaces $\mathcal{X}$ and $\mathcal{Z}$ are complex Hilbert spaces, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, and $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{Z}$ is densely defined and closed.

Definition 3.1: The operator pencil (E, A) is called dissipative, if

$$\|(\lambda E - A)x\|_{\mathcal{Z}} \geq \lambda \|Ex\|_{\mathcal{Z}}, \qquad \lambda > 0, x \in \mathcal{D}(A). \quad (5)$$

In the following three propositions we define dissipative operator pencils and prove some implications.

Proposition 3.2: The following statements are equivalent:

1) The operator pencil (E, A) is dissipative.
2) $\operatorname{Re}\langle Ax, Ex \rangle_{\mathcal{Z}} \leq 0$, $x \in \mathcal{D}(A)$,

Proof: Squaring both sides of (5) and simplifying yields that it is equivalent to

$$\frac{1}{2\lambda}\|Ax\|^2 \geq \operatorname{Re}\langle Ax, Ex \rangle.$$

This implies the statement of the proposition. □

Remark 3.3: If $\frac{d}{dt}Ex(t) = Ax(t)$, $t \geq 0$, has a classical solution and satisfies $\frac{d}{dt}\|Ex(t)\|^2 \leq 0$ for every classical solution x, then the operator pencil is dissipative. We note that $x : [0, T] \to \mathcal{D}(A)$ is a classical solution if $Ex \in C^1([0, T]; \mathcal{Z})$, $Ax \in C([0, T]; Z)$, and DAE (1) is satisfied.

Proposition 3.4: If the operator pencil (E, A) is dissipative, then the following are equivalent

1) $\ker A \cap \ker E = \{0\}$,
2) $\ker(\lambda E - A) = \{0\}$ for one $\lambda > 0$,
3) $\ker(\lambda E - A) = \{0\}$ for every $\lambda > 0$,

Proof: Clearly statement (3) implies (2) and (2) implies (1). Thus it remains to show that (1) implies (3). Assume that (3) does not hold, that is, there exists $\lambda > 0$ and $x \in \ker(\lambda E - A)$, $x \neq 0$. This implies

$$\lambda^2 \|Ex\|^2 - 2\lambda \operatorname{Re}\langle Ex, Ax\rangle + \|Ax\|^2 = 0.$$

Since (E, A) is dissipative, all terms on the right hand side are non-negative. Thus $Ex = 0$ and $Ax = 0$, which implies that (1) does not hold. Hence statement (1) implies (3). □

Proposition 3.5: If the operator pencil (E, A) is dissipative and $\varrho(E, A) \cap (0, \infty) \neq \emptyset$, then $(0, \infty) \subset \varrho(E, A)$.

Proof: Let $\lambda_0 \in \varrho(E, A) \cap (0, \infty)$. Proposition 3.4 implies that $\ker(\lambda E - A) = \{0\}$ for every $\lambda > 0$.

Next we show that $\operatorname{ran}(\lambda E - A)$ is dense in $\mathcal{Z}$ for every $\lambda > 0$. Let $\lambda > 0$ be arbitrary and $z \in \mathcal{Z}$ be orthogonal to $\operatorname{ran}(\lambda E - A)$. Since $\operatorname{ran}(\lambda_0 E - A) = \mathcal{Z}$ there exists $x \in \mathcal{D}(A)$ such that $z = (\lambda_0 E - A)x$. This implies

$$\begin{aligned} 0 &= \operatorname{Re}\langle z, (\lambda E - A)x\rangle \\ &= \operatorname{Re}\langle (\lambda_0 E - A)x, (\lambda E - A)x\rangle \\ &= \lambda_0 \lambda \|Ex\|^2 - (\lambda + \lambda_0) \operatorname{Re}\langle Ex, Ax\rangle + \|Ax\|^2. \end{aligned}$$

Because (E, A) is dissipative, all terms of the right hand side are non-negative. Thus $Ex = 0$ and $Ax = 0$, which implies $\lambda_0 Ex - Ax = 0$. Since $\ker(\lambda_0 E - A) = \{0\}$, it follows that $x = 0$ and therefore $z = 0$. Thus $\operatorname{ran}(\lambda E - A)$ is dense in $\mathcal{Z}$ for every $\lambda > 0$.

It remains to show that $\operatorname{ran}(\lambda E - A)$ is closed in $\mathcal{Z}$ for every $\lambda > 0$. We note that for an injective, closed and densely defined operator $T : \mathcal{D}(T) \subset \mathcal{X} \to \mathcal{Z}$ the following statements are equivalent

- $\operatorname{ran}(T)$ is closed in $\mathcal{Z}$.
- There exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in \mathcal{D}(T)$.

Thus there exists $c_{\lambda_0} > 0$ such that $\|(\lambda_0 E - A)x\| \geq c_{\lambda_0}\|x\|$ for every $x \in \mathcal{D}(A)$. This implies that for any $x \in \mathcal{D}(A)$ with $\|x\| = 1$:

$$\lambda_0^2 \|Ex\|^2 - 2\lambda_0 \operatorname{Re}\langle Ex, Ax\rangle + \|Ax\|^2 \geq c_{\lambda_0}. \quad (6)$$

It remains to show that for every $\lambda > 0$ there exists $c_\lambda > 0$ such that

$$\lambda^2 \|Ex\|^2 - 2\lambda \operatorname{Re}\langle Ex, Ax\rangle + \|Ax\|^2 \geq c_\lambda \quad (7)$$

for all $x \in \mathcal{D}(A)$ with $\|x\| = 1$. Assume that this is not true: that is, there exists $\lambda > 0$ and a sequence $(x_n)_n \subset \mathcal{D}(A)$ with $\|x_n\| = 1$ such that

$$\lambda^2 \|Ex_n\|^2 - 2\lambda \operatorname{Re}\langle Ex_n, Ax_n\rangle + \|Ax_n\|^2 \to 0$$

as $n \to \infty$. As (E, A) is dissipative, all three terms are non-negative and thus $\|Ex_n\|^2 \to 0$, $\|Ax_n\|^2 \to 0$ and $\langle Ex_n, Ax_n\rangle \to 0$ as $n \to \infty$, which implies that (6) does not hold. Thus, statement (7) holds. □

The first main result of this section is summarized in the following theorem.

Theorem 3.6: If $\lambda \in \varrho(E, A)$ for some $\lambda > 0$ and

$$\begin{aligned} \operatorname{Re}\langle Ax, Ex\rangle_{\mathcal{Z}} \leq 0, &\quad x \in \mathcal{D}(A), \\ \operatorname{Re}\langle A^*x, E^*x\rangle_{\mathcal{X}} \leq 0, &\quad x \in \mathcal{D}(A^*), \end{aligned}$$

then

1) $(0, \infty) \subset \varrho(E, A)$ and $(0, \infty) \subset \varrho(E^*, A^*)$.
2) $\|E(\lambda E - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$,
3) $\|(\lambda E - A)^{-1}E\| \leq \frac{1}{\lambda}$ for $\lambda > 0$,
4) A is E-radial.

Proof: The first statement follows from Proposition 3.5, and the second from Proposition 3.2. Moreover, Proposition 3.2 implies

$$\|E^*(\lambda E^* - A^*)^{-1}\| \leq \frac{1}{\lambda}$$

for $\lambda > 0$, which implies the third statement. Thus the system is weakly E-radial with $K \leq 1$, which implies (4). □

Theorem 3.6 is similar to the Lumer-Phillips Theorem, except that additional assumption on the dissipativeness of the adjoint system is needed. This reflects the non-equivalence of the left and right E-resolvents.

Theorem 3.6 and Theorem 2.5 imply the second result of this section.

Theorem 3.7: If the operator E has closed range, $\lambda \in \varrho(E, A)$ for some $\lambda > 0$ and

$$\begin{aligned} \operatorname{Re}\langle Ax, Ex\rangle_{\mathcal{Z}} \leq 0, &\quad x \in \mathcal{D}(A), \\ \operatorname{Re}\langle A^*x, E^*x\rangle_{\mathcal{X}} \leq 0, &\quad x \in \mathcal{D}(A^*), \end{aligned}$$

Then the Hilbert spaces $\mathcal{X}$, $\mathcal{Z}$ can be split as $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$ and $\mathcal{Z} = \mathcal{Z}^0 \oplus \mathcal{Z}^1$, where

$$\begin{aligned} \mathcal{X}^0 &= \ker E, & \mathcal{Z}^0 &= \{Ax \mid x \in \mathcal{D}(A) \cap \ker E\}, \\ \mathcal{X}^1 &= \overline{\operatorname{ran} R^E(\alpha, A)}, & \mathcal{Z}^1 &= \operatorname{ran} E \end{aligned}$$

for some (and hence every) $\alpha \in \rho(E, A)$. Further, $P : \mathcal{X} \to \mathcal{X}$ defined by $Px := \lim_{s\to\infty} sR^E(s, A)x$ is a projection onto $\mathcal{X}^1$ with $\ker P = \mathcal{X}^0$, and $Q : \mathcal{X} \to \mathcal{X}$ defined by $Qz := \lim_{s\to\infty} sL^E(s, A)z$ is a projection onto $\mathcal{Z}^1$ with $\ker Q = \mathcal{Z}^0$. Defining $\mathcal{D}(A_1) = \mathcal{X}^1 \cap \mathcal{D}(A)$,

$$E_1 = E|_{\mathcal{X}^1} : \mathcal{X}^1 \to \mathcal{Z}^1, \quad A_1 = A|_{\mathcal{D}(A_1)} : \mathcal{D}(A_1) \to \mathcal{Z}^1,$$

E_1 is boundedly invertible, and A_1 is closed and densely defined.

The equation $\frac{d}{dt}(Ex) = Ax$ is equivalent to

$$\frac{d}{dt}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & E_1^{-1}A_1 \end{bmatrix}\begin{bmatrix} x^0 \\ x^1 \end{bmatrix},$$

where $x = \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}$, and $E_1^{-1}A_1$ generates a contraction semigroup on $\mathcal{X}^1$.

IV. Coupled Systems

In this section we study an application of Theorem 2.5 to a class of differential-algebraic systems defined on a reflexive Banach space $\mathcal{Z}$,

$$\begin{aligned} \frac{d}{dt}x(t) &= A_1 x(t) + A_2 y(t) \\ 0 &= A_3 x(t) + A_4 y(t), \end{aligned}$$

where for $i = 1, \dots, 4$, $A_i : \mathcal{D}(A_i) \subset Z \to Z$ are closed and densely defined. This class of systems is of the form (1):

$$\frac{d}{dt}(\underbrace{\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]}_{E} x(t)) = \underbrace{\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right]}_{A} x(t), \qquad t > 0, \tag{8}$$

with $\mathcal{X} = \mathcal{Z} = Z \times Z$ and

$$\mathcal{D}(A) = (\mathcal{D}(A_1) \cap \mathcal{D}(A_3)) \times (\mathcal{D}(A_2) \cap \mathcal{D}(A_4)).$$

Although this is a very particular class of systems, a number of applications fit this class. For examples see [MO14] for piezo-electric beams with quasi-static magnetic effects, [CKC+10] for lithium-ion cell models and [HHOS07] for a model with convection-diffusion dynamics.

Throughout this section we make the following assumptions that will guarantee well-posedness. The notation $\overline{A}$ indicates the closure of an operator.

Assumption 4.1: (a) Let A_4 have a bounded inverse, $\mathcal{D}(A_4) \subset \mathcal{D}(A_2)$ and $\mathcal{D}(A_4^*) \subset \mathcal{D}(A_3^*)$. By [Tre08, Rem. 2.2.315], this implies that the operator $A_2 A_4^{-1} A_3 : \mathcal{D}(A_3) \to Z$ is well defined. Assume also $\overline{A_2 A_4^{-1} A_3} \in \mathcal{L}(Z)$.

(b) Let there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $s > \omega$, $s \in \varrho(A_1)$ and

$$\|(s - A_1)^{-n}\| \leq \frac{M}{(s-\omega)^n}, \quad s > \omega, n \in \mathbb{N}.$$

Remark 4.2: If $A_2, A_3 \in \mathcal{L}(Z)$, then Assumption 4.1a reduces to $0 \in \varrho(A_4)$.

Remark 4.3: By the Hille-Yosida Theorem, Assumption 4.1b is equivalent to assuming that A_1 generates a C_0-semigroup on Z.

For $\mu > \omega$ define the *Schur complement*

$$S_1(\mu) : \mathcal{D}(A_1) \subset Z \to Z$$

by

$$S_1(\mu) := \mu - A_1 + \overline{A_2 A_4^{-1} A_3}.$$

Because A_1 is closed and densely defined, the Schur complement $S_1(\mu)$ is closed and densely defined. Moreover, we can factor $S_1(\mu)$ as

$$S_1(\mu) := (\mu - A_1)[I + (\mu - A_1)^{-1}\overline{A_2 A_4^{-1} A_3}]. \tag{9}$$

A Neumann series argument yields that the operator $S_1(\mu)$ is invertible for $\mu > \omega_0 := \omega + M\|\overline{A_2 A_4^{-1} A_3}\|$ and also

$$\begin{aligned} \|S_1(\mu)^{-n}\| &\leq \|(\mu - A_1)^{-n}\| \|[I + (\mu - A_1)^{-1}\overline{A_2 A_4^{-1} A_3}]^{-1}\|^n \\ &\leq \frac{M}{(\mu-\omega)^n} \frac{1}{\left(1 - \frac{M\|\overline{A_2 A_4^{-1} A_3}\|}{\mu - \omega}\right)^n} \\ &= \frac{M}{(\mu - \omega_0)^n}. \end{aligned} \tag{10}$$

Application of [Tre08, Thm. 2.3.3] yields immediately the statement of the following proposition.

Proposition 4.4: The operator A is closable and for every $\mu > \omega_0$ we have that $\mu \in \varrho(E, \overline{A})$ and

$$\begin{aligned} (\mu E - \overline{A})^{-1} &= \begin{pmatrix} \mu - A_1 & -A_2 \\ -A_3 & -A_4 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I & 0 \\ -\overline{A_4^{-1} A_3} & I \end{pmatrix} \begin{pmatrix} S_1(\mu)^{-1} & 0 \\ 0 & -A_4^{-1} \end{pmatrix} \begin{pmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} S_1(\mu)^{-1} & -S_1(\mu)^{-1} A_2 A_4^{-1} \\ -\overline{A_4^{-1} A_3} S_1(\mu)^{-1} & * \end{pmatrix}. \end{aligned}$$

Using Proposition 4.4,

$$\begin{aligned} ((\mu E - A)^{-1} E)^n &= \begin{pmatrix} S_1(\mu)^{-n} & 0 \\ -\overline{A_4^{-1} A_3} S_1(\mu)^{-n} & 0 \end{pmatrix}, \\ &= \begin{pmatrix} I & 0 \\ -\overline{A_4^{-1} A_3} & 0 \end{pmatrix} \begin{pmatrix} S_1(\mu)^{-n} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{11}$$

$$\begin{aligned} (E(\mu E - A)^{-1})^n &= \begin{pmatrix} S_1(\mu)^{-n} & -S_1(\mu)^{-n} A_2 A_4^{-1} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} S_1(\mu)^{-n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -A_2 A_4^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{12}$$

These calculations together with (10) show that $A - \omega_0 E$ is E-radial. Further, $\operatorname{ran} E$ is closed. Thus Theorem 2.5 is applicable.

The projections P and Q will be explicitly calculated for this class of systems. Note that because of Assumption 4.1, A_1 generates a C_0-semigroup, and thus

$$\lim_{s \to \infty} s(S_1(s))^{-1} z = z.$$

This implies

$$\begin{aligned} P \begin{pmatrix} x \\ y \end{pmatrix} &= \lim_{s \to \infty} s(sE - \overline{A})^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix} \\ &= \lim_{s \to \infty} \begin{pmatrix} sS_1(s)^{-1} x \\ -\overline{A_4^{-1} A_3} s S_1(s)^{-1} x \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -\overline{A_4^{-1} A_3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

and similarly

$$Q \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & -A_2 A_4^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

V. Example: Dzektser equation

We consider the Dzektser equation

$$\frac{\partial}{\partial t}\left(1 + \frac{\partial^2}{\partial \zeta^2}\right) x(\zeta, t) = \left(\frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial^4}{\partial \zeta^4}\right) x(\zeta, t),$$

$t > 0$ and $\zeta \in (0, \pi)$, with boundary conditions

$$\begin{aligned} x(0, t) = x(\pi, t) = 0, \quad t > 0 \\ \frac{\partial^2 x}{\partial \zeta^2}(0, t) = \frac{\partial^2 x}{\partial \zeta^2}(\pi, t) = 0, \quad t > 0. \end{aligned}$$

Let $\mathcal{Z} = L^2(0, \pi)$ and $\mathcal{X} = H^2(0, \pi) \cap H_0^1(0, \pi)$ with $\|x\|_{\mathcal{X}} = \|x''\|_{\mathcal{Z}}$, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ and $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{Z}$

given by

$$\begin{aligned}Ex &= x + x'',\\ Ax &= x'' + 2x^{(4)},\\ \mathcal{D}(A) &= \{x \in H^4(0,\pi) \cap H_0^1(0,\pi) \mid x''(0) = x''(\pi) = 0\}.\end{aligned}$$

This system was shown in [GGZ20] to have a splitting (2) using the eigenfunction expansion. Here the system will be shown to be dissipative E-radial. For $x \in \mathcal{D}(A)$ we calculate

$$\begin{aligned}\operatorname{Re}\langle Ax, Ex\rangle_{\mathcal{Z}} &= \operatorname{Re}\int_0^\pi (x'' + 2x^{(4)})(\overline{x} + \overline{x}'')d\zeta\\ &= -\|x'\|^2_{L^2(0,\pi)} + \|x''\|^2_{L^2(0,\pi)}\\ &\quad - 2\|x^{(3)}\|^2_{L^2(0,\pi)} - 2\operatorname{Re}\int_0^\pi x^{(3)}\overline{x}' d\zeta\\ &\le \|x''\|^2_{L^2(0,\pi)} - \|x^{(3)}\|^2_{L^2(0,\pi)}\\ &\le 0,\end{aligned}$$

by the Poincaré inequality. It is easy to see that $\operatorname{ran}(E - A) = \mathcal{Z}$ and $\ker A \cap \ker E = \{0\}$. Next we calculate $A^* : \mathcal{D}(A^*) \subset \mathcal{Z} \to \mathcal{X}$.

Note that $S : \mathcal{X} \to \mathcal{Z}$ given by $Sf := f''$ is an isometric isomorphism with

$$(S^{-1}f)(x) = \int_0^x (x-t)f(t)dt - \frac{x}{\pi}\int_0^\pi (\pi - t)f(t)dt.$$

Then, for $x \in \mathcal{D}(A)$ and $z \in \mathcal{X}$

$$\begin{aligned}\langle Ax, z\rangle_{\mathcal{Z}} &= \int_0^\pi x''\overline{z} + 2x^{(4)}\overline{z} d\zeta\\ &= \int_0^\pi x''\overline{z} + 2x''\overline{z}'' d\zeta\\ &= \int_0^\pi x''(\overline{S^{-1}z + 2z})'' d\zeta\\ &= \langle x, A^* z\rangle_{\mathcal{X}}\end{aligned}$$

with $A^* z = S^{-1}z + 2z$ for $z \in \mathcal{X}$. For $x \in \mathcal{D}(A^*) = \mathcal{X}$ and $y = S^{-1}x$ we calculate

$$\begin{aligned}\operatorname{Re}\langle A^*x, E^*x\rangle_{\mathcal{X}} &= \operatorname{Re}\langle EA^*x, x\rangle_{\mathcal{Z}}\\ &= \operatorname{Re}\int_0^\pi (S^{-1}x + x + 2x + 2x'')\overline{x} d\zeta\\ &= \operatorname{Re}\int_0^\pi (y + y'' + 2y'' + 2y^{(4)})\overline{y''} d\zeta\\ &= -\|y'\|^2_{\mathcal{Z}} + \|y''\|^2_{\mathcal{Z}}\\ &\quad - 2\operatorname{Re}\int_0^\pi y'\overline{y^{(3)}} d\zeta - 2\|y^{(3)}\|^2_{\mathcal{Z}}\\ &= \|y''\|^2_{\mathcal{Z}} - \|y^{(3)}\|^2_{\mathcal{Z}}\\ &\le 0.\end{aligned}$$

The calculations are simpler than using the eigenfunction expansion. Furthermore, with this approach it can be concluded that not only is the system well-posed, but also that the dynamics are a contraction with respect to the $L^2(0,\pi)$-norm.

Conclusion

Conditions for the solvability of partial differential-algebraic equations on reflexive Banach spaces have been presented and also a generalization of the Lumer-Phillips generation theorem to dissipative differential-algebraic equations. The results were illustrated with some applications.

It is assumed that E is a linear bounded operator from $\mathcal{X}$ to $\mathcal{Z}$. In order to apply the results to some other applications such as dissipative water waves one has to deal with unbounded operators E. This will be the subject of future work.

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