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Strong stochastic Runge-Kutta–Munthe-Kaas methods for nonlinear Itô SDEs on manifolds

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Abstract

In this paper we present numerical methods for the approximation of nonlinear Itô stochastic differential equations on manifolds. For this purpose, we extend Runge-Kutta–Munthe-Kaas (RKMK) schemes for ordinary differential equations on manifolds to the stochastic case and analyse the strong convergence of these schemes. Since these schemes are based on the application of a stochastic Runge-Kutta (SRK) scheme in a corresponding Lie algebra, we address the question under which circumstances the stochastic RKMK method has the same strong order of convergence as the applied SRK scheme. To illustrate our answer to this question and the effectiveness of our schemes, we show some numerical results of applying these methods to a problem with an autonomous underwater vehicle.

Keywords: stochastic Runge-Kutta method, Runge-Kutta–Munthe-Kaas scheme, Casimir functions
2000 MSC: 60H10, 70G65, 91G80

1. Introduction

Let $(\Theta, \mathcal{F}, \mathbb{P})$ be a complete probability space and $W_t = (W_t^1, W_t^2, \ldots, W_t^m)$ an m-dimensional standard Brownian motion w.r.t. a filtration $\mathcal{F}_t$ for $t \geq$

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0 which satisfies the usual conditions. On a manifold $\mathcal{M}$ we consider the following stochastic differential equation (SDE) for $y(t) \in \mathcal{M}$,

$$dy = \Big(\big(\lambda_* f_0(y)\big)(y) + \frac{1}{2} \sum_{i=1}^{m} \Big(\frac{d}{dy} \big(\lambda_* f_i(y)\big)(y) \Big) \big(\lambda_* f_i(y)\big)(y) \Big)\, dt + \sum_{i=1}^{m} \big(\lambda_* f_i(y)\big)(y)\, dW_t^i, \quad y(0) = y_0, \quad (1)$$

where $f_i(y)$ are elements of a corresponding Lie algebra for $i = 0, 1, \ldots, m$ and λ_* maps these elements to $\mathfrak{X}(\mathcal{M})$, the set of all vector fields on the manifold $\mathcal{M}$. We consider (1) as a generic presentation of an Itô SDE on a manifold, which is in accordance with [6, 19], where corresponding Stratonovich presentations of this SDE can be found.

Our goal in this paper is the formulation of an algorithm for the numerical approximation of (1) such that the result will lie on the correct manifold $\mathcal{M}$ as it is well-known that conventional integrators such as stochastic Runge-Kutta (SRK) schemes would give approximations that drift off the manifold. Having dealt only with special cases of manifolds such as matrix Lie groups or the unit sphere in previous work [10, 11] we now consider a more generalized setting such that numerical approximations for (1) can be obtained on any smooth manifold $\mathcal{M}$. For this purpose, we extend the Runge-Kutta–Munthe-Kaas (RKMK) schemes for ordinary differential equations (ODEs) on manifolds [13] to stochastic Runge-Kutta–Munthe-Kaas (SRKMK) schemes, where the main idea is the application of SRK methods in a corresponding Lie algebra followed by a projection back to the manifold $\mathcal{M}$. Since the Lie algebra is a vector space the linear approximation obtained by the SRK scheme will not drift off and due to its projection to $\mathcal{M}$ we get results on the correct manifold by construction.

To our knowledge, this approach was first mentioned in [6]. However, a convergence analysis was not performed by the authors. A weak convergence analysis and an application to the stochastic Landau-Lifshitz equation can be found in [1]. Special cases of strong order 1 SRKMK schemes applied to Itô SDEs on Lie groups were presented in [16, 7, 9].

Our contribution to this area of research is a theorem that gives the conditions such that the SRKMK method applied to a nonlinear SDE on a manifold has the same strong order as the SRK scheme applied in the Lie algebra. This theorem can be considered as an extension of corresponding theorems in [10] and in [11], where the former only mentions linear SDEs on matrix Lie groups and the latter only deals with a nonlinear SDE involved in the rigid body modelling.

The rest of this paper is structured as follows. In Section 2 we derive in more detail the generic presentation of an SDE on a manifold and give examples of what this SDE looks like for commonly used manifolds. Then we present our main results on the strong numerical approximation of (1) in Section 3. Numerical examples are given in Section 4, followed by a conclusion and an outlook in Section 5.

2. Generic presentation of Itô stochastic differential equations on manifolds

Let $\mathcal{M}$ be a manifold and $T_p\mathcal{M}$ be the tangent space at $p \in \mathcal{M}$. A vector field is an assignment of a tangent vector to each point in $\mathcal{M}$. By $\mathfrak{X}(\mathcal{M})$ we denote the set of all vector fields on $\mathcal{M}$. In order to derive the SDE (1) we first give a brief overview of Lie groups and of some related notations.

2.1. Lie group and Lie algebra

A *Lie group* G is a differentiable manifold equipped with a continuous group product $\cdot\colon G \times G \to G$. Here, we will focus on matrix Lie groups, which are Lie groups that are also subgroups of the general linear group $\mathrm{GL}(n)$.

Let G be a matrix Lie group, then the tangent space at the identity I is called the *Lie algebra* $\mathfrak{g}$ of the Lie group G, i.e. $\mathfrak{g} = T_I G$. The Lie algebra is a vector space equipped with a bilinear, skew-symmetric bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is called the Lie bracket or the commutator on $\mathfrak{g}$ and satisfies the Jacobi identity

$$\big[A,[B,C]\big] + \big[C,[A,B]\big] + \big[B,[C,A]\big] = 0.$$

We denote by $\mathrm{ad}_\Omega\colon \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_\Omega(H) = [\Omega, H] = \Omega H - H\Omega$ the adjoint operator, which is used iteratively,

$$\mathrm{ad}_\Omega^0(H) = H, \quad \mathrm{ad}_\Omega^k(H) = \mathrm{ad}_\Omega\left(\mathrm{ad}_\Omega^{k-1}(H)\right) = [\Omega, \mathrm{ad}_\Omega^{k-1}(H)]$$

for $k \geq 1$.

For more information on Lie groups and Lie algebras we refer the interested reader to [3].

2.2. The exponential map

For a matrix Lie group G and its Lie algebra $\mathfrak{g}$, the matrix exponential given by $\exp(\Omega) = \sum_{k=0}^{\infty} \Omega^k/k!$ maps elements from the Lie algebra to the Lie group, i.e. $\exp\colon \mathfrak{g} \to G$. Furthermore, the exponential map is a local diffeomorphism in a neighbourhood of $\Omega = 0$.

The derivative of the matrix exponential is given by

$$\Big(\frac{d}{d\Omega} \exp(\Omega)\Big) H = d\exp_{\Omega}(H) \exp(\Omega)$$

where

$$d\exp_{\Omega}(H) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \operatorname{ad}_{\Omega}^{k}(H). \tag{2}$$

According to the classical Lemma of Baker (1905, see e.g. [2, p. 84]) an inverse of $d\exp_{\Omega}(H)$ exists, if the eigenvalues of $\operatorname{ad}_{\Omega}$ are different from $2\ell\pi i$ with $\ell \in \{\pm 1, \pm 2, \dots\}$. Let B_k denote the Bernoulli numbers defined by $\sum_{k=0}^{\infty} (B_k/k!) x^k = x/(e^x - 1)$, then we have

$$d\exp_{\Omega}^{-1}(H) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega}^{k}(H), \tag{3}$$

which converges for $\|\Omega\| < \pi$.

2.3. Lie group actions and Lie algebra actions

A (left) *Lie group action* is a map $\Lambda\colon G \times \mathcal{M} \to \mathcal{M}$ which satisfies

1. $\Lambda(I, p) = p$,
2. $\Lambda\big(g_1, \Lambda(g_2, p)\big) = \Lambda(g_1 \cdot g_2, p)$ for $g_1, g_2 \in G$.

If Λ is a Lie group action then a (left) *Lie algebra action* $\lambda\colon \mathfrak{g} \times \mathcal{M} \to \mathcal{M}$ is defined by $\lambda(v, p) = \Lambda\big(\exp(v), p\big)$.

For the formulation of a generic presentation of an Itô SDE on a manifold we use that each element of the Lie algebra $\mathfrak{g}$ generates a vector field on the manifold $\mathcal{M}$. Let $\lambda_*\colon \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ be

$$(\lambda_* v)(p) = \frac{d}{dt} \lambda(tv, p)\Big|_{t=0} \tag{4}$$

for $v \in \mathfrak{g}, p \in \mathcal{M}$ (see [13]).

Now, assuming that there exist functions $f_i \colon \mathcal{M} \to \mathfrak{g}$ for $i = 0, 1, \ldots, m$ and applying (4) to the image of these functions at $y(t) \in \mathcal{M}$ we recover the drift and diffusion coefficients of the SDE (1). In order to solve this SDE numerically we first derive a related SDE in the Lie algebra $\mathfrak{g}$.

Theorem 2.1. *Let $\lambda \colon \mathfrak{g} \times \mathcal{M} \to \mathcal{M}$ be a Lie algebra action and $f_i \colon \mathcal{M} \to \mathfrak{g}$ for $i = 0, 1, \ldots, m$. Assume that an Itô SDE for $y(t) \in \mathcal{M}$ is given by (1). For t small enough the solution of this SDE is given by $y(t) = \lambda(\Omega(t), y_0)$ where $\Omega(t) \in \mathfrak{g}$ satisfies*

$$d\Omega = d\exp_{\Omega}^{-1}\Big(f_0\big(\lambda(\Omega, y_0)\big)\Big)dt + \sum_{i=1}^{m} d\exp_{\Omega}^{-1}\Big(f_i\big(\lambda(\Omega, y_0)\big)\Big)dW_t^i,\ \Omega(0) = 0. \tag{5}$$

We consider this Theorem as an extension of [13, Corollary 9] from ODEs to SDEs on manifolds and as an Itô version of [6, Theorem 5.1], where a proof in Stratonovich notation can be found.

2.4. Examples

In the following we specify Lie algebra actions and the corresponding representation of an Itô SDE for common manifolds.

1. $\mathcal{M} = \mathbb{R}^n$: In this case the Lie algebra action is given by $\lambda(v, p) = v + p$ with $(\lambda_* v)(p) = v$. Therefore, we obtain

$$dy = \left(f_0(y) + \frac{1}{2}\sum_{i=1}^{m} f_i'(y) f_i(y)\right) dt + \sum_{i=1}^{m} f_i(y)\, dW_t^i,$$

for (1), where well-known SRK methods can be applied for the numerical approximation.

2. $\mathcal{M} = G$: If the considered manifold is a matrix Lie group we can choose $\lambda(v, p) = \exp(v)p$ and $(\lambda_* v)(p) = vp$. An SDE with a solution evolving on G can then be formulated as

$$dy = \left(f_0(y)y + \frac{1}{2}\sum_{i=1}^{m} \Big(\frac{d}{dy} f_i(y)y\Big) f_i(y)y\right) dt + \sum_{i=1}^{m} f_i(y)y\, dW_t^i.$$

3. $\mathcal{M} = \mathrm{Sym}(n)$: Let $\mathcal{M}$ be the space of symmetric matrices and $G = \mathrm{SO}(n)$ the special orthogonal group with $\mathfrak{g} = \mathfrak{so}(n)$, the space of skew-symmetric matrices. Then, we have $\lambda(v, p) = \exp(v)p\exp(-v)$ with

$(\lambda_* v)(p) = [v, p]$ and

$$dy = \left([f_0(y), y] + \frac{1}{2} \sum_{i=1}^{m} \left(\frac{d}{dy}[f_i(y), y] \right) [f_i(y), y] \right) dt + \sum_{i=1}^{m} [f_i(y), y] \, dW_t^i,$$

is an isospectral flow on $\mathrm{Sym}(n)$. An example on how this isospectral flow can be used to approximate correlation matrices is presented in [12].

4. $\mathcal{M} = \mathfrak{se}(3)^*$: Suppose the manifold is given by the dual of the Lie algebra of the special Euclidean group $\mathrm{SE}(3) \cong \mathrm{SO}(3) \times \mathbb{R}^3$ such that $G = \mathrm{SE}(3)$ and $\mathfrak{g} = \mathfrak{se}(3)$. An element of the Lie group SE(3) can be identified with a 4×4-matrix

$$g = \begin{pmatrix} R & r \\ \mathbf{O} & 1 \end{pmatrix},$$

where $R \in \mathrm{SO}(3)$, $r \in \mathbb{R}^3$ and $\mathbf{O} = (0, 0, 0)$. In the following we will use the shorthand notation $g = (R, r)$ to represent elements of SE(3). To denote an arbitrary element v of the Lie algebra $\mathfrak{se}(3) \cong \mathfrak{so}(3) \times \mathbb{R}^3$ we use the notation $v = (w, u)$ with $w, u \in \mathbb{R}^3$, where we make use of the fact that $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$. The Lie group action $\Lambda\colon \mathrm{SE}(3) \times \mathfrak{se}(3)^* \to \mathfrak{se}(3)^*$ can then be specified by $\Lambda(g, y) = \big(R\pi + r \times (R\rho), R\rho\big)$ for $y = (\pi, \rho) \in \mathfrak{se}(3)^*$ and $g = (R, r) \in \mathrm{SE}(3)$. A more detailed investigation of this manifold will be provided in Section 4.

Since transitive Lie algebra actions can always be found at least locally, any SDE on a manifold $\mathcal{M}$ can be written in the form of (1) (see [13]). More examples can be found in [6].

3. The stochastic Runge-Kutta–Munthe-Kaas (SRKMK) scheme

Inspired by the RKMK schemes for ODEs on manifolds [13] we use the SDE in the Lie algebra (5) to formulate a numerical approximation method for the SDE on the manifold (1).

Algorithm 3.1 (SRKMK)**.** *Divide the time interval $[0, T]$ uniformly into J subintervals $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, J-1$ and define $\Delta = t_{j+1} - t_j$. The following steps are repeated until $t_{j+1} = T$.*

1. ***Initialization step:*** *Let y_j be the approximation of y_t at time $t = t_j$.*

2. ***Numerical method step:*** *Compute* $\Omega_1 \approx \Omega_\Delta$ *by applying a stochastic Runge-Kutta method to Eq.* (5)*.*
3. ***Projection step:*** *Define a numerical solution of Eq.* (1) *as* $y_{j+1} = \lambda(\Omega_1, y_j)$*.*

We recall that an approximating process $\hat{X}_t$ is said to *converge in a strong sense with order* $\gamma > 0$ to the Itô process X_t if there exists a finite constant K and a $\Delta' > 0$ such that

$$\mathbb{E}[|X_T - \hat{X}_T|] \leq K\Delta^\gamma \tag{6}$$

for any time discretization with maximum step size $\Delta \in (0, \Delta')$ [5].
As the SRKMK scheme requires the evaluation of the infinite series (3) the question arises of how many summands of this series have to be computed in order to obtain a scheme of strong order γ.

Theorem 3.2. *Let* q *denote the truncation index in the approximation of* (3)*,*

$$d\exp_\Omega^{-1}(H) \approx \sum_{k=0}^{q} \frac{B_k}{k!}\,\mathrm{ad}_\Omega^k(H) = H - \frac{1}{2}[\Omega, H] + \frac{1}{12}[\Omega, [\Omega, H]] + \dots, \tag{7}$$

and let the stochastic Runge-Kutta scheme applied to Eq. (5) *be of strong order* γ*. Furthermore, assume that* $(f_i \circ \lambda_{y_0})\colon \mathfrak{g} \to \mathfrak{g}$ *fulfills a linear growth condition, i.e.*

$$\|(f_i \circ \lambda_{y_0})(\Omega_s)\|_F \leq a_i + b_i\|\Omega_s\|_F \quad \text{for } a_i, b_i < \infty, \tag{8}$$

where we use the notation $\lambda_{y_0}\colon \mathfrak{g} \to \mathcal{M}$*,* $\lambda_{y_0}(\Omega) = \lambda(\Omega, y_0) = \Lambda(\exp(\Omega), y_0)$ *and* $i = 0, 1, \dots, m$*. If the truncation index* q *satisfies* $q \geq 2\gamma - 2$*, then the SRKMK scheme for solving Eq.* (1) *is also of strong order* γ*.*

Before proving this theorem we provide the following remarks:

1. The linear growth condition (8) is also an assumption which is needed to show the existence and uniqueness of the solution of (5) (see [14, Theorem 5.2.1]).
2. Since $\Omega_1 \mapsto y_{j+1} = \lambda(\Omega_1, y_j)$ (see the last step of Algorithm 3.1) is a smooth mapping it is sufficient to show that the SRK scheme applied to (5) is of the strong order $\gamma = (q + 2)/2$.

3. The proof of this theorem can be conducted very similar to the proof of Theorem 3.2 in [11] with the main difference being the usage of the linear growth condition instead of using properties of the unit sphere, which was the considered manifold in the rigid body problem. Therefore, we only state the main results of this proof with correspondingly made adaptations.

Proof. We denote by Ω_Δ the exact solution of (5) after one time step at $t = \Delta$ and by Ω_Δ^q the exact solution of the truncated version of (5), where the drift and diffusion coefficients are replaced by approximations (7).
Considering the mean-squared error,

$$\begin{aligned}\mathbb{E}[\|\Omega_\Delta - \Omega_1\|_F] &\leq \Big(\mathbb{E}[\|\Omega_\Delta - \Omega_1\|_F^2]\Big)^{1/2} \\ &\leq \Big(\mathbb{E}\big[\|\Omega_\Delta - \Omega_\Delta^q\|_F^2\big]\Big)^{1/2} + \Big(\mathbb{E}\big[\|\Omega_\Delta^q - \Omega_1\|_F^2\big]\Big)^{1/2},\end{aligned}$$

we see that further steps of the proof have only be conducted for the modelling error since the numerical error has strong order γ by construction.
Hence, we use the Itô isometry, calculate the Frobenius norm of the adjoint operator and apply Taylor's theorem to $F(x) = x\big(1 - \cot(x/2)\big)/2 + 2$ with $|F^{q+1}(\xi)| \leq M_q$ for some $M_q < \infty$ and for ξ between 0 and $x = 2\|\Omega_s\|_F$ to

get the following estimation

$$
\begin{aligned}
&\Big(\mathbb{E}\big[\|\Omega_\Delta - \Omega_\Delta^q\|_F^2\big]\Big)^{1/2} \\
&\le \sum_{i=0}^{m} \left(\int_0^\Delta \mathbb{E}\Big[\Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!} \|\operatorname{ad}_{\Omega_s}^k \Big(f_i\big(\lambda_{y_0}(\Omega_s)\big)\Big)\|_F\Big)^2\Big] ds \right)^{1/2} \\
&\le \sum_{i=0}^{m} \left(\int_0^\Delta \mathbb{E}\Big[\Big\|f_i\big(\lambda_{y_0}(\Omega_s)\big)\Big\|_F^2 \Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!} 2^k \|\Omega_s\|_F^k\Big)^2\Big] ds \right)^{1/2} \\
&\le \frac{2^{q+1} M_q}{(q+1)!} \sum_{i=0}^{m} \left(\int_0^\Delta \mathbb{E}\Big[\big\|f_i\big(\lambda_{y_0}(\Omega_s)\big)\big\|_F^2 \|\Omega_s\|_F^{2(q+1)}\Big] ds \right)^{1/2} \\
&\le \frac{2^{q+1} M_q}{(q+1)!} \sum_{i=0}^{m} \left(\int_0^\Delta \Big(a_i^2 \mathbb{E}\Big[\big\|\Omega_s\big\|_F^{2(q+1)}\Big] + 2a_i b_i \mathbb{E}\Big[\big\|\Omega_s\big\|_F^{2(q+3/2)}\Big] + b_i^2 \mathbb{E}\Big[\big\|\Omega_s\big\|_F^{2(q+2)}\Big]\Big) ds \right)^{1/2} \\
&\le \frac{2^{q+1} M_q}{(q+1)!} \sum_{i=0}^{m} \left(\int_0^\Delta \mathcal{O}(s^{q+1}) ds \right)^{1/2} = \mathcal{O}\left(\Delta^{(q+2)/2}\right),
\end{aligned}
$$

where the last line is obtained by applying the Itô-Taylor expansion according to [5, Proposition 5.9.1]. □

4. Numerical Example

We consider an autonomous underwater vehicle, more precisely, an ellipsoidal rigid body submerged in an ideal fluid (see e.g. [4]). Assuming that the vehicle is perturbed by a Wiener process, the dynamics can be described by (1) with $m = 1$ and $y = (\pi, \rho) \in \mathfrak{se}(3)^*$, where $\pi \in \mathfrak{so}(3)^*$ represents the angular momentum and $\rho \in (\mathbb{R}^3)^*$ the linear momentum [6]. The considered manifold $\mathfrak{se}(3)^*$ is the dual of the Lie algebra $\mathfrak{se}(3) \cong \mathfrak{so}(3) \times \mathbb{R}^3$ of the group of rigid body motions. We utilise the isomorphism between the $\mathfrak{so}(3)$ and $\mathbb{R}^3$ via the *hat map*, $\hat{\cdot} \colon \mathbb{R}^3 \to \mathfrak{so}(3)$,

$$
\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \mapsto \hat{\theta} = \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix},
$$

such that $\hat{\theta} z = \theta \times z$ for $\theta, z \in \mathbb{R}^3$. A closed-form expression for the exponential map $\exp_{\mathfrak{so}(3)}$ from $\mathfrak{so}(3)$ to the corresponding Lie group SO(3) is given by the Rodrigues formula (see e.g. [8, p. 291]). It can also be used to compute the exponential map $\exp_{\mathfrak{se}(3)} \colon \mathfrak{se}(3) \to \mathrm{SE}(3)$ for $\Omega = (\theta, \zeta)$,

$$\exp_{\mathfrak{se}(3)}(\Omega) = \begin{pmatrix} \Theta & \frac{1}{\|\theta\|^2}\big((I - \Theta)(\theta \times \zeta) + \theta\theta^\top \zeta\big) \\ \mathbf{O} & 1 \end{pmatrix},$$

where $\Theta = \exp_{\mathfrak{so}(3)}(\hat{\theta})$ [15].

Let $f_i \colon \mathfrak{se}(3)^* \to \mathfrak{se}(3)$, $f_i(y) = (w_i(y), u_i(y))$, be given by the angular velocity $w_i(y) = I_i^{-1}\pi$ and the linear velocity $u_i(y) = M_i^{-1}\rho$, where $I_i = \mathrm{diag}(\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$ is the moment of inertia and $M_i = \mathrm{diag}(\beta_{i1}, \beta_{i2}, \beta_{i3})$ is the mass matrix for $i = 0, 1$. Based on the Lie group action Λ in the fourth example in Section 2.4 the vector fields or rather the coefficients of (1) read

$$\big(\lambda_* f_i(y)\big)(y) = (\pi \times w_i + \rho \times u_i, \rho \times w_i), \quad i = 0, 1.$$

To solve (1) numerically with these coefficients we implemented some SRKMK methods in the software package MATLAB, where we used the Euler-Maruyama scheme, Rößler's scheme of strong order 1 [18] and Rößler's scheme of strong order 1.5 [17] in the second step of Algorithm 3.1 in order to compute an approximation for (5). We chose the initial value $y_0 = (\pi_0, \rho_0)$ with $\pi_0 = (\sqrt{2}, \sqrt{2}, 0)^\top$ and $\rho_0 = (0, \sqrt{2}, \sqrt{2})^\top$, the moments of inertia $I_0 = \mathrm{diag}(3, 1, 2)$ and $I_1 = \mathrm{diag}(1, 0.5, 1.5)$ and the mass matrices $M_0 = \mathrm{diag}(20, 55, 101)$ and $M_1 = \mathrm{diag}(55, 78, 120)$. For the implementation of the different SRK schemes in the second step of Algorithm 3.1 we followed the conditions in Theorem 3.2, i.e. we set the truncation index $q = 0$ for the Euler-Maruyama and Rößler's strong order 1 scheme and $q = 1$ for the strong order 1.5 scheme by Rößler. The estimation of the absolute error between a reference solution and the approximations $\hat{y}_T$ obtained with the SRKMK methods using the step sizes $\Delta = 2^{-\ell}$ for $\ell = 14, 13, 12, 11, 10, 9, 8, 7$ can be viewed as a log-log plot in Figure 1. It shows that the strong convergence order of the SRK scheme is preserved although only the first summands of (7) are evaluated. For the reference solution y_T^{ref} we used Rößler's strong order 1.5 scheme with a step size of $\Delta = 2^{-16}$ and we used a closed-form expression for (3) applied in $\mathfrak{se}(3)$ which can be found in [15].

As in [6] we evaluate the Casimir functions $C_1 = \pi^\top \rho$ and $C_2 = |\rho|^2$ to indicate how far the trajectories stray from the manifold $\mathfrak{se}(3)^*$. Figure 2

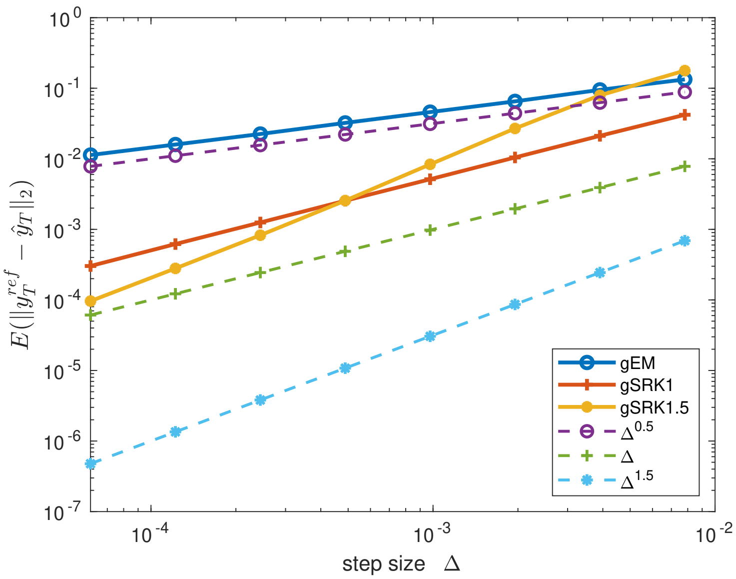

Figure 1: Simulation of the strong convergence order for $M = 1000$ paths. *Geometry-preserving* versions of the Euler-Maruyama (gEM), Rößler's strong order 1 SRK (gSRK1) and Rößler's strong order 1.5 scheme (gSRK1.5) are shown with solid lines while the dashed lines are corresponding reference slopes.

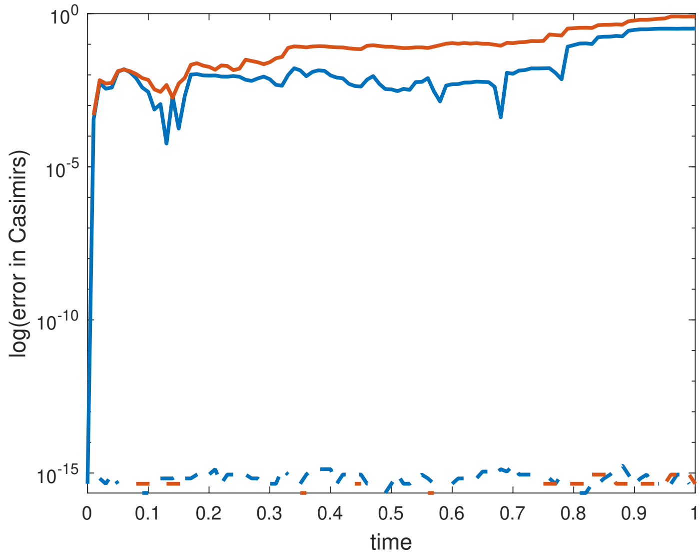

Figure 2: Distance of a sample path of Rößler's order 1 scheme (SRK1, solid lines) applied directly to (1) and its geometry-preserving counterpart (gSRK1, dashed lines). Blue lines correspond to the error in C_1 while red lines indicate the error in C_2.

shows that the SRKMK scheme preserves the Casimir functions within machine precision while the corresponding SRK method applied directly to (1) clearly violates the conserved quantities already after the first time steps.

5. Conclusion

In this work we extended Munthe-Kaas methods such that they can be applied to solve nonlinear Itô SDEs on manifolds. Furthermore, we formulated conditions for these SRKMK schemes to inherit the strong convergence order of the underlying SRK scheme in the Lie algebra. We specified the considered representation of an Itô SDE for some commonly used manifolds and analysed the application of SRKMK schemes more thoroughly for the manifold $\mathfrak{se}(3)^*$. The numerical results confirm our theorem on the strong convergence order and show that SRKMK schemes preserve conserved quantities of the underwater vehicle problem, namely the Casimir functions, whereas SRK schemes fail to conserve these quantities.

In future work we would like conduct a more detailed investigation of the weak convergence of SRKMK schemes since this work only covers the strong convergence.

References

[1] M. Ableidinger, E. Buckwar, *Weak stochastic Runge-Kutta Munthe-Kaas methods for finite spin ensembles*. Appl. Numer. Math. 118 (2017), 50–63.

[2] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration*. Springer Series in Computational Mathematics, Vol. 31. Springer Verlag, 2nd edition (2006).

[3] B.C. Hall, *Lie Groups, Lie Algebras, and Representations*. Graduate Texts in Mathematics, Vol. 222. Springer Verlag, 2nd edition (2015).

[4] P. Holmes, J. Jenkins, N. E. Leonard, *Dynamics of the Kirchhoff Equations I: coincident centers of gravity and bouyancy*. Phys. D 118 (1998), 311—342.

[5] P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin (1992).

[6] S.J.A. Malham, A. Wiese, *Stochastic Lie group Integrators*. SIAM J. Sci. Comput. 30(2) (2008), 597–617.

[7] G. Marjanovic, V. Solo, *Numerical Methods for Stochastic Differential Equations in Matrix Lie Groups Made Simple*. IEEE Trans. Auto. Contr. 63(12) (2018), 4035–4050.

[8] J.E. Marsden, T.S. Ratiu, *Introduction to mechanics and symmetry*. Springer Verlag, 2nd edition (1999).

[9] M. Muniz, M. Ehrhardt, M. Günther, *Approximating correlation matrices using stochastic Lie group methods*, Mathematics 9(1) (2021), 94.

[10] M. Muniz, M. Ehrhardt, M. Günther, R. Winkler, *Higher Strong Order Methods for linear Itô SDEs on matrix Lie Groups*, BIT Numer. Math. (2022).

[11] M. Muniz, M. Ehrhardt, M. Günther, R. Winkler, *Stochastic Runge-Kutta–Munthe-Kaas Methods in the Modelling of Perturbed Rigid Bodies*, Adv. Appl. Math. Mech. 14 (2) (2022), 528–538.

[12] M. Muniz, M. Ehrhardt, M. Günther, *Correlation Matrices driven by Stochastic Isospectral Flows*, in: M. Ehrhardt, M. Günther, W. Schilders, *Progress in Industrial Mathematics at ECMI 2021*, The European Consortium for Mathematics in Industry, Springer, 2022 (in press).

[13] H. Munthe-Kaas, *High order Runge-Kutta methods on manifolds*. Appl. Numer. Math. 29 (1999), 115–127.

[14] B. Øksendal, *Stochastic Differential Equations - An Introduction with Applications*, Fifth Edition. Springer, Berlin (1998).

[15] J. Park, W.-K. Chung, *Geometric Integration on Euclidean Group With Application to Articulated Multibody Systems*. IEEE Trans. Robotics 21(5) (2005), 850–863.

[16] M.J. Piggott, V. Solo, *Geometric Euler-Maruyama schemes for stochastic differential equations in SO(n) and SE(n)*. SIAM J. Numer. Anal. 54(4) (2016), 2490–2516.

[17] A. Rößler, *Explicit Order 1.5 Schemes for the Strong Approximation of Itô Stochastic Differential Equations*. PAMM 5(1) (2005), 817–818.

[18] A. Rößler, Strong and Weak Approximation Methods for Stochastic Differential Equations - Some Recent Developments. In: L. Devroye, B. Karasözen, M. Kohler, R. Korn (Eds.), *Recent Developments in Applied Probability and Statistics*, Physica-Verlag HD, Heidelberg, 2010, pp. 127–153.

[19] Z. Wang, Q. Ma, Z. Yao, X. Ding, *The Magnus Expansion for Stochastic Differential Equations*. J. Nonlin. Sci. 30 (2020), 419–447.