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ERROR ESTIMATES FOR A SPLITTING INTEGRATOR FOR SEMILINEAR BOUNDARY COUPLED SYSTEMS*

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Abstract. We derive a numerical method, based on operator splitting, to abstract parabolic semilinear boundary coupled systems. The method decouples the linear components which describe the coupling and the dynamics in the bulk and on the surface, and treats the nonlinear terms by approximating the integral in the variation of constants formula. The convergence proof is based on estimates for a recursive formulation of the error, using the parabolic smoothing property of analytic semigroups and a careful comparison of the exact and approximate flows. Numerical experiments, including problems with dynamic boundary conditions, reporting on convergence rates are presented.

Key words. Lie splitting, error estimates, boundary coupling, semilinear problems

AMS subject classifications. 47D06, 47N40, 34G20, 65J08, 65M12, 65M15

1. Introduction.

In this paper we derive a Lie-type splitting integrator for abstract *semilinear* boundary coupled systems, and prove first order error estimates for the time integrator by extending the results of [8] from the linear case. The main idea of our algorithm is to decouple the two nonlinear problems appearing in the original coupled system, while maintaining stability of the boundary coupling. More precisely, we combine the splitting scheme presented in [8] with the appropriate handling of the nonlinear terms. We use techniques from operator semigroup theory to prove the first-order convergence in the following abstract setting.

We consider the abstract semilinear boundary coupled systems of the form:

$$
\left\{
\begin{array}{lll}
\dot u(t) = A_m u(t) + \mathcal{F}_1(u(t), v(t)) & \text{for } 0 < t \le t_{\max}, & u(0) = u_0 \in E, \\
\dot v(t) = B v(t) + \mathcal{F}_2(u(t), v(t)) & \text{for } 0 < t \le t_{\max}, & v(0) = v_0 \in F, \\
L u(t) = v(t) & \text{for } 0 \le t \le t_{\max}, &
\end{array}
\right.
\tag{1.1}
$$

where A_m, B are linear operators on the Banach spaces E and F, respectively, $\mathcal{F}_1$, $\mathcal{F}_2$ are suitable functions, and the two unknown functions u and v are related via the linear coupling operator L acting between (subspaces of) E and F. A typical setting would be that $L : E \to F$ is a *trace-type operator* between the space E (for the bulk dynamics) and the *boundary* space F (for the surface dynamics). The precise setting and assumptions for (1.1) will be described below.

This abstract framework simultaneously includes problems which have been analysed on their own as well. For instance, abstract *boundary feedback* systems, see [9], [10], [6] and the references therein, fit into the above abstract framework where the equations in E and F representing the bulk and boundary equations. Such examples arise, for instance, for the boundary control of partial differential equation systems,

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see [27, 28], and [26], [13, Section 3], and [1, Section 3]. These problems usually involve a bounded feedback operator acting on u, which can be easily incorporated into the nonlinear term $\mathcal{F}_2$ above. We further note, that semilinear parabolic equations with *dynamic boundary conditions*, see [46, 12, 16, 7, 44, 29, 39, 15, 25], etc., and diffusion processes on *networks* with boundary conditions satisfying ordinary differential equations in the vertices, see [33, 34, 40, 36, 35], etc., both formally fit into this setting. In both cases, however, the feedback operator is unbounded.

In this paper we propose, as a first step into this direction, a Lie splitting scheme for abstract semilinear boundary coupled systems, where the semilinear term $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is locally Lipschitz (and might include feedback). An important feature of our splitting method is that it *separates the flows on E and F*, i.e. separates the bulk and surface dynamics. This could prove to be a considerable computational advantage if the bulk and surface dynamics are fundamentally different (e.g. fast and slow reactions, linear–nonlinear coupling, etc.). In general, splitting methods simplify (or even make possible) the numerical treatment of complex systems. If the operator on the right-hand side of the initial value problem can be written as a sum of at least two suboperators, the numerical solution is obtained from a sequence of simpler subproblems corresponding to the suboperators. We will use the Lie splitting, introduced in [4], which, from the functional analytic viewpoint, corresponds to the Lie–Trotter product formula, see [43], [14, Corollary III.5.8]. Splitting methods have been widely used in practice and analysed in the literature, see for instance the survey article [31], and see also, e.g., [41, 24, 42, 22], etc. In particular, for semilinear partial differential equations (PDEs) with dynamic boundary conditions, two bulk–surface splitting methods were proposed in [25]. The numerical experiments of Section 6.3 therein illustrate that both of the proposed splitting schemes suffer from order reduction. Recently, in [3], a first-order convergent bulk–surface Lie splitting scheme was proposed and analysed.

In the present work we start by the variation of constants formula and apply the Lie splitting to approximate the appearing linear operator semigroups. More precisely, we will identify three linear suboperators: two describing the dynamics in the bulk and on the surface, respectively, and one corresponding to the coupling. Then, either the solutions to the linear subproblems are known explicitly, or can be efficiently obtained numerically. We will show that the proposed method is first-order convergent for boundary coupled semilinear problems. The proposed method does not suffer from order reduction, and is therefore suitable for PDEs with dynamic boundary conditions, cf. [25], see the experiment in Section 5.2. However, due to the *unbounded* boundary feedback operator, our present results do not apply to this case *directly*. Nevertheless, we strongly believe that the developed techniques presented in this work provide further insight into the behaviour of operator splitting schemes of such problems. This is strengthened by our numerical experiments.

The convergence result is based on studying stability and consistency, using the procedure called Lady Windermere's fan from [21, Section II.3], however, these two issues cannot be separated as in most convergence proofs, since this would lead to sub-optimal error estimates. Instead, the error is rewritten using recursion formula which, using the parabolic smoothing property (see, e.g., [14, Theorem 4.6 (c)]), leads to an induction process to ensure that the numerical solution stays within a strip around the exact solution. A particular difficulty lies in the fact that the numerical method for the linear subproblems needs to approximate a convolution term in the exact flow [8], therefore the stability of these approximations cannot be merely established based on semigroup properties. Estimates from [8] together with new technical results yield an abstract first-order error estimate for semilinear problems (with a logarithmic factor in the time step), under suitable (local Lipschitz-type) conditions on the nonlinearities. By this analysis within the abstract setting we gain a deep operator theoretical understanding of these methods, which are applicable for all specific models (e.g. mentioned above) fitting into the framework of (1.1). Numerical experiments illustrate the proved error estimates, and an experiment for dynamic boundary conditions complement our theoretical results.

The paper is organised as follows.

In Section 2 we introduce the used functional analytic framework, and derive the proposed numerical method. We also state our main result, namely, the first-order convergence, the proof of which along with error estimates takes up Sections 3 and 4.

Section 5 presents numerical experiments illustrating and complementing our theoretical results.

2. Setting and the numerical method.

We consider two Banach spaces E and F, sometimes referred to as the bulk and boundary space, respectively, over the complex field $\mathbb{C}$. The product space $E \times F$ is endowed with the sum norm, or any other equivalent norm, rendering it a Banach space and the coordinate projections bounded. Elements in the product space will be denoted by boldface letters, e.g. $\boldsymbol{u} = (u, v)$ for $u \in E$ and $v \in F$. We first discuss a convenient framework established in [6] to treat linear boundary coupled problems. Then we treat the nonlinearities, derive the numerical method, and present the main result of the paper.

General framework. We will now define the abstract setting for *linear* boundary coupled systems, established in [6], i.e. for (1.1) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. We will also list all our assumptions on the linear operators in (1.1).

The following general conditions—collected using Roman numerals—will be assumed throughout the paper:

(i) The operator $A_m : \operatorname{dom}(A_m) \subseteq E \to E$ is linear.
(ii) The linear operator $L : \operatorname{dom}(A_m) \to F$ is surjective and bounded with respect to the graph norm of A_m on $\operatorname{dom}(A_m)$.
(iii) The restriction A_0 of A_m to $\ker(L)$ generates a strongly continuous semigroup T_0 on E.
(iv) The operator B generates a strongly continuous semigroup S on F.
(v) The operator matrix $\binom{A_m}{L} : \operatorname{dom}(A_m) \to E \times F$ is closed.

We recall from [6, Lemma 2.2] that $L|_{\ker(A_m)}$ is invertible, and its inverse, often called the *Dirichlet operator*, given by

$$D_0 := L|_{\ker(A_m)}^{-1} : F \to \ker(A_m) \subseteq E, \tag{2.1}$$

is bounded, and that

$$\operatorname{dom}(A_m) = \operatorname{dom}(A_0) \oplus \ker(L).$$

Let us briefly recall the following example from [8] (see Examples 2.7 and 2.8 therein), which is also one of the main motivating examples of [6]; we refer also to [19, 18, 5] for facts concerning Lipschitz domains.

Example 2.1 (Bounded Lipschitz domains). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $E = \mathrm{L}^2(\Omega)$ and $F = \mathrm{L}^2(\partial\Omega)$.

(a) Consider the following operators: $A_m = \Delta_\Omega$ with domain $\operatorname{dom}(A_m) := \{f : f \in \mathrm{H}^{1/2}(\Omega) \text{ with } \Delta_\Omega f \in \mathrm{L}^2(\Omega)\}$, and $Lf = f|_{\partial\Omega}$ the Dirichlet trace of $f \in \operatorname{dom}(A_m)$

on $\partial\Omega$ (see, e.g., [32, pp. 89–106]). Then L is surjective and actually has a bounded right-inverse D_0, which is the harmonic extension operator, i.e. for any $v \in \mathrm{L}^2(\partial\Omega)$ the function $u = D_0 v$ solves (uniquely) the Poisson problem $\Delta_\Omega u = 0$ with inhomogeneous Dirichlet boundary condition $Lu = v$. The operator A_0 is strictly positive and self-adjoint operator generating the Dirichlet-heat semigroup T_0 on E.

(b) One can also consider the Laplace–Beltrami operator $B := \Delta_{\partial\Omega}$ on $\mathrm{L}^2(\partial\Omega)$, which (with an appropriate domain) is also a strictly positive, self-adjoint operator, see [19, Theorem 2.5] or [17] for details.

In summary, we see that the abstract framework of [6], hence of this paper, covers interesting cases of boundary coupled problems on bounded Lipschitz domains.

We now turn our attention towards the semigroup, and its generator, corresponding to the linear problem. Consider the linear operator

$$\mathcal{A} := \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix} \quad \text{with} \quad \mathrm{dom}(\mathcal{A}) := \Big\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathrm{dom}(A_m) \times \mathrm{dom}(B) : Lx = y \Big\}. \tag{2.2}$$

For $y \in \mathrm{dom}(B)$ and $t \geq 0$ define the convolution

$$Q_0(t)y := -\int_0^t T_0(t-s) D_0 S(s) B y \,\mathrm{d}s. \tag{2.3}$$

For all $y \in \mathrm{dom}(B)$ we also define $Q(t)y$, and using integration by parts, see [6], we immediately write

$$Q(t)y := -A_0 \int_0^t T_0(t-s) D_0 S(s) y \,\mathrm{d}s = Q_0(t)y + D_0 S(t) y - T_0(t) D_0 y. \tag{2.4}$$

We see that $Q_0(t) : \mathrm{dom}(B) \to E$ and $Q(t) : \mathrm{dom}(B) \to E$ are both linear operators on $\mathrm{dom}(B)$ and bounded when $\mathrm{dom}(B)$ is endowed with the graph norm.

The next result, recalled from [6], characterizes the generator property of $\mathcal{A}$, which in turn is in relation with the well-posedness of (1.1), see Section 1.1 in [34].

THEOREM 2.2 ([6, Theorem 2.7]). *Within this setting, let the operators $\mathcal{A}$, D_0 be as defined in* (2.2) *and* (2.1)*, and suppose that A_0 is invertible. The operator $\mathcal{A}$ is the generator of a C_0-semigroup if and only if for each $t \geq 0$ the operator $Q(t)$ extends as a bounded linear operator to F and satisfies*

$$\limsup_{t \downarrow 0} \|Q(t)\| < \infty. \tag{2.5}$$

The semigroup $\mathcal{T}$ generated by $\mathcal{A}$ is then given as

$$\mathcal{T}(t) = \begin{pmatrix} T_0(t) & Q(t) \\ 0 & S(t) \end{pmatrix}. \tag{2.6}$$

In other words, if the conclusion of Theorem 2.2 holds, then the linear problem $\dot{\boldsymbol{u}} = \mathcal{A}\boldsymbol{u}$ is well-posed and the solution with initial value $\boldsymbol{u}_0 = (u_0, v_0)$ is given by the semigroup as $\mathcal{T}(t)\boldsymbol{u}_0$.

We further add to the list of general conditions (i)–(v) by further assuming:

(vi) The operators A_0 and B are invertible.

(vii) The operators A_0 and B generate bounded analytic semigroups.

Remark 2.3. (a) By Corollary 2.8 in [6] the assumption in (vii) implies that $\mathcal{A}$ is the generator of an analytic C_0-semigroup on $E \times F$.

(b) The invertibility of A_0 or B is merely a technical assumption which slightly simplifies the proofs and assumptions, avoiding a shifting argument.

(c) In principle, one can drop the assumption of B being the generator of an analytic semigroup. In this case minor additional assumptions on the nonlinearity $\mathcal{F}$ are needed, and the error bound for the numerical method will look slightly differently. We will comment on this in Remark 4.1 below, after the proof of the main theorem.

(d) The fact that A_0 generates a bounded analytic semigroup T_0 implies the bound $\sup_{t\geq 0} \|tA_0T_0(t)\| \leq M$, see, e.g., [14, Theorem 4.6 (c)].

For further details on analytic semigroups we refer to the monographs [38, 30, 14, 20].

The abstract semilinear problem. We now turn our attention to semilinear boundary coupled problems (1.1). In particular we will give our precise assumptions related to the solutions of the semilinear problem, and to the nonlinearity $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\colon \mathcal{D} \to E \times F$.

Assumptions 2.4. The function $\boldsymbol{u} := (u, v)\colon [0, t_{\max}] \to E \times F$, $t_{\max} > 0$, is a mild solution of the problem (1.1), written on $E \times F$ as

$$\dot{\boldsymbol{u}} = \mathcal{A}\boldsymbol{u} + \mathcal{F}(\boldsymbol{u}), \tag{2.7}$$

i.e. it satisfies the variation of constant formula:

$$\boldsymbol{u}(t) = \mathcal{T}(t)\boldsymbol{u}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\boldsymbol{u}(s))\,\mathrm{d}s. \tag{2.8}$$

We further assume that the exact solution $\boldsymbol{u}$ has the following properties:

(1) The function $\mathcal{F} : \Sigma \to E \times F$ is Lipschitz continuous on the strip
$$\Sigma := \{\boldsymbol{v} \in E \times F : \|\boldsymbol{u}(t) - \boldsymbol{v}\| \leq R \text{ for some } t \in t_{\max}\} \subseteq \mathcal{D}$$
around the exact solution with constant ℓ_Σ.

(2) The second component $\mathcal{F}_2 : \Sigma \to \mathrm{dom}(B)$ is Lipschitz continuous on Σ, with constant $\ell_{\Sigma,B}$.

(3) For each $t \in [0, t_{\max}]$ $v(t) = \boldsymbol{u}(t)|_2 \in \mathrm{dom}(B^2)$, and $\sup_{t\in[0,t_{\max}]} \|B^2 v(t)\| < \infty$.

(4) The second component along the solution satisfies $\mathcal{F}_2(\boldsymbol{u}(t)) \in \mathrm{dom}(B^2)$ for each $t \in [0, t_{\max}]$, and $\sup_{t\in[0,t_{\max}]} \|B^2\mathcal{F}_2(\boldsymbol{u}(t))\| < \infty$.

(5) Furthermore, $\mathcal{F} \circ \boldsymbol{u}$ is differentiable and $(\mathcal{F} \circ \boldsymbol{u})' \in \mathrm{L}^1([0, t_{\max}]; E \times F)$.

The numerical method. We are now in the position to derive the numerical method. For a time step $\tau > 0$, for all $t_n = n\tau \in [0, t_{\max}]$, we define the numerical approximation $\boldsymbol{u}_n = (u_n, v_n)$ to $\boldsymbol{u}(t_n) = (u(t_n), v(t_n))$ via the following steps.

Step 1. We approximate the integral in (2.8) by an appropriate quadrature rule.

Step 2. We approximate the semigroup operators $\mathcal{T}$ by using an operator splitting method. Due to its special form (2.6), this includes the approximation of the convolution Q_0, defined in (2.4), by an operator V. The choice of V is determined by the used splitting method, see [8, Section 3] and below.

In what follows we describe the numerical method by using first-order approximations in *Steps 1–2*, and show its first-order convergence. We note here that the application of a correctly chosen exponential integrator could be inserted as a preliminary step, see [23]. Since it eliminates the integral's dependence on $\boldsymbol{u}(s)$, the quadrature rule simplifies in *Step 1*. This approach, however, leads to the same numerical method as *Steps 1–2*.

Before proceeding as proposed, for all $\tau > 0$, we rewrite formula (2.8) at $t = t_n = t_{n-1} + \tau$ as

$$\boldsymbol{u}(t_n) = \mathcal{T}(\tau)\boldsymbol{u}(t_{n-1}) + \int_0^\tau \mathcal{T}(\tau - s)\mathcal{F}(\boldsymbol{u}(t_{n-1} + s))\,\mathrm{d}s. \tag{2.9}$$

Now, according to *Step 1*, we approximate the integral by the left rectangle rule leading to

$$\boldsymbol{u}(t_n) \approx \mathcal{T}(\tau)\boldsymbol{u}(t_{n-1}) + \tau\mathcal{T}(\tau)\mathcal{F}(\boldsymbol{u}(t_{n-1})) = \mathcal{T}(\tau)\Big(\boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))\Big),$$

for any $t_n = n\tau \in (0, t_{\max}]$.

In *Step 2*, we apply the Lie splitting, which, according to [8], results in the approximation of the convolution operator $Q_0(t)$ by an appropriate $V(t)$ (to be specified later). Altogether, we approximate the semigroup operators $\mathcal{T}(\tau)$ by

$$\boldsymbol{T}(\tau) = \begin{pmatrix} T_0(\tau) & V(\tau) + D_0S(\tau) - T_0(\tau)D_0 \\ 0 & S(\tau) \end{pmatrix}. \tag{2.10}$$

We remark that $\boldsymbol{T}(\tau) = \mathcal{R}_0^{-1}\mathbb{T}(\tau)\mathcal{R}_0$ holds with the notations introduced in [8]:

$$\mathbb{T}(\tau) = \begin{pmatrix} T_0(\tau) & V(\tau) \\ 0 & S(\tau) \end{pmatrix} \quad \text{and} \quad \mathcal{R}_0 = \begin{pmatrix} I & -D_0 \\ 0 & I \end{pmatrix}.$$

This leads to the numerical method approximating $\boldsymbol{u}$ at time $t_n = n\tau \in [0, t_{\max}]$:

$$\boldsymbol{u}_n := \boldsymbol{L}(\tau)(\boldsymbol{u}_{n-1}) := \boldsymbol{T}(\tau)\big(\boldsymbol{u}_{n-1} + \tau\mathcal{F}(\boldsymbol{u}_{n-1})\big), \tag{2.11}$$

with $\boldsymbol{u}_0 := (u_0, v_0)$.

The actual form of operator $V(\tau)$ depends on the underlying splitting method. Here, we will use the Lie splitting of the operator $\mathcal{A}_0 := \mathcal{R}_0\mathcal{A}\mathcal{R}_0^{-1}$, proposed in [8, Section 3]. Namely, we split up the operator $\mathcal{A}_0 =: \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$ with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

and $\mathrm{dom}(\mathcal{A}_1) = \mathrm{dom}(A_0) \times F$, $\mathrm{dom}(\mathcal{A}_2) = E \times \mathrm{dom}(B)$, $\mathrm{dom}(\mathcal{A}_3) = E \times \mathrm{dom}(B)$. It was shown in [8, Prop. 3.2.] that the operator parts $\mathcal{A}_1|_{E\times\mathrm{dom}(B)}$, $\mathcal{A}_2$ and $\mathcal{A}_3|_{E\times\mathrm{dom}(B)}$ generate the strongly continuous semigroups

$$\mathcal{T}_1(\tau) = \begin{pmatrix} T_0(\tau) & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{T}_2(\tau) = \begin{pmatrix} I & -\tau D_0B \\ 0 & I \end{pmatrix}, \quad \mathcal{T}_3(\tau) = \begin{pmatrix} I & 0 \\ 0 & S(\tau) \end{pmatrix},$$

respectively, on $E \times \mathrm{dom}(B)$. Then the application of the Lie splitting as $\boldsymbol{T}(\tau) = \mathcal{R}_0^{-1}\mathcal{T}_1(\tau)\mathcal{T}_2(\tau)\mathcal{T}_3(\tau)\mathcal{R}_0$ leads to the formula (2.10) with

$$V(\tau) = -\tau T_0(\tau)D_0BS(\tau). \tag{2.12}$$

Thus, the Lie splitting transfers the coupled linear problem into the sequence of simpler ones. First we solve the equation $\dot{v} = Bv$ on $\mathrm{dom}(B)$ by using the original initial condition v_0, then we propagate the solution by $\mathcal{T}_2(\tau)$, which serves as an initial condition to the homogeneous problem $\dot{u} = A_0 u$ on E. To get an approximation at $t_n = n\tau$, the semilinear expressions and the terms coming from the "diagonalisation" should be treated. Then the whole process needs to be cyclically performed n times.

We note that the approximation $Q_0(\tau) \approx V(\tau) = -\tau T_0(\tau) D_0 B S(\tau)$ can also be obtained by using an appropriate convolution quadrature, i.e. by approximating $T_0(\tau - \xi)$ from the left (at $\xi = 0$) and $S(\xi)$ from the right (at $\xi = \tau$).

Upon plugging in the splitting approximation (2.12) into the convolution $Q_0(\tau)$, and by introducing the intermediate values

$$\begin{aligned} \widetilde{u}_n &= u_{n-1} + \tau \mathcal{F}_1(u_{n-1}, v_{n-1}), \\ \widetilde{v}_n &= v_{n-1} + \tau \mathcal{F}_2(u_{n-1}, v_{n-1}), \end{aligned}$$

the method (2.11) reads componentwise as

$$\begin{aligned} u_n &= T_0(\tau)\Big(\widetilde{u}_{n-1} - D_0\big(\widetilde{v}_{n-1} + \tau B v_n\big)\Big) + D_0 v_n, \\ v_n &= S(\tau)\widetilde{v}_n. \end{aligned} \tag{2.13}$$

This formulation only requires two applications of the Dirichlet operator D_0 per time step. We point out that the two terms with the Dirichlet operator can be viewed as correction terms which correct the boundary values of the bulk-subflow along the splitting method.

The main result. We are now in the position to state the main result of this paper, which asserts first order (up to a logarithmic factor) error estimates for the approximations obtained by the splitting integrator (2.11) (with (2.12)) separating the bulk and surface dynamics in E and F.

THEOREM 2.5. *In the above setting, let $\boldsymbol{u} : [0, t_{\max}] \to E \times F$ be the solution of* (1.1) *subject to the conditions in Assumptions* 2.4 *and consider the approximations $\boldsymbol{u}_n$ at time t_n determined by the splitting method* (2.11) *(with* (2.12)*). Then there exists a $\tau_0 > 0$ and $C > 0$ such that for any time step $\tau \le \tau_0$ we have at time $t_n = n\tau \in [0, t_{\max}]$ the error estimate*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n\| \le C\,\tau\,|\log(\tau)|. \tag{2.14}$$

The constant $C > 0$ is independent of n and $\tau > 0$, but depends on $t_{\max}$, on constants related to the semigroups T_0 and S, as well as on the exact solution $\boldsymbol{u}$.

The proof of this result will be given in Section 4 below. In the next section we state and prove some preparatory and technical results needed for the error estimates.

Recall that the splitting method (2.11), written componentwise (2.13), decouples the bulk and surface flows, which can be extremely advantageous if the two subsystems behave in a substantially different manner. We remind that, when applied to PDEs with dynamic boundary conditions, naive splitting schemes suffer from order reduction, see [25, Section 6], and a correction in [3].

We make the following remark about the logarithmic factor in the above error estimate. Inequality (2.14) implies that for any $\varepsilon \in (0, 1)$ we have $\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n\| \le C'\,\tau^{1-\varepsilon}$ with another constant C'. This amounts to saying that the proposed method has convergence order arbitrarily close to 1, and in fact this is also what the numerical experiments show. Indeed, numerical experiments in Section 5 illustrate the first-order error estimates of Theorem 2.5, including an example with dynamic boundary conditions, Section 2.5, without any order reductions.

3. Preparatory results. In this section we collect some general technical results which will be used later on in the convergence proof. After a short calculation, or by using the results in Section 3 of [8], we obtain

$$\boldsymbol{T}(\tau)^k = \begin{pmatrix} T_0(k\tau) & -T_0(k\tau)D_0 + D_0 S(k\tau) + V_k(\tau) \\ 0 & S(k\tau) \end{pmatrix}, \tag{3.1}$$

$$\text{where} \qquad V_k(\tau)y = \sum_{j=0}^{k-1} T_0((k-1-j)\tau)V(\tau)S(j\tau)y,$$

see [8, equation (3.9)]. Now we are in the position to prove exponential bounds for the powers of $\boldsymbol{T}(\tau)$.

LEMMA 3.1. *There exist a constant $M>0$ such that for $\tau>0$ and $\boldsymbol{T}(\tau)$ defined in (2.10) (with (2.12)), and for any $(x,y)\in E\times \mathrm{dom}(B)$ and $k\in\mathbb{N}$ with $k\tau\in[0,t_{\max}]$*

$$\|\boldsymbol{T}(\tau)^k \tbinom{x}{y}\| \le M\|\tbinom{x}{y}\| + M\|By\|.$$

Moreover, if S is a bounded analytic semigroup, then we have

$$\|\boldsymbol{T}(\tau)^k \tbinom{x}{y}\| \le M(1+\log(k))\|\tbinom{x}{y}\|.$$

Proof. From the sum norm on the product space $E\times F$, we have

$$\begin{aligned} \|\boldsymbol{T}(\tau)^k \tbinom{x}{y}\| &= \|T_0(k\tau)x + T_0(k\tau)D_0 y + D_0 S(k\tau)y + V_k(\tau)y\| + \|S(k\tau)y\| \\ &\le \|T_0(k\tau)x\| + \|T_0(k\tau)D_0 y\| + \|D_0 S(k\tau)y\| + \|V_k(\tau)y\| + \|S(k\tau)y\|. \end{aligned}$$

The exponential boundedness of the semigroups T_0 and S, and the boundedness of D_0 directly yield

$$\begin{aligned} \|T_0(k\tau)x\| + \|T_0(k\tau)D_0 y\| + \|D_0 S(k\tau)y\| &\le M\big(\|x\|+\|y\|\big), \\ \text{and} \quad \|S(k\tau)y\| &\le M\|y\|. \end{aligned}$$

It remains to bound the term $V_k(\tau)y$. We obtain

$$\begin{aligned} \|V_k(\tau)y\| &\le \tau\sum_{j=0}^{k-1} \|T_0((k-1-j)\tau)T_0(\tau)D_0 BS(\tau)S(j\tau)y\| \\ &\le \tau\sum_{j=0}^{k-1} \|T_0((k-j)\tau)D_0 S((j+1)\tau)By\| \\ &\le \tau\sum_{j=0}^{k-1} M\|By\| \le M\|By\|, \end{aligned}$$

which completes the proof of the first statement.

If S is a bounded analytic semigroup, then we improve the last estimate to

$$\begin{aligned} \|V_k(\tau)\| &= \sum_{j=0}^{k-1} \|T_0\big((k-1-j)\tau\big)V(\tau)S(j\tau)\| \\ &= \tau\sum_{j=0}^{k-1} \|T_0\big((k-j)\tau\big)\|\,\|D_0 BS(\tau)S(j\tau)\| \end{aligned}$$

$$\leq M_1 M_2 \|D_0\|\, \tau \sum_{j=0}^{k-1} \frac{1}{(j+1)\tau} \leq M(1+\log(k)).$$

By putting the estimates together, the assertions follows. ∎

We recall the following lemma from [8].

Lemma 3.2 ([8, Lemma 4.4]). *There is a $C \geq 0$ such that for every $\tau \in [0, t_{\max}]$, for any $s_0, s_1 \in [0,\tau]$, and for every $y \in \mathrm{dom}(B^2)$ we have*

$$\Big\| \int_0^\tau T_0(\tau-s) A_0^{-1} D_0 S(s) By \,\mathrm{d}s - \tau T_0(\tau - s_0) A_0^{-1} D_0 S(s_1) By \Big\| \leq C\tau^2 (\|By\| + \|B^2 y\|).$$

Using the above quadrature estimate we prove the following approximation lemma.

Lemma 3.3. *For $(x,y) \in E \times \mathrm{dom}(B^2)$ and $j \in \mathbb{N} \setminus \{0\}$ we have*

$$\big\| \boldsymbol{T}(\tau)^j \big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big) \tbinom{x}{y} \big\| \leq C\tau^2 \|A_0 T_0(j\tau)\| \,\big(\|By\| + \|B^2 y\|\big).$$

Proof. Using the formula (3.1) for $\boldsymbol{T}(\tau)^j$ and a direct computation for the difference $\mathcal{T}(\tau) - \boldsymbol{T}(\tau)$, we obtain

$$\begin{aligned}
&\boldsymbol{T}(\tau)^j \big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big) \tbinom{x}{y} \\
&= \boldsymbol{T}(\tau)^j \left(\int_0^\tau T_0(\tau-\xi) D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) D_0 B S(\tau) y \ ,\ 0 \right)^\top \\
&= \left(T_0(\tau)^j \left(\int_0^\tau T_0(\tau-\xi) D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) D_0 B S(\tau) y \right) \ ,\ 0 \right)^\top
\end{aligned}$$

for all $(x,y) \in E \times \mathrm{dom}(B)$. We can further rewrite the first component as

$$\begin{aligned}
&T_0(j\tau) \left(\int_0^\tau T_0(\tau-\xi) D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) D_0 B S(\tau) y \right) \\
&= A_0 T_0(j\tau) \left(\int_0^\tau T_0(\tau-\xi) A_0^{-1} D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) A_0^{-1} D_0 B S(\tau) y \right).
\end{aligned}$$

We have

$$\begin{aligned}
&\big\| \boldsymbol{T}(\tau)^j \big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big) \tbinom{x}{y} \big\| \\
&= \left\| A_0 T_0(j\tau) \left(\int_0^\tau T_0(\tau-\xi) A_0^{-1} D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) A_0^{-1} D_0 B S(\tau) y \right) \right\| \\
&\leq \|A_0 T_0(j\tau)\| \left\| \int_0^\tau T_0(\tau-\xi) A_0^{-1} D_0 B S(\xi) y \,\mathrm{d}\xi - \tau T_0(\tau) A_0^{-1} D_0 B S(\tau) y \right\|,
\end{aligned}$$

therefore an application of Lemma 3.2 with $s_0 = 0$ and $s_1 = \tau$ proves the assertion. ∎

Lemma 3.4. *For $t, s \in [0, t_{\max}]$ we have*

$$\|A_0^{-1} T_0(t) - A_0^{-1} T_0(s)\| \leq M|t-s|.$$

Proof. Resorting to the Taylor expansion we have for $x \in E$ that

$$A_0^{-1} T_0(t) x - A_0^{-1} T_0(s) x = \int_s^t T_0(r) A_0^{-1} A_0 x \,\mathrm{d}r = \int_s^t T_0(r) x \,\mathrm{d}r,$$

which readily implies $\|A_0^{-1} T_0(t) x - A_0^{-1} T_0(s) x\| \leq M\|x\| |t-s|$, and hence the assertion. ∎

LEMMA 3.5. *Let $f\colon [0,t_{\max}] \to E$ be Lipschitz continuous and consider*

$$(T_0 * f)(t) := \int_0^t T_0(t-r)f(r)\,\mathrm{d}r, \quad t \in [0,t_{\max}].$$

Then for all $t,s \in [0,t_{\max}]$ we have

$$\|(T_0 * f)(t) - (T_0 * f)(s)\| \le C|t-s|\|f\|_{\mathrm{Lip}}.$$

Proof. For $t,s \in [0,t_{\max}]$, we have

$$\begin{aligned}
&\big\|(T_0 * f)(t) - (T_0 * f)(s)\big\| = \Big\|\int_0^t T_0(r)f(t-r)\,\mathrm{d}r - \int_0^s T_0(r)f(s-r)\,\mathrm{d}r\Big\| \\
&\le \int_0^s \|T_0(r)(f(t-r)-f(s-r))\|\,\mathrm{d}r + \int_s^t \|T_0(r)f(t-r)\|\,\mathrm{d}r \\
&\le C_1|t-s|s\|f\|_{\mathrm{Lip}} + C_1|t-s|\|f\|_\infty \le C|t-s|\|f\|_{\mathrm{Lip}}.
\end{aligned}$$

∎

Let $|_1$ and $|_2$ denote the projection onto the first and second coordinate in $E \times F$.

LEMMA 3.6. *For $t_{\max} > 0$ there is a $C \ge 0$ such that for every $(x,y) \in E \times \mathrm{dom}(B)$, $t,s \in [0,t_{\max}]$ we have*

$$\begin{aligned}
&\big\|\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_1\big\| \le C\,(\|x\| + \|y\| + \|By\|), \\
\text{and}\quad &\big\|\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_2\big\| \le C\,|t-s|\|By\|.
\end{aligned}$$

Proof. We have

$$\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_2 = \int_s^t S(r)By\,\mathrm{d}r$$

and the second asserted inequality follows at once.

On the other hand, for the first component

$$\begin{aligned}
&\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_1 = T_0(t)x - T_0(s)x + Q(t)y - Q(s)y \\
&= T_0(t)x - T_0(s)x + D_0S(t)y - D_0S(s)y - T(t)D_0y + T(s)D_0y - Q_0(t)y + Q_0(s)y,
\end{aligned}$$

and we obtain

$$\big\|\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_1\big\| = 2M\|x\| + 4M\|D_0\|\|y\| + |t-s|M^2\|D_0\|\|By\|,$$

and the first inequality is also proved. ∎

LEMMA 3.7. *For $t_{\max} > 0$ there is a $C \ge 0$ such that for every $(x,y) \in E \times \mathrm{dom}(B^2)$, $t,s \in [0,t_{\max}]$, $\tau > 0$, $0 \le j\tau \le t_{\max}$ we have*

$$\begin{aligned}
\big\|\boldsymbol{T}(\tau)^j\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}\big\| \le &C\,|t-s|\|A_0T_0(j\tau)\|(\|x\| + \|y\| + \|By\|) \\
&+ C\,|t-s|(\|y\| + \|By\| + \|B^2y\|).
\end{aligned}$$

Proof. From (3.1) we obtain

$$\boldsymbol{T}(\tau)^j\big(\mathcal{T}(t) - \mathcal{T}(s)\big)\tbinom{x}{y}|_2 = \int_s^t S(j\tau + r)By\,\mathrm{d}r \quad \text{and}$$

$$
\begin{aligned}
\boldsymbol{T}(\tau)^j\big(\mathcal{T}(t)-\mathcal{T}(s)\big)\tbinom{x}{y}|_1 &= T_0(j\tau)\big(T_0(t)x - T_0(s)x + Q(t)y - Q(s)y\big)\\
&\quad - T_0(j\tau)D_0\int_s^t S(r)By\,\mathrm{d}r + D_0S(j\tau)\int_s^t S(r)By\,\mathrm{d}r + V_j(\tau)\int_s^t S(r)By\,\mathrm{d}r\\
&= I_1+I_2+I_3+I_4,
\end{aligned}
$$

where $I_1,\dots,I_4$ denote the four terms in the order of appearance. By Lemma 3.4

$$
\begin{aligned}
\|I_1\| &\le \|A_0T_0(j\tau)\|\Big(\|A_0^{-1}(T_0(t)-T_0(s))\|\|x\| + \|A_0^{-1}(Q(t)-Q(s))y\|\Big)\\
&\le C\|A_0T_0(j\tau)\||t-s|\|x\| + \|A_0T_0(j\tau)\|\|A_0^{-1}(Q(t)-Q(s))y\|,
\end{aligned}
$$

so we need to estimate $\|A_0^{-1}(Q(t)-Q(s))y\|$. Since $A_0^{-1}Q$ has the appropriate convolution form, Lemma 3.5 implies

$$
\|A_0^{-1}(Q(t)-Q(s))y\| = \Big\|(T_0 * D_0S)(t) - (T_0 * D_0S)(s)\Big\| \le C_1|t-s|\|D_0\|\|By\|.
$$

Altogether we obtain

$$
\|I_1\| \le C_2|t-s|\|A_0T_0(j\tau)\|(\|x\|+\|y\|+\|By\|).
$$

For I_2 and I_3 we have

$$
\|I_2\| + \|I_3\| \le C_3|t-s|\|By\|.
$$

To estimate I_4 we recall from the proof of Lemma 3.1 that $\|V_j(\tau)z\| \le C_4\|Bz\|$ (for $j\tau \le [0,t_{\max}]$), so that

$$
\|I_4\| \le C_4\Big\|B\int_s^t S(r)By\,\mathrm{d}r\Big\| \le C_5|t-s|\|B^2y\|.
$$

Finally, the estimates for $I_1,\dots,I_4$ together yield the assertion. ∎

4. Proof of Theorem 2.5. The proof of our main result is based on a recursive expression for the global error, which involves the local error and some nonlinear error terms. The recursive formula is obtained using a procedure which is sometimes called Lady Windermere's fan [21, Section II.3]; our approach is inspired by [37], [45, Chapter 3]. The local errors are weighted by $\boldsymbol{T}(\tau)^j$, therefore a careful accumulation estimate—heavily relying on the parabolic smoothing property—is required. In order to estimate the locally Lipschitz nonlinear terms we have to ensure that the numerical solution remains in the strip Σ (see Assumptions 2.4). This will be shown using an induction process, which is outlined as follows:

- We shall find $\tau_0>0$ and a constant $C>0$ such that for any $0<\tau\le\tau_0$ if $\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n-1}$ belong to the strip Σ and $t_n = n\tau \le t_{\max}$, then
$$
\|\boldsymbol{u}(t_n)-\boldsymbol{u}_n\| \le C\tau|\log(\tau)|.
$$
- Since $C>0$ is a constant independent of n and τ, we can take $\tau_0>0$ sufficiently small such that for each $\tau\le\tau_0$ we have $C\tau|\log(\tau)|\le R$, the width of the strip Σ, therefore by the previous step we have $\boldsymbol{u}_n\in\Sigma$.
- Since $\boldsymbol{u}_0$ belongs to the strip and since τ_0 and $C>0$ are independent of n, the proof can be concluded by induction.

Within the proof we will use the following conventions: The positive constant M comes from bounds for any of the analytic semigroups T_0, S, or $\mathcal{T}$: For each $t \in (0, t_{\max}]$

$$\|T_0(t)\|, \|S(t)\|, \|\mathcal{T}(t)\| \leq M, \qquad \text{and} \qquad \|t\, A_0\, T_0(t)\| \leq M. \tag{4.1}$$

Here the last estimate is usually referred to as the parabolic smoothing property of analytic semigroups, cf. Remark 2.3 (c). By $C > 0$ we will denote a constant that is independent of the time step, but may depend on other constants (e.g. parameters of the problem) and on the exact solution (hence on the initial condition). Within a proof we shall indicate a possible increment of such appearing constants by a subscript: $C_1, C_2, \ldots$, etc.

Proof of Theorem 2.5. For the local Lipschitz continuity of the nonlinearity $\mathcal{F}$, we will prove that the numerical solution remains in the strip Σ around the exact solution $\boldsymbol{u}(t)$ using an induction argument.

We estimate the global error $\boldsymbol{u}(t_n) - \boldsymbol{u}_n$, at time $t_n = n\tau \in (0, t_{\max}]$, by expressing it using the local error $\mathrm{e}_n^{\mathrm{loc}} = \boldsymbol{u}(t_n) - \boldsymbol{L}(\tau)(\boldsymbol{u}(t_{n-1}))$ as follows:

$$\begin{aligned}
\boldsymbol{u}(t_n) - \boldsymbol{u}_n &= \boldsymbol{u}(t_n) - \boldsymbol{L}(\tau)\big(\boldsymbol{u}(t_{n-1})\big) + \boldsymbol{L}(\tau)\big(\boldsymbol{u}(t_{n-1})\big) - \boldsymbol{L}(\tau)\big(\boldsymbol{u}_{n-1}\big) \\
&= \mathrm{e}_n^{\mathrm{loc}} + \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))\big) - \boldsymbol{T}(\tau)\big(\boldsymbol{u}_{n-1} + \tau\mathcal{F}(\boldsymbol{u}_{n-1})\big) \\
&= \mathrm{e}_n^{\mathrm{loc}} + \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-1}) - \boldsymbol{u}_{n-1}\big) + \tau\boldsymbol{T}(\tau)\varepsilon_{n-1}^{\mathcal{F}},
\end{aligned}$$

with the nonlinear difference term $\varepsilon_n^{\mathcal{F}} = \mathcal{F}(\boldsymbol{u}(t_n)) - \mathcal{F}(\boldsymbol{u}_n)$. By resolving the recursion we obtain

$$\begin{aligned}
\boldsymbol{u}(t_n) - \boldsymbol{u}_n &= \mathrm{e}_n^{\mathrm{loc}} + \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-1}) - \boldsymbol{u}_{n-1}\big) + \tau\boldsymbol{T}(\tau)\varepsilon_{n-1}^{\mathcal{F}} \\
&= \mathrm{e}_n^{\mathrm{loc}} + \boldsymbol{T}(\tau)\mathrm{e}_{n-1}^{\mathrm{loc}} + \boldsymbol{T}(\tau)^2\big(\boldsymbol{u}(t_{n-2}) - \boldsymbol{u}_{n-2}\big) + \tau\boldsymbol{T}(\tau)^2\varepsilon_{n-2}^{\mathcal{F}} + \tau\boldsymbol{T}(\tau)\varepsilon_{n-1}^{\mathcal{F}} \\
&\ \ \vdots \\
&= \mathrm{e}_n^{\mathrm{loc}} + \sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\mathrm{e}_{n-j}^{\mathrm{loc}} + \tau\sum_{j=1}^{n}\boldsymbol{T}(\tau)^j\varepsilon_{n-j}^{\mathcal{F}} + \boldsymbol{T}(\tau)^n\big(\boldsymbol{u}(0) - \boldsymbol{u}_0\big).
\end{aligned} \tag{4.2}$$

Since we have $\boldsymbol{u}_0 = \boldsymbol{u}(0)$, the last term vanishes.

We now start the induction process. Let us assume that the error estimate (2.14) holds for all $k \leq n-1$ with $n\tau \leq t_{\max}$, i.e., for a $K > 0$ independent of τ and n, we have

$$\text{for } k = 0, \ldots, n-1, \qquad \|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\| \leq K\,\tau\,|\log(\tau)|. \tag{4.3}$$

Below, we will show that the same error estimate also holds for n as well. We note that, via $\boldsymbol{u}_0 = \boldsymbol{u}(0)$, the assumed error estimate trivially holds for $n - 1 = 0$.

We will now estimate the remaining terms of (4.2) in parts (i)–(iii), respectively. The estimates (4.3) for the past values for k only appear in part (iii).

(i) We rewrite the local error $\mathrm{e}_n^{\mathrm{loc}}$ by using the forms (2.9) and (2.11) of the exact and approximate solutions, respectively, and by Taylor's formula and (5) as

$$\begin{aligned}
\mathrm{e}_n^{\mathrm{loc}} &= \boldsymbol{u}(t_n) - \boldsymbol{L}\big(\boldsymbol{u}(t_{n-1})\big) \\
&= \mathcal{T}(\tau)\boldsymbol{u}(t_{n-1}) + \int_0^\tau \mathcal{T}(\tau - s)\mathcal{F}(\boldsymbol{u}(t_{n-1} + s))\,\mathrm{d}s - \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))\big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{T}(\tau)\boldsymbol{u}(t_{n-1}) + \int_0^\tau \mathcal{T}(\tau-s)\mathcal{F}(\boldsymbol{u}(t_{n-1}))\,\mathrm{d}s \\
&\quad + \int_0^\tau \mathcal{T}(\tau-s)\int_0^s (\mathcal{F}\circ\boldsymbol{u})'(t_{n-1}+\xi)\,\mathrm{d}\xi\,\mathrm{d}s - \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))\big) \\
&= \big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big)\big(\boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))\big) + \int_0^\tau \big(\mathcal{T}(\tau-s) - \mathcal{T}(\tau)\big)\mathcal{F}(\boldsymbol{u}(t_{n-1}))\,\mathrm{d}s \\
&\quad + \int_0^\tau \mathcal{T}(\tau-s)\int_0^s (\mathcal{F}\circ\boldsymbol{u})'(t_{n-1}+\xi)\,\mathrm{d}\xi\,\mathrm{d}s. \qquad (4.4)
\end{aligned}
$$

In what follows we will estimate the three terms separately.

We will bound the first term by using the boundedness of the semigroups T_0 and S. Denote $(x,y) = \boldsymbol{u}(t_{n-1}) + \tau\mathcal{F}(\boldsymbol{u}(t_{n-1}))$ and write

$$
\begin{aligned}
\big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big)\tbinom{x}{y} &= \begin{pmatrix} 0 & Q_0(\tau) - V(\tau) \\ 0 & 0 \end{pmatrix}\tbinom{x}{y} \\
&= Q_0(\tau)y - V(\tau)y = -\int_0^\tau T_0(\tau-\xi)D_0BS(\xi)y\,\mathrm{d}\xi + \tau T_0(\tau)D_0BS(\tau)y.
\end{aligned}
$$

Whence we conclude

$$
\big\|\big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big)\tbinom{x}{y}\big\| \le \tau 2M^2\|D_0\|\|By\| \le C_1\tau\|B(v(t_{n-1}) + \tau\mathcal{F}_2(\boldsymbol{u}(t_{n-1})))\|.
$$

The second term in (4.4) can be estimated by Lemma 3.6, and using (4), as

$$
\int_0^\tau \Big\|\big(\mathcal{T}(\tau-s) - \mathcal{T}(\tau)\big)\mathcal{F}(\boldsymbol{u}(t_{n-1}))\Big\|\,\mathrm{d}s \le C_2\tau\big(\|\mathcal{F}(\boldsymbol{u}(t_{n-1}))\| + \|B\mathcal{F}_2(\boldsymbol{u}(t_{n-1}))\|\big).
$$

While, using the exponential boundedness of $\mathcal{T}$ and (5), the third term in (4.4) is directly bounded by

$$
\begin{aligned}
\int_0^\tau\int_0^s \Big\|\mathcal{T}(\tau-s)(\mathcal{F}\circ\boldsymbol{u})'(t_{n-1}+\xi)\Big\|\,\mathrm{d}\xi\,\mathrm{d}s &\le M\tau\|(\mathcal{F}\circ\boldsymbol{u})'\|_{\mathrm{L}^1([t_{n-1},t_n])} \\
&\le M\tau\|(\mathcal{F}\circ\boldsymbol{u})'\|_{\mathrm{L}^1([0,t_{\max}])}.
\end{aligned}
$$

Therefore, we finally obtain for the local error that

$$
\|\mathrm{e}_n^{\mathrm{loc}}\| \le C_3\tau. \qquad (4.5)
$$

(ii) Since in each time step the local error is $\mathcal{O}(\tau)$ and we have $\mathcal{O}(1/\tau)$ time steps, a more careful analysis is needed for the the second term in (4.2). We first rewrite this term by the variation of constants formula (2.9) and the numerical method in the form (2.11):

$$
\begin{aligned}
\sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\mathrm{e}_{n-j}^{\mathrm{loc}} &= \sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\Big(\boldsymbol{u}(t_{n-j}) - \boldsymbol{T}(\tau)\big(\boldsymbol{u}(t_{n-j-1}) - \tau\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\big)\Big) \\
&= \sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\big(\mathcal{T}(\tau) - \boldsymbol{T}(\tau)\big)\boldsymbol{u}(t_{n-j-1}) \\
&\quad + \sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\Big(\int_0^\tau \mathcal{T}(\tau-s)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}+s))\,\mathrm{d}s - \tau\boldsymbol{T}(\tau)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\Big). \qquad (4.6)
\end{aligned}
$$

We rewrite the second term on the right-hand side of (4.6) using Taylor's formula:

$$
\begin{aligned}
&\boldsymbol{T}(\tau)^j\Big(\int_0^\tau \mathcal{T}(\tau-s)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}+s))\,\mathrm{d}s-\tau\boldsymbol{T}(\tau)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\Big)\\
&=\boldsymbol{T}(\tau)^j\int_0^\tau \Big(\mathcal{T}(\tau-s)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}+s))-\boldsymbol{T}(\tau)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\Big)\,\mathrm{d}s\\
&=\boldsymbol{T}(\tau)^j\Big(\int_0^\tau (\mathcal{T}(\tau-s)-\boldsymbol{T}(\tau))\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\\
&\qquad+\int_0^\tau \mathcal{T}(\tau-s)\int_0^s(\mathcal{F}\circ\boldsymbol{u})'(t_{n-j-1}+\xi)\Big)\,\mathrm{d}\xi\,\mathrm{d}s\\
&=\int_0^\tau\boldsymbol{T}(\tau)^j\big(\mathcal{T}(\tau-s)-\mathcal{T}(\tau)\big)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\,\mathrm{d}s+\tau\boldsymbol{T}(\tau)^j\big(\mathcal{T}(\tau)-\boldsymbol{T}(\tau)\big)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\\
&\qquad+\int_0^\tau\int_0^s\boldsymbol{T}(\tau)^j\mathcal{T}(\tau-s)(\mathcal{F}\circ\boldsymbol{u})'(t_{n-j-1}+\xi)\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}
$$

Combining the two identities above, for (4.6) we obtain:

$$
\begin{aligned}
&\sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j\mathrm{e}^{\mathrm{loc}}_{n-j}=\sum_{j=1}^{n-1}\big(\delta_{1,j}+\delta_{2,j}+\delta_{3,j}\big)\\
&\quad\text{with}\qquad \delta_{1,j}=\boldsymbol{T}(\tau)^j\big(\mathcal{T}(\tau)-\boldsymbol{T}(\tau)\big)\Big(\boldsymbol{u}(t_{n-j-1})+\tau\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\Big),\\
&\qquad\qquad\quad \delta_{2,j}=\int_0^\tau\boldsymbol{T}(\tau)^j\big(\mathcal{T}(\tau-s)-\mathcal{T}(\tau)\big)\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\,\mathrm{d}s,\\
&\qquad\qquad\quad \delta_{3,j}=\int_0^\tau\int_0^s\boldsymbol{T}(\tau)^j\mathcal{T}(\tau-s)(\mathcal{F}\circ\boldsymbol{u})'(t_{n-j-1}+\xi)\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}
\tag{4.7}
$$

For the term $\delta_{1,j}$, upon setting $(x,y)=\boldsymbol{u}(t_{n-j-1})+\tau\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))$ in Lemma 3.3 and (3), (4), we obtain the following estimate for $j=1,\dots,n-1$:

$$
\begin{aligned}
\|\delta_{1,j}\|\le C_4\tau^2\|A_0T_0(j\tau)\|\Big(&\big\|B\big(v(t_{n-j-1})+\tau\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\big)\big\|\\
&+\big\|B^2\big(v(t_{n-j-1})+\tau\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\big)\big\|\Big).
\end{aligned}
\tag{4.8}
$$

For the term $\delta_{2,j}$, setting $(x,y)=\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))$ in Lemma 3.7 and (4), we obtain the estimate for $j=1,\dots,n-1$:

$$
\begin{aligned}
\|\delta_{2,j}\|\le{}& C_5\tau^2\|A_0T_0(j\tau)\|\Big(\|\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\|+\|B\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|\Big)\\
&+C_6\tau^2\Big(\|\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|+\|B\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|+\|B^2\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|\Big).
\end{aligned}
\tag{4.9}
$$

The term $\delta_{3,j}$ is directly estimated by using Lemma 3.1 and (5), for $j=1,\dots,n-1$, as

$$
\begin{aligned}
\|\delta_{3,j}\|&\le\int_0^\tau\int_0^s C_7(1+\log(j))\Big\|\mathcal{T}(\tau-s)(\mathcal{F}\circ\boldsymbol{u})'(t_{n-j-1}+\xi)\Big\|\,\mathrm{d}\xi\,\mathrm{d}s\\
&\le MC_7(1+\log(j))\int_0^\tau\int_0^s\|(\mathcal{F}\circ\boldsymbol{u})'(t_{n-j-1}+\xi)\|\,\mathrm{d}\xi\,\mathrm{d}s\\
&\le\tau MC_7(1+\log(j))\|(\mathcal{F}\circ\boldsymbol{u})'\|_{\mathrm{L}^1([t_{n-j-1},t_{n-j}])}.
\end{aligned}
\tag{4.10}
$$

Finally, we combine the bounds (4.8), (4.9), (4.10), respectively, for $\delta_{k,j}$, $k = 1,2,3$, then collecting the terms we obtain

(4.11)
$$\Big\|\sum_{j=1}^{n-1}\boldsymbol{T}(\tau)^j \mathrm{e}_{n-j}^{\mathrm{loc}}\Big\| \le \sum_{j=1}^{n-1}\Big(\|\delta_{1,j}\| + \|\delta_{2,j}\| + \|\delta_{3,j}\|\Big)$$
$$\le C_8\tau\sum_{j=1}^{n-1}\frac{1}{j}\Big(\|Bv(t_{n-j-1})\| + \|B^2v(t_{n-j-1})\|\Big)$$
$$+ C_8\tau\sum_{j=1}^{n-1}\frac{1}{j}\Big(\|\mathcal{F}(\boldsymbol{u}(t_{n-j-1}))\| + \|B\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|\Big)$$
$$+ C_9\tau^2\sum_{j=1}^{n-1}\Big(\|\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\| + \|B\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\| + \|B^2\mathcal{F}_2(\boldsymbol{u}(t_{n-j-1}))\|\Big)$$
$$+ C_{10}\tau\log(n)\|(\mathcal{F}\circ u)'\|_{\mathrm{L}^1([0,t_{\max}])}$$
$$\le C_{11}(1+\log(n))\tau + C_{12}\tau \;\le\; C_{13}\tau\log(n+1),$$

where we have used the parabolic smoothing property (4.1) of the analytic semigroup T_0 to estimate the factor by $\|A_0T_0(j\tau)\| \le M/(j\tau)$.

(iii) The errors in the nonlinear terms are estimated by using Lemma 3.1 and the local Lipschitz continuity of $\mathcal{F}$ in the appropriate spaces ((1) and (2)), in combination with the bounds (4.3) for the past, as

(4.12)
$$\Big\|\tau\sum_{j=1}^{n}\boldsymbol{T}(\tau)^j\varepsilon_{n-j}^{\mathcal{F}}\Big\| \le \tau\sum_{j=1}^{n}\Big\|\boldsymbol{T}(\tau)^j\big(\mathcal{F}(\boldsymbol{u}(t_{n-j})) - \mathcal{F}(\boldsymbol{u}_{n-j})\big)\Big\|$$
$$\le \tau\sum_{j=1}^{n}M\big\|\mathcal{F}(\boldsymbol{u}(t_{n-j})) - \mathcal{F}(\boldsymbol{u}_{n-j})\big\| + \tau\sum_{j=1}^{n}M\big\|B\big(\mathcal{F}_2(\boldsymbol{u}(t_{n-j})) - \mathcal{F}_2(\boldsymbol{u}_{n-j})\big)\big\|$$
$$\le \tau\sum_{k=0}^{n-1}M\ell_\Sigma\|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\| + \tau\sum_{k=0}^{n-1}M\ell_{\Sigma,B}\|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\| \le C_{14}\tau\sum_{k=0}^{n-1}\|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\|,$$

recalling that ℓ_Σ and $\ell_{\Sigma,B}$ are the Lipschitz constants on Σ, see Assumptions 2.4 (1) and (2). For the last inequality, we used here that $(\boldsymbol{u}_k)_{k=0}^{n-1}$ belongs to the strip Σ so that the Lipschitz continuity of $\mathcal{F}$ can be used, see (1) and (2).

The global error (4.2) is bounded by the combination of the estimates (4.5), (4.11), and (4.12) from (i)–(iii), which altogether yield

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n\| \le C_3\tau + C_{13}\log(n+1)\tau + C_{14}\tau\sum_{k=0}^{n-1}\|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\|$$
(4.13)
$$\le C_{15}\log(n+1)\tau + C_{14}\tau\sum_{k=0}^{n-1}\|\boldsymbol{u}(t_k) - \boldsymbol{u}_k\|.$$

A discrete Gronwall inequality then implies

(4.14)
$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n\| \le C_{15}\mathrm{e}^{C_{14}t_{\max}}\log(n+1)\tau \le C|\log(\tau)|\tau,$$

for $t_n = \tau n \in [0, t_{\max}]$, with the constant $C := 2C_{15}\mathrm{e}^{C_{14}t_{\max}} > 0$. Then for a $\tau_0 > 0$ sufficiently small such that for each $\tau \le \tau_0$ we have $C|\log(\tau)|\tau \le R$, then $\boldsymbol{u}_n \in \Sigma$

and the error estimate (2.14) is satisfied for n as well. Hence (4.3) holds even up to n instead of $n-1$. Therefore, by induction, the proof of the theorem is complete. □

Remark 4.1. (a) Theorem 2.5 remains true, with an almost verbatim proof as above, if B is merely assumed to be the generator of a C_0-semigroup. This requires the following additional condition:

(5′) The function $B \circ \mathcal{F}_2 \circ \boldsymbol{u}$ is differentiable and $(B \circ \mathcal{F}_2 \circ \boldsymbol{u})' \in \mathrm{L}^1([0, t_{\max}]; F)$.

This is relevant only for the term $\delta_{3,j}$ in the inequality (4.10) when one applies the stability estimate from Lemma 3.1.

(b) Time-dependent nonlinearities can also be allowed and the same error bound holds without essential modification of the previous proof. Of course, the conditions (1), (2), (4) and (5) in Assumption 2.4, involving $\mathcal{F}$ and $\mathcal{F}_2$ need to be suitably modified. For example the functions $\mathcal{F}(t, \cdot)$ need to be uniformly Lipschitz for $t \in [0, t_{\max}]$ (and even this can be relaxed a little), and the function f defined by $f(t) := \mathcal{F}(t, \boldsymbol{u}(t))$ needs to be differentiable, etc.

(c) The assumptions (3) and (4) involving the domain $\mathrm{dom}(B^2)$ may seem a little restrictive. However, in some applications these conditions are naturally satisfied: For example if F is finite dimensional (such is the case for finite networks, see [36] or [40]). At the same time, these conditions seem to be optimal in this generality, and play a role only in the local error estimate of the Lie splitting, i.e., in Lemma 3.2 and its applications. Indeed, at other places the conditions involving $\mathrm{dom}(B^2)$ are not needed.

5. Numerical experiments. We have performed numerical experiments for Example 2.1: Let Ω be the unit disk with boundary $\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_2 = 1\}$, with γ denoting the trace operator, and ν denoting the outward unit normal field. Let us consider the boundary coupled semilinear parabolic partial differential equation (PDE) system $u : \overline{\Omega} \times [0, t_{\max}] \to \mathbb{R}$ and $v \times [0, t_{\max}] : \Gamma \to \mathbb{R}$ satisfying

$$\begin{cases} \partial_t u = \Delta u + \mathcal{F}_1(u, v) + \varrho_1 & \text{in } \Omega, \\ \partial_t v = \Delta_\Gamma v + \mathcal{F}_2(u, v) + \varrho_2 & \text{on } \Gamma, \\ \gamma u = v & \text{on } \Gamma, \end{cases} \tag{5.1}$$

where the two nonlinearities are $\mathcal{F}_1(u, v) = u^2$ and $\mathcal{F}_2(u, v) = v\gamma u$, and where the two inhomogeneities ϱ_1 and ϱ_2 are chosen such that the exact solutions are known to be $u(x, t) = \exp(-t)x_1^2 x_2^2$ and $v(x, t) = \exp(-t)x_1^2 x_2^2$ (which naturally satisfy $\gamma u = v$). The boundary coupled PDE system (5.1) fits into the abstract framework (1.1) in the sense of Example 2.1. We note that Theorem 2.5 still holds for (5.1) with the time-dependent inhomogeneities ϱ_i, see Remark 4.1 (c).

We performed numerical experiments using the splitting method (2.11), written componentwise (2.13), which is applied to the bulk–surface finite element semi-discretisation, see [11, 25], of the weak form of (5.1). The bulk–surface finite element semi-discretisation is based on a quasi-uniform triangulation Ω_h of the continuous domain Ω, such that the discrete boundary $\Gamma_h = \partial\Omega_h$ is also a sufficient good approximation of Γ. By this construction the traces of the finite element basis functions in Ω_h naturally form a basis on the boundary Γ_h, i.e. $\{\gamma_h \phi_j\}$ forms a boundary element basis on Γ_h. For more details we refer to [11, Section 4 and 5], or [25, Section 3]. Altogether this yields the matrix–vector formulation of the semi-discrete problem, for the nodal vectors $\mathbf{u}(t) \in \mathbb{R}^{N_\Omega}$ and $\mathbf{v}(t) \in \mathbb{R}^{N_\Gamma}$,

$$(5.2) \qquad \begin{cases} \mathbf{M}_\Omega \dot{\mathbf{u}} + \mathbf{A}_\Omega \mathbf{u} = \boldsymbol{\mathcal{F}}_1(\mathbf{u},\mathbf{v}) + \boldsymbol{\varrho}_1, \\ \mathbf{M}_\Gamma \dot{\mathbf{v}} + \mathbf{A}_\Gamma \mathbf{v} = \boldsymbol{\mathcal{F}}_2(\mathbf{u},\mathbf{v}) + \boldsymbol{\varrho}_2, \\ \boldsymbol{\gamma}\mathbf{u} = \mathbf{v}, \end{cases}$$

where $\mathbf{M}_\Omega$ and $\mathbf{A}_\Omega$ are the mass-lumped mass matrix and stiffness matrix for Ω_h, and similarly $\mathbf{M}_\Gamma$ and $\mathbf{A}_\Gamma$ for the discrete boundary Γ_h, while the nonlinearities $\boldsymbol{\mathcal{F}}_i$ and the inhomogeneities $\boldsymbol{\varrho}_i$ are defined accordingly. The discrete trace operator $\boldsymbol{\gamma} \in \mathbb{R}^{N_\Gamma \times N_\Omega}$ extracts the nodal values at the boundary nodes. For all these quantities we have used quadratures of sufficiently high order such that the quadrature errors are negligible compared to all other spatial errors. For mass lumping in this context, and for its spatial approximation properties, we refer to [25, Section 3.6].

The two semigroups in (2.13) are known, and are computed using the `expmv` Matlab package of Al-Mohy and Higham [2], in the above matrix–vector formulation (5.2) the (diagonal) mass matrices are transformed to the identity, i.e. $\widetilde{\mathbf{A}}_\Omega = \mathbf{M}_\Omega^{-1}\mathbf{A}_\Omega$, and similarly for $\widetilde{\mathbf{A}}_\Gamma$, and all other terms. The numerical experiments were performed for this transformed system. In this setting the operator D_0 in (2.1) corresponds to the harmonic extension operator, which we compute here by solving a Poisson problem with inhomogeneous Dirichlet boundary conditions.

5.1. A convergence experiment. We performed a convergence experiment for the above boundary coupled PDE system. Using the splitting integrator (2.11), in the form (2.13), we have solved the transformed system (5.2) for a sequence of time steps $\tau_k = \tau_{k-1}/2$ (with $\tau_0 = 0.2$) and a sequence of meshes with mesh width $h_k \approx h_{k-1}/\sqrt{2}$.

In Figure 1 we report on the $\mathrm{L}^\infty(\mathrm{L}^2(\Omega))$ and $\mathrm{L}^\infty(\mathrm{L}^2(\Gamma))$ error of the two components, comparing the (nodal interpolation of the) exact solutions and the numerical solutions. In the log-log plot we can observe that the temporal convergence order matches the predicted convergence rate $\mathcal{O}(\tau|\log(\tau)|)$ of Theorem 2.5, note the dashed reference line $\mathcal{O}(\tau)$ (the factor $|\log(\tau)|$ is naturally not observable). In the figures each line (with different marker and colour) corresponds to a fixed mesh width h, while each marker on the lines corresponds to a time step size τ_k. The precise time steps and degrees of freedom values are reported in Figure 1.

5.2. A convergence experiment with dynamic boundary conditions. We performed the same convergence experiment for a partial differential equation with *dynamic boundary conditions*, cf. [25], let $u : \overline{\Omega} \times [0, t_{\max}] \to \mathbb{R}$ solve the problem

$$(5.3) \qquad \begin{cases} \partial_t u = \Delta u + f_\Omega(u) + \varrho_1 & \text{in } \Omega, \\ \partial_t u = \Delta_\Gamma u - \partial_\nu u + f_\Gamma(u) + \varrho_2 & \text{on } \Gamma, \end{cases}$$

using the same domain, exact solution, nonlinearities, etc. as above.

Problem (5.3) is equivalently rewritten as a boundary coupled PDE system (5.1), where the two nonlinearities are given by

$$\mathcal{F}_1(u,v) = f_\Omega(u) = u^2 \quad \text{and} \quad \mathcal{F}_2(u,v) = -\partial_\nu u + f_\Gamma(u) = -\partial_\nu u + (\gamma u)^2.$$

That is, the the nonlinear term $\mathcal{F}_2$ incorporates the coupling through the Neumann trace $-\partial_\nu u$. The numerical method (2.11), written componentwise (2.13), is applied to this formulation with the nonlinearity $\mathcal{F}_2$ containing the Neumann trace operator.

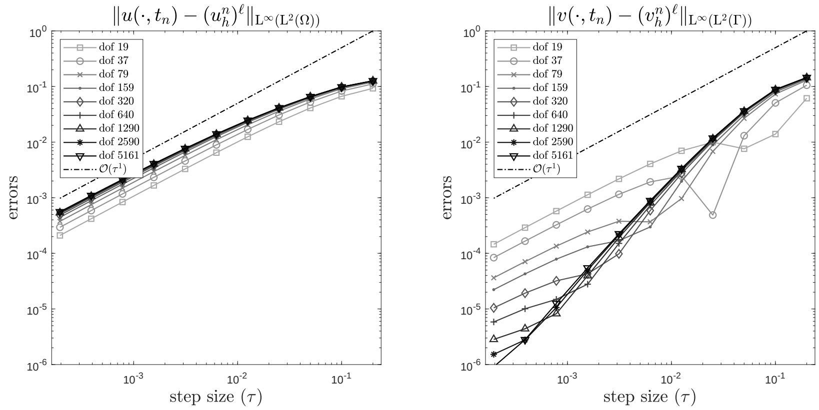

FIG. 1. *Temporal convergence plot for the splitting scheme* (2.13) *applied to the boundary coupled PDE system* (5.1), $\mathrm{L}^\infty(\mathrm{L}^2)$*-norms of u and v components on the left- and right-hand sides, respectively.*

In Figure 2 we report on the $\mathrm{L}^\infty(\mathrm{L}^2(\Omega))$ and $\mathrm{L}^\infty(\mathrm{L}^2(\Gamma))$ error of the bulk and surface errors, comparing the (nodal interpolation of the) exact solutions and the numerical solutions. (Figure 2 is obtained exactly as it was described for Figure 1, the precise time steps and degrees of freedom values can be read off from Figure 2.) Although in this case, due to the unboundedness of the Neumann trace operator in $\mathcal{F}_2(u,v) = -\partial_\nu u + f_\Gamma(u)$, the conditions of Theorem 2.5 are not satisfied, in Figure 2 we still observe a convergence rate $\mathcal{O}(\tau)$ (note the reference lines). Qualitatively we obtain the same plots for $\mathrm{L}^\infty(\mathrm{H}^1(\Omega))$ and $\mathrm{L}^\infty(\mathrm{H}^1(\Gamma))$ norms.

Note that our splitting method does not suffer from any type of order reduction, in contrast to the splitting schemes proposed in [25], see Figure 1 and 2 therein. In [3] the same order reduction issue was overcome by a different approach, using a correction term.

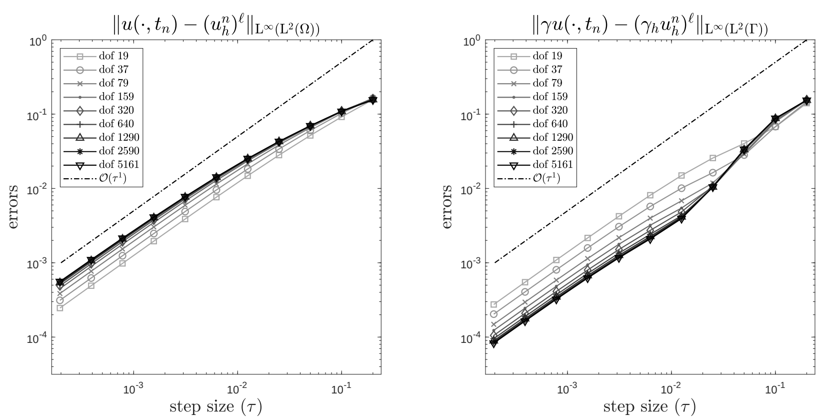

FIG. 2. *Temporal convergence plot for the splitting scheme applied to the PDE with dynamic boundary conditions* (5.3), $\mathrm{L}^\infty(\mathrm{L}^2)$*-norms of u and γu components on the left- and right-hand sides, respectively.*

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