Bergische Universität Wuppertal

Fakultät für Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and
Computational Mathematics (IMACM)

Preprint BUW-IMACM 21/30

Birgit Jacob, Jonathan R. Partington, Sandra Pott, Eskil
Rydhex, Felix L. Schwenninger

Laplace-Carleson embeddings and infinity-norm admissibility

September 24, 2021

http://www.imacm.uni-wuppertal.de

Laplace–Carleson embeddings and infinity-norm admissibility

Birgit Jacob* Jonathan R. Partington† Sandra Pott‡

Eskil Rydhe§ Felix L. Schwenninger¶

Abstract

New results on the boundedness of Laplace–Carleson embeddings on L^∞ and Orlicz spaces are proved. These findings are crucial for characterizing admissibility of control operators for linear diagonal semigroup systems in a variety of contexts. A particular focus is laid on essentially bounded inputs.

Keywords. Admissibility, Laplace–Carleson embeddings, semigroup systems.
2000 Subject Classification. 93D25, 93B28, 47D06, 46E15.

1 Introduction

This article is centred around the boundedness of linear operators

$$\Theta : Z(0, t_0; U) \to X, \quad t_0 > 0.$$

of the following convolution type form

$$\Theta(u) = \int_0^{t_0} T(t_0 - s)Bu(s)\mathrm{d}s.$$

*Fakultät für Mathematik und Naturwissenschaften, IMACM, Bergische Universität Wuppertal, Gaußstraße 20, 42119 Wuppertal, Germany, bjacob@uni-wuppertal.de

†School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K. J.R.Partington@leeds.ac.uk

‡Centre for Mathematics, Faculty of Science, Lund University, Sölvegatan 18, 22100 Lund, Sweden, Sandra.Pott@math.lu.se

§Centre for Mathematics, Faculty of Science, Lund University, Sölvegatan 18, 22100 Lund, Sweden, Eskil.Rydhe@math.lu.se.

¶Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands and Department of Mathematics, University of Hamburg, Bundesstraße 55, 20146 Hamburg, Germany, f.l.schwenninger@utwente.nl

Here $(T(t))_{t\geq 0}$ is a C_0-semigroup on X with infinitesimal generator A and $B : U \to D(A^*)'$ is a bounded linear operator, where $D(A^*)'$ is the completion of X with respect to the norm $\|x\|_{D(A^*)'} = \|(\beta - A)^{-1}x\|_X$ for some $\beta \in \rho(A)$. Here, $Z(0, t_0; U)$ refers to a U-valued function Lebesgue or, more generally, an Orlicz space on the interval $(0, t_0)$, and U, X are Banach spaces. The case $Z = \mathrm{L}^p$, and in particular the case $p = 2$, are commonly studied in abstract systems theory within the context of admissible operators. As it is well-known, the semigroup $(T(t))_{t\geq 0}$ has a unique extension to $X_{-1} := D(A^*)'$, which we again denote by $(T(t))_{t\geq 0}$. We recall that B is a Z-*admissible operator* if $\Theta : Z(0, t_0; U) \to X_{-1}$ is well-defined and bounded from $Z(0, t_0; U)$ to X. Note that Θ formally corresponds to the input-to-state map $u \mapsto x(t_0)$ of the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0, \quad t \geq 0. \tag{1}$$

The purpose of this paper can be described as follows. First some new results for general classes of semigroups are discussed. In particular, for left invertible C_0-semigroups on Hilbert spaces we show that L^∞- and L^2-admissibility are equivalent notions. Our main objective is to establish Laplace–Carleson embeddings from a class of function spaces Z to $\mathrm{L}^q(\mathbb{C}_+, \mu)$. The boundedness of these embeddings is naturally linked with admissibility for diagonal semigroups. Again, we investigate L^∞-admissibility and show that L^∞-admissibility even implies admissibility with respect to some Orlicz space. Furthermore, we study when the bound of such operators Θ goes to 0 as $t_0 \to 0^+$, which is expressed by the notion of *zero-class* L^∞-admissibility.

Unbounded admissible operators, that is, operators B not bounded as a mapping from U to X, naturally appear in the study of boundary control of evolution equations. The most commonly studied case in the literature is $Z = \mathrm{L}^2$ and we refer to the survey [6] and the book [14] for the basic background to admissibility in the context of well-posed and boundary control systems. The general case was already studied in the seminal works by Weiss [15, 16], where the notion of "admissibility" was coined, although it had appeared earlier, e.g. [13]. See also [3], where several results previously known for $p = 2$ were generalized. Admissible operators with respect to Orlicz spaces, $Z = \mathrm{L}^\Phi$, were studied in [7] and we refer to that paper for elementary facts of Z-admissible operators.

It is easy to see that the property of admissibility does not depend on the choice of t_0, which justifies the fact that we omit the reference to t_0 in the operator Θ. Let us fix the following notation for a semigroup generator A on X:

$$\mathfrak{B}_Z(A, U) = \{B \in L(U, X_{-1}) \colon B \text{ is an } Z\text{-admissible control operator for } A\}.$$

The inclusions

$$\mathfrak{B}_{\mathrm{L}^1}(A, U) \subseteq \mathfrak{B}_Z(A, U) \subseteq \mathfrak{B}_{\mathrm{L}^\infty}(A, U), \quad p \in [1, \infty], \tag{2}$$

are clear by the nesting properties of Orlicz spaces. A question in which we are particularly interested in is when $\mathfrak{B}_{\mathrm{L}^p}(A,U) = \mathfrak{B}_{\mathrm{L}^\infty}(A,U)$ for some $p \in [1,\infty)$. This is non-trivial as examples of semigroups are known for which all inclusions in (2) are strict (for all $p \in [1,\infty)$), see [7, Example 5.2] and [9]. One should note that these are examples on Hilbert spaces X, whereas the following, simpler, example shows that the situation on Banach spaces only becomes worse.

Example 1.1. *Let $p \in (1,\infty)$ and let $(T_p(t))_{t\geq 0}$ be the right-shift semigroup with generator A_p on $\mathrm{L}^p(0,\infty)$ defined by $T_p(t)f(x) = f(t+x)$ for $t, x \geq 0$. Define B through its dual $B^*f = f(0)$ acting on $D(A_p)$. It is easy to see that B is L^p-admissible, but not L^q-admissible for any $q < p$. In particular, this shows that $\mathfrak{B}_{\mathrm{L}^q}(A_2,\mathbb{C}) \subsetneq \mathfrak{B}_{\mathrm{L}^2}(A_2,\mathbb{C})$ for $q < 2 = p$.*

We show that this example is sharp in the following sense. For any left-invertible semigroup generator A and Hilbert spaces X, U, it holds that $\mathfrak{B}_{\mathrm{L}^q}(A,U) = \mathfrak{B}_{\mathrm{L}^2}(A,U)$ for all $q \geq 2$, Theorem 2.2. The underlying reason is a consequence of the Paley–Wiener theorem in the context of admissible operators [17]. This result can be seen as the point of departure for the analysis of variants of this statement and our aim to characterise L^∞-admissibility for classes of semigroups relevant in applications. We invoke more structure on the semigroup and on the input space, such as being diagonal and assuming U to be finite-dimensional, while weakening the condition that X is a Hilbert space. The key tool for handling this combination is by using the relation between admissibility and Laplace–Carleson embeddings, which have already been used in prior works [8, 9]. Hence, in order to answer the above mentioned questions, we prove new embedding theorems. Since our focus is in particular on $p = \infty$, this extends previously derived results, where this case was not considered. The set-up of the paper is as follows: In Section 2 we present several results formulated in the language of admissible operators while Section 3 is devoted to Laplace–Carleson embeddings.
For the rest of the paper, A will always denote the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and the space X_{-1} is defined as above. Further assumptions on the semigroup may be imposed in the respective sections. The spaces X and U will generally refer to general complex Banach spaces, unless specified otherwise. The space of bounded linear operators from U to X will be denoted by $L(U,X)$. Let I be an interval and $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, that is, an increasing, continuous and convex function such that $\lim_{x\to 0} \frac{\Phi(x)}{x} = \lim_{x\to\infty} \frac{x}{\Phi(x)} = 0$. By $\mathrm{L}^\Phi(I;U)$ we denote space of U-valued measurable functions $f : I \to U$ such that $\Phi(k^{-1}\|f(\cdot)\|_U)$ is integrable for some $k > 0$. This space is equipped with the norm

$$\|f\|_{\mathrm{L}^\Phi} = \inf\{k > 0\colon \int_I \Phi(k^{-1}\|f(s)\|_U)\,\mathrm{d}s \leq 1\} \tag{3}$$

and called the Orlicz space corresponding to the Young function Φ. If $U = \mathbb{C}$, we may write $\mathrm{L}^{\Phi}(I)$.
With a slight abuse of notation, we will also regard $\mathrm{L}^1(I;U)$ and $\mathrm{L}^\infty(I;U)$ as "Orlicz spaces" with their natural norm. Therefore, if we say that "Z is an Orlicz space", we mean that either $Z = \mathrm{L}^1$, $Z = \mathrm{L}^\infty$ or $Z = \mathrm{L}^\Phi$ for some Young function Φ. Note that for the special case $\Phi(s) = s^p$, $p \in (1,\infty)$, L^Φ is isomorphic to L^p. In analogy to Hölder conjugates for L^p-spaces, given a Young function Φ we can define the *complementary Young function* Φ^c by $\Phi^c(s) = \max_{t\geq 0}(st - \Phi(t))$, which indeed defines a Young function and a corresponding Orlicz space. For details on Orlicz spaces, we refer the reader to textbooks such as [1, Chapter 4.8] or the appendix of [7], where they appeared in the context of admissible operators for the first time.

2 Admissible operators

By a closed graph argument and the semigroup property, the notion of an admissible operator B can be rephrased as follows.

Definition 2.1. *Let Z be an Orlicz space. An operator $B \in L(U, X_{-1})$ is called Z-*admissible *(for $(T(t))_{t\geq 0}$ or A), if for all $t_0 > 0$ and all $u \in Z(0, t_0; U)$ it holds that*

$$\Theta u = \Theta_{t_0} u = \int_0^{t_0} T(t_0 - s)Bu(s)\,\mathrm{d}s \in X.$$

Furthermore, we define the following two refinements of admissibility. We say that

- *B is* zero-class *Z-admissible, if $\lim_{t_0\to 0^+} \|\Theta_{t_0}\|_{\mathcal{L}(Z(0,t_0;U),X)} = 0$, and*
- *B is* infinite-time *Z-admissible, if $\sup_{t_0>0} \|\Theta_{t_0}\|_{\mathcal{L}(Z(0,t_0;U),X)} < \infty$.*

Note that B is infinite-time Z-admissible if and only if the operator

$$Z(0,\infty;U) \to X, u \mapsto \int_0^\infty T(s)Bu(s)\mathrm{d}s$$

is bounded. We further mention that admissibility may be studied for other choices of function spaces Z, such as weighted L^p-spaces [4] and Sobolev spaces [9]. The interest in Orlicz spaces arises in the connection of admissibility to (integral) input-to-state stability for infinite-dimensional systems, see [7].

2.1 Left-invertible semigroups on Hilbert spaces

In [16], it was (implicitly) shown that $\mathfrak{B}_{\mathrm{L}^\infty}(A,\mathbb{C}) = \mathfrak{B}_{\mathrm{L}^2}(A,\mathbb{C})$ for A being the periodic left-shift semigroup on $\mathrm{L}^2(0,2\pi)$, corresponding to the control of a one-dimensional wave equation. It turns out that this result holds true in a much more general setting. This is a rather direct consequence of another result by G. Weiss, which was derived in the context of what later became known as the Weiss conjecture, [17].

Theorem 2.2. *Let A generate a left-invertible semigroup on a Hilbert space X. Then for any Hilbert space U it holds that*

$$\mathfrak{B}_{\mathrm{L}^\infty}(A,U) = \mathfrak{B}_{\mathrm{L}^2}(A,U).$$

Proof. This basically follows by [17, Theorem 4.1] which is a slight generalization of an older result by Hansen and Weiss [5]. In fact, let $B \in \mathfrak{B}_{\mathrm{L}^\infty}(A,U)$. Then, it is follows by the definition of L^∞-admissibility and the Laplace transform that

$$\sup_{\operatorname{Re}\lambda>\alpha} \|(\lambda - A)^{-1}B\| < \infty$$

for some $\alpha \in \mathbb{R}$. By (the dual version of) [17, Theorem 4.1], see also [18], this implies that B is L^2-admissible. □

Example 1.1 shows that the assumption that X is a Hilbert space in Theorem 2.2 cannot be dropped in general, even in the specific case of $U = \mathbb{C}$. However, for the specific case of $A_{r,\mathrm{per}}$ being the generator of the periodic left-shift semigroup on $\mathrm{L}^r(0,2\pi)$, characterizations of $\mathfrak{B}_{\mathrm{L}^p}(A_{r,\mathrm{per}},\mathbb{C})$ can be derived from results on Fourier multipliers, [16, Proposition 5.2]; more precisely,

$$\begin{aligned}&\mathfrak{B}_{\mathrm{L}^p}(A_{r,\mathrm{per}},\mathbb{C})\\ &\quad= \quad \{b \in S_{\mathrm{per}}[0,2\pi] \colon f \mapsto \sum_{k\in\mathbb{Z}} \hat{b}(k)\hat{f}(k)\mathrm{e}^{ikt} \in \mathcal{L}(\mathrm{L}^p(0,2\pi),\mathrm{L}^r(0,2\pi))\},\end{aligned}$$

where $\hat{h}(k)$ denotes the k-th Fourier coefficient and $S_{\mathrm{per}}[0,2\pi]$ the periodic distributions on $[0,2\pi]$. By known facts on multipliers, this implies in particular that

$$\mathfrak{B}_{\mathrm{L}^p}(A_{r,\mathrm{per}},\mathbb{C}) = \mathfrak{B}_{\mathrm{L}^\infty}(A_{r,\mathrm{per}},\mathbb{C}) \qquad \text{for all } p \geq 2 \text{ and } r \leq 2,$$

which generalizes the assertion of Theorem 2.2 in the situation of this special generator. This observation motivates studying the relation of the sets $\mathfrak{B}_{\mathrm{L}^p}(A,\mathbb{C})$ for more general group generators of *diagonal* form. This is treated in the next section. Also note that the facts on Fourier multipliers used above give a glimpse on why the relation between the sets $\mathfrak{B}_{\mathrm{L}^p}(A,U)$ for different p is non-trivial in general. We shall see a related result in Theorem 2.4 below.

2.2 Diagonal C_0-semigroups

In this section we assume that the semigroup generator A is diagonal with respect to a (Schauder) basis of X. More precisely, fix $1 \leq q < \infty$ and a q-Riesz basis $(\phi_k)_{k\in\mathbb{Z}}$ of X, i.e., for some $C_1, C_2 > 0$ we have that for all finite sequences $(a_k)_k$,

$$C_1 \sum_k |a_k|^q \leq \| \sum_k a_k \phi_k \|^q \leq C_2 \sum_k |a_k|^q.$$

Let $A : D(A) \subset X \to X$ be an operator such that $\phi_n \in D(A)$ for all $n \in \mathbb{Z}$ and $A\phi_n = \lambda_n \phi_n$ for a complex sequence $(\lambda_n)_n$ in a left-half plane of $\mathbb{C}$. This implies that A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ with $T(t)\phi_n = \mathrm{e}^{\lambda_n t}\phi_n$ for all $n \in \mathbb{Z}$, $t \geq 0$.

In the above situation we say that A *generates a diagonal semigroup with respect to the q-Riesz basis* $(\phi_n)_n$. If the sequence $(\lambda_n)_n$ lies in a vertical strip of the complex plane, we say that A generates a *diagonal group* with respect to the q-Riesz basis.

Note that the eigenvalues $(\lambda_n)_n$ lie in the open left-half plane $\mathbb{C}_-$ if and only if $(T(t))_{\geq 0}$ is strongly stable, i.e. $\lim_{t\to\infty} \|T(t)x\| = 0$ for all $x \in X$. Without loss of generality we may set $X = \ell^q$ and choose ϕ_n to be the n-th canonical basis vector of ℓ^q. Further, we assume $U = \mathbb{C}$. It follows that every operator $B \in L(U, X_{-1})$ can be represented by a sequence $(b_n)_{n\in\mathbb{Z}}$ in

$$X_{-1} \cong \left\{ b \in \mathbb{C}^{\mathbb{Z}} \colon \left(\frac{b_n}{\lambda_n - \lambda} \right)_{n\in\mathbb{Z}} \in \ell^q \right\}$$

for some λ in the resolvent set $\rho(A)$ of A. We use the analogous notation if the index set $\mathbb{Z}$ is replaced by $\mathbb{N}$.

We can link admissibility with the boundedness of Laplace–Carleson embeddings: the following result was proved in [9] only for $Z = \mathrm{L}^p$ with $1 \leq p < \infty$, but the case $p = \infty$ and even $Z = \mathrm{L}^\Phi$ for some Young function Φ follows analogously.

Proposition 2.3 (Theorem 2.1 in [9]). *Let $q \geq 2$ and let $A : D(A) \subset X \to X$ generate a strongly stable diagonal semigroup $(T(t))_{t\geq 0}$ with respect to a q-Riesz basis of X. Let Z be an Orlicz space. The operator $B \in L(U, X_{-1})$ is infinite-time Z-admissible for $(T(t))_{t\geq 0}$ if and only if the Laplace–Carleson embedding*

$$\mathcal{L}f(z) := \int_0^\infty e^{-zt} f(t)dt, \qquad z \in \mathbb{C}_+$$

induces a continuous mapping from $Z(0,\infty)$ into $\mathrm{L}^q(\mathbb{C}_+, d\mu)$, where μ is the measure $\sum |b_k|^q \delta_{-\lambda_k}$.

Hence, in order to answer the above mentioned questions, we prove new embeddings theorem for the Laplace–Carleson embedding. These new Laplace–Carleson embeddings are proved in the following section and of independent interest, but already applied here to obtain new admissibility results.

The main results of this section are the following.

Theorem 2.4. *Let $q \geq 2$. If $A : D(A) \subset X \to X$ generates a diagonal group with respect to a q-Riesz basis on X, then*

$$\mathfrak{B}_{\mathrm{L}^\infty}(A, \mathbb{C}^n) = \mathfrak{B}_{\mathrm{L}^{q/(q-1)}}(A, \mathbb{C}^n).$$

Clearly the case $p = 2$ in Theorem 2.4 is already covered Theorem 2.2.

Proof. We first mention, that it suffices to prove the results for $n = 1$ (and apply e.g. [10, Prop. 4] in the general). The statement then follows directly from Proposition 2.3 and Theorem 3.9. □

As explained in the introduction, we cannot expect that $\mathfrak{B}_{\mathrm{L}^\infty}(A, \mathbb{C}^n)$ equals $\mathfrak{B}_{\mathrm{L}^p}(A, \mathbb{C}^n)$ for some $p < \infty$ in general. The following result, however, shows that for diagonal semigroup generators A, at least every element B in $\mathfrak{B}_{\mathrm{L}^\infty}(A, \mathbb{C}^n)$ is contained in $\mathfrak{B}_{\mathrm{L}^\Phi}(A, \mathbb{C}^n)$ for some Young function Φ depending on B.

Theorem 2.5. *Let $q \geq 2$ and $A : D(A) \subset X \to X$ be the generator of a strongly stable diagonal semigroup $(T(t))_{t \geq 0}$ with respect to a q-Riesz basis and eigenvalues $(\lambda_n)_{n \in \mathbb{Z}}$. Then the operator $B \in L(\mathbb{C}, X_{-1})$ is infinite-time* L^∞*-admissible if and only if*

$$\sum_{n \in \mathbb{Z}} \sup_{\substack{I \subset i\mathbb{R} \\ I \text{ interval}}} \frac{\mu(Q_I \cap S_n)}{|I|^q} < \infty, \tag{4}$$

where $\mu = \sum |b_k|^q \delta_{-\lambda_k}$, $Q_I := \{z = x + iy \mid iy \in I, 0 < x < |I|\}$ and $S_n := \{x + iy \mid y \in \mathbb{R}, 2^n \leq x < 2^{n+1}\}$. In this case there exists a Young function Φ such that B is infinite-time L^Φ*-admissible.*
Moreover, B is zero-class L^∞*-admissible for $(T(t))_{t \geq 0}$.*

Proof. The statement about the equivalence follows from Proposition 2.3 and 3.3, whereas the existence of a suitable Young function is guaranteed by Theorem 3.10. Finally the zero-class L^∞-admissibility follows directly from the L^Φ-admissibility by Hölder's inequality for Orlicz spaces. □

Since (finite-time) admissibility remains invariant under for the shifted generator $A - cI$, $c \in \mathbb{R}$, we obtain the following consequence.

Corollary 2.6. *Let $q \geq 2$ and $A : D(A) \subset X \to X$ be the generator of a diagonal semigroup with respect to a q-Riesz basis. Then for every $B \in \mathfrak{B}_{\mathrm{L}^\infty}(A, \mathbb{C}^n)$ there exists a Young function Φ such that $B \in \mathfrak{B}_{\mathrm{L}^\Phi}(A, \mathbb{C}^n)$.*

Finally we can formulate a characterization for L^Φ-admissible operators for the specific Young function $\Phi_{\exp}(t) = \exp(t) - t - 1$. This complements existing characterizations of L^p-admissible operators for diagonal semigroups, [9].

Theorem 2.7. *Let $q \geq 2$ and $A : D(A) \subset X \to X$ be the generator of a strongly stable diagonal semigroup $(T(t))_{t\geq 0}$ with respect to a 2-Riesz basis and eigenvalues $(\lambda_n)_{n\in\mathbb{Z}}$. Then*

$$\begin{aligned}&\mathfrak{B}_{\mathrm{L}^{\Phi_{\exp}}}(A,\mathbb{C})\\ &= \{(b_k)_{k\in\mathbb{N}} \in L(\mathbb{C}, X_{-1}) \mid \sum_{n=1}^{\infty} n^2 \sup_{\substack{I\subset i\mathbb{R}\\ I \text{ interval}}} \frac{\mu(Q_I \cap S_n)}{|I|^2} + \sup_{\substack{I\subset i\mathbb{R}\\ I \text{ interval}\\ |I|=2}} \mu(Q_I) < \infty\},\end{aligned}$$

where μ, Q_I, S_n are defined as in Theorem 2.5.

Proof. Since $\mathfrak{B}_{\mathrm{L}^{\Phi_{\exp}}}(A,\mathbb{C}) = \mathfrak{B}_{\mathrm{L}^{\Phi_{\exp}}}(A-cI,\mathbb{C})$ for any $c \in \mathbb{R}$, we can assume that A has only eigenvalues with real part less than -2. The result now follows by Proposition 2.3 and Theorem 3.13, which is proved later. □

Remark 2.8.

1. *Theorems 2.5 and 2.7 can be used to formulate analogous results for finite-dimensional input spaces, i.e. $B \in L(\mathbb{C}^n, X_{-1})$ for $n \in \mathbb{N}$, by considering every "component" of B separately, see also [10, Prop. 4].*

2. *Theorem 2.5 generalizes [7, Thm. 4.1] where the case of analytic diagonal semigroups was considered and thus condition* (4) *is satisfied for all $B \in L(\mathbb{C}^n, X_{-1})$. Also note that in those references, q may more generally be chosen from* $[1,\infty)$. *On the other hand note that [7, Thm. 4.1] was generalized to more general analytic semigroups which are not necessarily diagonal in [10].*

3. *Corollary 2.6 also relates to the concept of input-to-state stability. More precisely, following the results in [7], it shows that for linear systems described by diagonal semigroups with respect to a q-Riesz basis, the notions of input-to-state stability and integral input-to-state stability are equivalent.*

3 Laplace–Carleson embeddings

Let μ be a positive regular Borel measure on the complex right half-plane $\mathbb{C}_+ = \{z = x + iy \mid y \in \mathbb{R}, x > 0\}$. In this section, we only consider scalar-valued Orlicz spaces Z on the interval $(0,\infty)$, that is $Z = Z(0,\infty;\mathbb{C})$ in our notation above. We will omit the reference to the interval here for the sake of brevity. Formally, what we mean by a *Laplace–Carleson embedding* is a map of the form $\mathcal{L}\colon Z \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ given by

$$\mathcal{L}f(z) := \int_0^\infty e^{-zt} f(t)dt, \qquad z \in \mathbb{C}_+.$$

Since convergence of a sequence in Z implies pointwise convergence of the corresponding sequence of Laplace transforms, any set inclusion of the form $\mathcal{L}Z \subseteq \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is automatically continuous by the closed graph theorem. We will require the notions of *Hardy spaces* and *reproducing kernels*:

Let $F\colon \mathbb{C}_+ \to \mathbb{C}$ be analytic. We say that F belongs to the *Hardy space* $\mathrm{H}^p(\mathbb{C}_+)$ whenever

$$\|F\|^p_{\mathrm{H}^p(\mathbb{C}_+)} := \sup_{\epsilon>0} \int_{y\in\mathbb{R}} |F(\epsilon + iy)|^p \, dy < \infty.$$

For a shifted half-plane $\mathbb{C}_{+,\alpha} = \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$, we have accordingly the Hardy space $\mathrm{H}^p(\mathbb{C}_{+,\alpha})$ of all analytic functions on $\mathbb{C}_{+,\alpha}$ such that

$$\|F\|^p_{\mathrm{H}^p(\mathbb{C}_{+,\alpha})} = \sup_{\epsilon>0} \int_{y\in\mathbb{R}} |F(\epsilon + \alpha + iy)|^p \, dy < \infty$$

For $F \in \mathrm{H}^p(\mathbb{C}_+)$ and $F_\epsilon(iy) = F(\epsilon + iy)$, the limit $F(iy) = \lim_{\epsilon\to 0^+} F_\epsilon(iy)$ exists for Lebesgue a.e. y. Moreover, $F_\epsilon \to F$ in $\mathrm{L}^p(i\mathbb{R})$. This makes $\mathrm{H}^p(\mathbb{C}_+)$ isometrically isomorphic to a closed subspace of $\mathrm{L}^p(i\mathbb{R})$. A good reference on Hardy spaces is [2, Chapter II].
For $\lambda \in \mathbb{C}_+$ and $t > 0$, let $k_\lambda(t) = \frac{1}{2\pi}\exp(-\bar{\lambda}t)$. Note that $\|k_\lambda\|^p_{\mathrm{L}^p} = \frac{1}{p(2\pi)^p \operatorname{Re}\lambda}$. The so-called *reproducing kernel* is the analytic function

$$K_\lambda\colon z \mapsto \mathcal{L}k_\lambda(z) = \frac{1}{2\pi}\frac{1}{z+\bar{\lambda}},$$

defined at least for $\operatorname{Re} z \geq 0$. If $p < \infty$ and $F \in \mathrm{H}^p(\mathbb{C}_+)$, then

$$F(\lambda) = \int_{y\in\mathbb{R}} F(iy)\overline{K_\lambda(iy)}\, dy, \tag{5}$$

which follows essentially from Cauchy's theorem.

3.1 Laplace–Carleson embeddings and Carleson intensities

The *Carleson square* associated to an interval $I \subset i\mathbb{R}$ is the set

$$Q_I = \{z = x + iy \in \mathbb{C}_+ \mid iy \in I, 0 < x < |I|\}.$$

These are related to reproducing kernels by the fact that if $\bar{\lambda}$ is the centre of Q_I, so that in particular $\operatorname{Re}\lambda = |I|/2$, then

$$\frac{1}{\sqrt{10}\pi|I|} \leq |K_\lambda(z)| \leq \frac{1}{\pi|I|} \quad \text{for } z \in Q_I.$$

With p' denoting the Hölder conjugate of $p \in [1, \infty]$, the above inequalities imply that if $\mathcal{L}\colon \mathrm{L}^p \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded, then

$$\mu(Q_I) \lesssim |I|^{q/p'} \quad \text{for all intervals} \quad I \subset i\mathbb{R}, \tag{6}$$

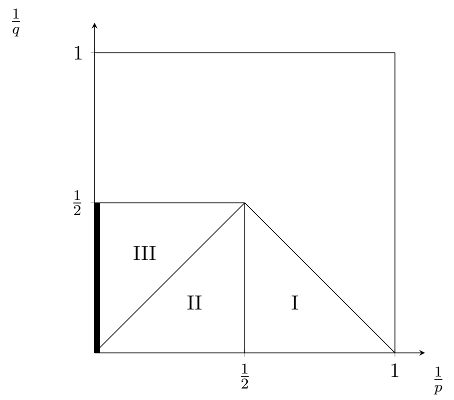

Figure 1: Relation between condition (6) and the boundedness of $\mathcal{L}\colon \mathrm{L}^p \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$. If $1 \leq p \leq 2$, $p' \leq q < \infty$ (region I), or $2 < p \leq q < \infty$ (region II), then (6) is necessary and sufficient for the embedding to be bounded. If $2 \leq q < p \leq \infty$ (region III), then (6) is necessary and sufficient under the additional assumption that μ has support on a vertical strip. For general measures, (6) is necessary but not sufficient. The bold edge to the left corresponds to the hypothesis of Theorem 3.3.

see [8, Proposition 3.1]. It is a remarkable fact that in a variety of situations, condition (6) is also sufficient for $\mathcal{L}\colon \mathrm{L}^p \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ to be bounded. For $1 \leq p \leq 2$, and $p' \leq q < \infty$ (this corresponds to the region I in Figure 1), $\mathcal{L}\colon \mathrm{L}^p \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ if and only if (6) holds, see [8, Theorem 3.2]. In [12, Theorem 1.1], this result was extended to $2 < p \leq q < \infty$ (region II in Figure 1). For $2 \leq q < p < \infty$ (region III in Figure 1), (6) is sufficient if μ has support on a vertical strip, but not if μ has support on a sector, see [8, Theorem 3.6 and Theorem 3.5]. The thick line in Figure 1 corresponds to the hypothesis of Theorem 3.3 below. This new result characterizes the class of μ such that $\mathcal{L}\colon \mathrm{L}^\infty \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ for $q \geq 2$.
Motivated by the significance of (6), we state the concept *α-Carleson intensity.*

Definition 3.1. *Let μ be a positive regular Borel measure on $\mathbb{C}_+$ and $\alpha > 0$. Then the* α-Carleson intensity $\mathcal{C}_\alpha[\mu]$ *is given by*

$$\mathcal{C}_\alpha[\mu] = \sup_{\substack{I \subset i\mathbb{R} \\ I \text{ interval}}} \frac{\mu(Q_I)}{|I|^\alpha}.$$

Obviously, (6) holds if and only if $\mathcal{C}_{q/p'}[\mu] < \infty$.
Measures supported on vertical strips will play an important role in the investigation below. The next definition is essentially a notational convention that will be used henceforth.

Definition 3.2. *Let μ be a positive regular Borel measure on $\mathbb{C}_+$. For each $n \in \mathbb{Z}$, consider the* dyadic strip

$$S_n := \left\{x + iy \mid y \in \mathbb{R}, 2^n \le x < 2^{n+1}\right\},$$

and define the measure μ_n on $\mathbb{C}_+$ by $\mu_n \colon E \mapsto \mu(E \cap S_n)$.

If $2 \le q < p \le \infty$, and μ is supported on a vertical strip, then $\mathcal{L} \colon \mathrm{L}^p \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ if and only if (6) holds. For general μ, this fails. We partially address this in the following result.

Theorem 3.3. *Let μ be a positive regular Borel measure on $\mathbb{C}_+$, and $2 \le q < \infty$. Then $\mathcal{L} \colon \mathrm{L}^\infty(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded if and only if*

$$\sum_{n\in\mathbb{Z}} \mathcal{C}_q[\mu_n] < \infty. \tag{7}$$

Furthermore, the above sum is comparable to $\|\mathcal{L} \colon \mathrm{L}^\infty(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)\|^q$.

For the proof of Theorem 3.3 we need the following lemma and propositions.

Lemma 3.4. *Let $\alpha \ge 1$.*

(*i*) *There exists an interval $I \subset i\mathbb{R}$ such that $|I| = 2^{n+1}$ and*

$$\mathcal{C}_\alpha[\mu_n] \le 2^{\alpha+1} \frac{\mu_n(Q_I)}{|I|^\alpha}.$$

(*ii*) *If $\beta \ge 1$, then*

$$\mathcal{C}_\alpha[\mu_n] \le 2^{\beta+n(\beta-\alpha)} \mathcal{C}_\beta[\mu_n] \le 2^{\alpha+\beta} \mathcal{C}_\alpha[\mu_n],$$

i.e. $\mathcal{C}_\alpha[\mu_n] \approx 2^{n(\beta-\alpha)}\mathcal{C}_\beta[\mu_n]$, where the constants of comparison depend only on α and β.

(*iii*) *If one defines the shifted measure $\tilde{\mu}_n \colon E \mapsto \mu_n(E + 2^{n-1})$, then*

$$\mathcal{C}_\alpha[\tilde{\mu}_n] \le 2^\alpha \mathcal{C}_\alpha[\mu_n].$$

Proof. To prove (*i*), introduce the auxiliary quantity

$$\widetilde{\mathcal{C}_\alpha}[\mu_n] = \sup_{|I|=2^{n+1}} \frac{\mu_n(Q_I)}{|I|^\alpha}.$$

If $|I| \ge 2^n$, then there exists a finite collection of intervals $\{J_k\}_{k=1}^N$, where $N \le 2^{-n}|I|$, each $|J_k| = 2^{n+1}$, and $I \subseteq \bigcup_{k=1}^N J_k$. Since also $Q_I \cap S_n \subset \bigcup_{k=1}^N Q_{J_k}$,

$$\begin{aligned} \mu_n(Q_I) &\le \sum_{k=1}^N \mu_n(Q_{J_k}) \le \sum_{k=1}^N \widetilde{\mathcal{C}_\alpha}[\mu_n] \left(2^{n+1}\right)^\alpha \le 2^\alpha \sum_{k=1}^N \widetilde{\mathcal{C}_\alpha}[\mu_n] \left(\frac{|I|}{N}\right)^\alpha \\ &\le 2^\alpha |I|^\alpha \widetilde{\mathcal{C}_\alpha}[\mu_n]. \end{aligned}$$

For smaller intervals, $\mu_n(Q_I) = 0$. From this, $\mathcal{C}_\alpha[\mu_n] \le 2^\alpha \widetilde{\mathcal{C}_\alpha}[\mu_n]$, and since there clearly exists I with $|I| = 2^{n+1}$ such that $\widetilde{\mathcal{C}_\alpha}[\mu_n] \le 2\frac{\mu(Q_I)}{|I|^\alpha}$, (i) follows. For the proof of (ii), it is immediate from the definition that $\widetilde{\mathcal{C}_\alpha}[\mu_n] = 2^{(n+1)(\beta-\alpha)}\widetilde{\mathcal{C}_\beta}[\mu_n]$. Since $\widetilde{\mathcal{C}_\beta}[\mu_n] \le \mathcal{C}_\beta[\mu_n]$, and we just proved that $\mathcal{C}_\alpha[\mu_n] \le 2^\alpha \widetilde{\mathcal{C}_\alpha}[\mu_n]$, this establishes the first inequality in (ii). The second inequality follows by interchanging α and β.
To prove (iii), note that $\mu_n(Q_I + 2^{n-1}) = 0$ when $|I| < 2^{n-1}$, whereas if $|I| \ge 2^{n-1}$, then $Q_I + 2^{n-1} \subseteq Q_{2I}$. □

The necessity of (7) for the boundedness of the Laplace–Carleson embedding even extends to the case $1 \le q < \infty$.

Theorem 3.5. *If $1 \le q < \infty$, then*

$$\sum_{n\in\mathbb{Z}} \mathcal{C}_q[\mu_n] \lesssim \|\mathcal{L}\colon \mathrm{L}^\infty \to \mathrm{L}^q(\mathbb{C}_+, d\mu)\|^q.$$

Proof. For $n \in \mathbb{Z}$, chose I_n with $|I_n| = 2^{n+1}$, and $\mathcal{C}_q[\mu_n] \le 2^{q+1}\frac{\mu_n(T_n)}{|I_n|^q}$, where T_n denotes the right-hand half of the square Q_{I_n}. With ic_n denoting the mid-point of I_n, define $f_n(t) = \chi_{(2^{-n-1},2^{-n}]}(t)e^{ic_n t}$, and $F_n = \mathcal{L}f_n$. The proof now proceeds through three different steps.

Step 1: We first show that there exists positive real constants c and C such that:

(i) If $n \in \mathbb{Z}$ and $z \in T_n$, then $|F_n(z)| \ge c2^{-n}$.

(ii) If $m, n \in \mathbb{Z}$ and $z \in T_n$, then $|F_m(z)| \le C2^{-n-|n-m|}$.

Proof. (i) If $z = x + iy \in T_n$ and $2^{-n-1} \le t \le 2^{-n}$, then $|t(y - c_n)| \le 1$. Hence,

$$|F_n(z)| \ge \operatorname{Re} F_n(z) = \int_{t=2^{-n-1}}^{2^{-n}} e^{-tx}\cos\left(t\left(y - c_n\right)\right)\, dt \ge e^{-2}\cos(1)2^{-n-1}.$$

(ii) By the triangle inequality,

$$|F_m(z)| \le \int_{t=2^{-m-1}}^{2^{-m}} e^{-xt}\, dt.$$

Since the above integral is less than $2^{-m} = 2^{-n-(m-n)}$, our inequality is immediate for $m \ge n$. For $m < n$, we use instead that

$$\begin{aligned} |F_m(z)| &\le \int_{t=2^{-m-1}}^{\infty} e^{-xt}\, dt \\ &= \frac{e^{-2^{-m-1}x}}{x} \\ &\le \frac{e^{-2^{n-m-1}}}{2^n} = 2^{-n}2^{n-m}e^{-2^{n-m-1}}2^{m-n}. \end{aligned}$$

Since $2ae^{-a}$ is bounded for $a \geq 0$, the conclusion follows. □

Step 2: With c and C as in Step 1, chose an integer N such that $C2^{3-N} \leq c$. For $k \in \{1, 2, \ldots, N\}$, define $g_k = \sum_{m\in\mathbb{Z}} f_{mN+k}$, and $G_k = \mathcal{L}g_k$. We now show that if $n \in \mathbb{Z}$ and $z \in T_{nN+k}$, then $|G_k(z)| \geq \frac{1}{2}|F_{nN+k}(z)|$.

Proof. By the properties in Step 1,

$$\begin{aligned} |G_k(z) - F_{nN+k}(z)| &\leq \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} |F_{mN+k}(z)| \leq C2^{-nN-k} \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} 2^{-N|n-m|} \\ &= \frac{C2^{-nN-k+1-N}}{1-2^{-N}} \leq C2^{-nN-k+2-N} \leq \frac{c}{2} 2^{-nN-k} \leq \frac{1}{2}|F_{nN+k}(z)|. \end{aligned}$$

The result now follows from the reverse triangle inequality. □

Step 3: We are now ready to complete the proof of Theorem 3.5. With N as above,

$$\sum_{n\in\mathbb{Z}} \mathcal{C}_q[\mu_n] \leq 2^{q+1} \sum_{n\in\mathbb{Z}} \frac{\mu_n(T_n)}{2^{(n+1)q}} = 2^{q+1} \sum_{k=1}^{N} \sum_{n\in\mathbb{Z}} 2^{-(nN+k+1)q} \mu_n(T_{nN+k}).$$

According to the previous steps,

$$2^{-(nN+k+1)q} \lesssim |F_{nN+k}(z)|^q \lesssim |G_k(z)|^q,$$

whenever $z \in T_{nN+k}$. Hence,

$$2^{-(nN+k+1)q} \mu_n(T_{nN+k}) \lesssim \int_{T_{nN+k}} |G_k|^q \, d\mu,$$

and

$$\sum_{n\in\mathbb{Z}} \mathcal{C}_q[\mu_n] \lesssim \sum_{k=1}^{N} \sum_{n\in\mathbb{Z}} \int_{T_{nN+k}} |G_k|^q \, d\mu \leq \sum_{k=1}^{N} \int_{\mathbb{C}_+} |G_k|^q \, d\mu.$$

Since $\|g_k\|_{\mathrm{L}^\infty} = 1$, $\sum_{n\in\mathbb{Z}} \mathcal{C}_q[\mu_n] \lesssim \|\mathcal{L} \colon \mathrm{L}^\infty \to \mathrm{L}^q(\mathbb{C}_+, d\mu)\|^q$, with implied constants only depending on q. □

The sufficiency of (7) can be extended to the situation where L^p is replaced by certain Orlicz spaces L^Φ. For Orlicz spaces, we have the following variant of sufficiency of (7), generalizing the sufficiency part of Theorem 3.3.

Theorem 3.6. *Assume $q \geq 2$. Let Φ be a Young function of the form $\Phi(t) = \tilde{\Phi}(t^{q'})$, where $\tilde{\Phi}$ is another Young function. Then it holds that*

$$\|\mathcal{L} \colon \mathrm{L}^\Phi(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)\|^q \lesssim \sum_n \left(2^n \| \exp^{-q' 2^{n-1}} \|_{\mathrm{L}^{\tilde{\Phi}^c}}\right)^{q-1} \mathcal{C}_q[\mu_n]. \quad (8)$$

The above inequality remains true for $\mathrm{L}^\Phi = \mathrm{L}^\infty$, in which case $\mathrm{L}^{\tilde{\Phi}^c} = \mathrm{L}^1$.

Remark 3.7. *It is clear that if $\tilde{\Phi}$ is a Young function, then so is $\Phi\colon t \mapsto \tilde{\Phi}(t^{q'})$. The converse is not true. The present construction ensures that Φ not too "small" relative to $t \mapsto t^{q'}$.*

Proof. To prove (8), we need two main tools. The first is the classical Hausdorff–Young theorem: Given $1 \leq p \leq 2$, the Fourier transform is a bounded map from $\mathrm{L}^p(\mathbb{R})$ to $\mathrm{L}^{p'}(\mathbb{R})$. This readily implies boundedness of $\mathcal{L}\colon \mathrm{L}^p(0,\infty) \to \mathrm{H}^{p'}(\mathbb{C}_+)$. The second tool is the Carleson embedding theorem, e.g. [2, Theorem II.3.9], which states that $\|\mathrm{H}^q(\mathbb{C}_+) \hookrightarrow \mathrm{L}^q(\mathbb{C}_+, d\mu)\|^q$ is comparable to $\mathcal{C}_1[\mu]$.
Let $f\colon (0,\infty) \to \mathbb{C}$ be such that $F = \mathcal{L}f$ is well-defined as an analytic function on $\mathbb{C}_+$. The following calculations will yield that this is always the case when $f \in \mathrm{L}^\Phi$.
It holds that

$$\begin{aligned}\int_{\mathbb{C}_+} |F|^q \, d\mu &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{C}_+} |F|^q \, d\mu_n \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{C}_+} |F(z+2^{n-1})|^q \, d\tilde{\mu}_n(z) \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{C}_+} |\mathcal{L}(f \exp^{-2^{n-1}})|^q \, d\tilde{\mu}_n,\end{aligned}$$

where $\tilde{\mu}_n$ is the shifted measure $E \mapsto \mu_n(E+2^{n-1})$ appearing in Lemma 3.4. In combination with Carleson's theorem and the Hausdorff–Young theorem, we obtain

$$\begin{aligned}\int_{\mathbb{C}_+} |\mathcal{L}(f \exp^{-2^{n-1}})|^q \, d\tilde{\mu}_n &\lesssim \mathcal{C}_1[\tilde{\mu}_n] \|\mathcal{L}(f \exp^{-2^{n-1}})\|^q_{\mathrm{H}^q} \\ &\lesssim \mathcal{C}_1[\tilde{\mu}_n] \, \|f \exp^{-2^{n-1}}\|^q_{\mathrm{L}^{q'}} \\ &= \mathcal{C}_1[\tilde{\mu}_n] \, \||f|^{q'} \exp^{-q'2^{n-1}}\|^{q/q'}_{\mathrm{L}^1}\end{aligned}$$

Appealing to Lemma 3.4, $\mathcal{C}_1[\tilde{\mu}_n] \lesssim 2^{n(q-1)}\mathcal{C}_q[\mu_n]$. We now apply Hölder's inequality for Orlicz spaces to control

$$\left\| |f|^{q'} \exp^{-q'2^{n-1}} \right\|_{\mathrm{L}^1} \left\| |f|^{q'} \right\|_{\mathrm{L}^{\tilde{\Phi}}} \left\| \exp^{-q'2^{n-1}} \right\|_{\mathrm{L}^{\tilde{\Phi}^c}}.$$

By the dominated convergence theorem, $\|\exp^{-q'2^{n-1}}\|_{\mathrm{L}^{\tilde{\Phi}^c}} < \infty$ for any Young function $\tilde{\Phi}^c$. This shows in particular that $f \exp^{-2^{n-1}} \in \mathrm{L}^{q'}$, so $F(z) = \mathcal{L}f(z)$ is well-defined for $\operatorname{Re} z > 2^{n-1}$. As n is arbitrary, $F\colon \mathbb{C}_+ \to \mathbb{C}$ is well-defined and analytic. It also holds that $\||f|^{q'}\|^{1/q'}_{\mathrm{L}^{\tilde{\Phi}}} = \|f\|_{\mathrm{L}^\Phi}$. Piecing all of this together, we obtain (8). □

Proof of Theorem 3.3. By Theorem 3.5, $\mathcal{L}\colon \mathrm{L}^\infty \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ implies (7). To see that (7) is also sufficient, apply Theorem 3.6 to the case where $\mathrm{L}^\Phi = \mathrm{L}^\infty$, in which $\mathrm{L}^{\tilde{\Phi}^c} = \mathrm{L}^1$. □

In general, applying Theorem 3.6 with $\Phi(t) = t^p$, and computing the norms $\| \exp^{-q'2^{n-1}} \|_{L^{\tilde{\Phi}^c}} = \| \exp^{-q'2^{n-1}} \|_{L^{(p/q')'}}$, we obtain the following result, which we state for the sake of being explicit.

Proposition 3.8. *Let $q \geq 2$ and $p \geq q'$. With μ_n as in Theorem 3.3, it then holds that*

$$\|\mathcal{L}\colon L^p(0,\infty) \to L^q(\mathbb{C}_+, d\mu)\|^q \lesssim \sum_n 2^{nq/p} \mathcal{C}_q[\mu_n]. \tag{9}$$

For $p < \infty$, condition (9) is not necessary for $\mathcal{L}\colon L^p(0,\infty) \to L^q(\mathbb{C}_+, d\mu)$ to be bounded, as can be seen from [8, Theorem 3.5]. Next we will show that if μ has support on a vertical strip, then (9) reduces to (6). This is the content of Theorem 3.9, which is basically a reformulation of [8, Thm. 3.6], but allowing specifically for the case $p = \infty$.

Theorem 3.9. *Let μ be a positive regular Borel measure supported in a strip $\mathbb{C}_{\alpha_1,\alpha_2} = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \alpha_2\}$ for some $\alpha_2 \geq \alpha_1 > 0$, and let $1 \leq p' \leq q < \infty$ and $q \geq 2$. Then the following assertions are equivalent:*

(i) The embedding $\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu)$ is well-defined and bounded.

(ii) There exists a constant $C > 0$ such that

$$\mu(Q_I) \leq C|I|^{q/p'} \text{ for all intervals } I \subset i\mathbb{R}. \tag{10}$$

In this case, the bound in (i) depends only on C and α_2/α_1.

Proof. Condition (ii) is a reformulation of $\mathcal{C}_{q/p'}[\mu] < \infty$. The implication $(i) \implies (ii)$ was proved already in relation to (6). To obtain the reverse implication, assume instead that $\mathcal{C}_{q/p'}[\mu] < \infty$. Since μ is supported on a vertical strip, $\mu = \sum_{n=M}^{N} \mu_n$ for some integers M, N, with $N - M$ only depending on α_2/α_1. Hence,

$$\sum_n 2^{nq/p} \mathcal{C}_q[\mu_n] = \sum_{n=M}^{N} 2^{nq/p} \mathcal{C}_q[\mu_n].$$

By Lemma 3.4, $2^{nq/p}\mathcal{C}_q[\mu_n] \approx \mathcal{C}_{q/p'}[\mu_n]$. Moreover, it's clear that $\mathcal{C}_{q/p'}[\mu_n] \leq \mathcal{C}_{q/p'}[\mu]$. Thus, the above sum is finite, and (i) follows from Proposition 3.8. □

3.2 Laplace–Carleson embeddings on Orlicz spaces

In addition to Theorem 3.3, we derive the following consequence of Theorem 3.5 and Theorem 3.6.

Theorem 3.10. *Assume that $q \geq 2$, and that $\mathcal{L}\colon \mathrm{L}^\infty(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded. Then there exists a Young function $\Phi\colon [0,\infty) \to [0,\infty)$ for which $\mathcal{L}\colon \mathrm{L}^\Phi(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded.*

We need some further lemmata to prove this result.

Lemma 3.11. *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function with left-continuous derivative ϕ. For $\alpha, C > 0$ it then holds that*

$$\int_0^\infty \Phi\left(\frac{e^{-\alpha t}}{C}\right) dt = \frac{1}{\alpha C}\int_0^1 \phi\left(\frac{s}{C}\right)\log\left(\frac{1}{s}\right) ds.$$

Proof. Changing the order of integration,

$$\begin{aligned}\int_0^\infty \Phi\left(\frac{e^{-\alpha t}}{C}\right) dt &= \int_{t=0}^\infty \int_{s=0}^{\frac{e^{-\alpha t}}{C}} \phi(s)\, ds dt \\ &= \int_{s=0}^{1/C} \phi(s) \int_{t=0}^{\frac{1}{\alpha}\log\left(\frac{1}{Cs}\right)} dt ds \\ &= \frac{1}{\alpha}\int_{s=0}^{1/C} \phi(s)\log\left(\frac{1}{Cs}\right) ds.\end{aligned}$$

All that remains is the change of variables $Cs = s'$. □

Lemma 3.12. *Let $q' \geq 1$ and $(\gamma_n)_{n\in\mathbb{Z}}$ be a positive sequence such that $\gamma_n \geq 1$ for all $n \in \mathbb{Z}$, and $\gamma_n \to \infty$ as $|n| \to \infty$. Then there exists a Young function $\tilde{\Phi}^c$ such that*

$$2^n \| \exp^{-q' 2^{n-1}} \|_{\mathrm{L}^{\tilde{\Phi}^c}} \leq \gamma_n \qquad (n \in \mathbb{Z}).$$

Proof. Let $\phi^c\colon [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\phi^c(0) = 0$ and

$$\phi^c(2^n) \leq \frac{q'}{2}\frac{\gamma_n}{\int_0^1 \log\left(\frac{1}{s}\right) ds}$$

for all n. Such a function exists, since $\gamma_n \to \infty$ as $|n| \to \infty$. Define the Young function $\tilde{\Phi}^c\colon t \mapsto \int_0^t \phi^c(s)\, ds$. Using that each $\gamma_n \geq 1$, together with monotonicity,

$$\int_{s=0}^1 \phi^c\left(\frac{2^n s}{\gamma_n}\right)\log\left(\frac{1}{s}\right) ds \leq \phi^c\left(2^n\right)\int_{s=0}^1 \log\left(\frac{1}{s}\right) ds \leq \frac{q'}{2}\gamma_n.$$

By Lemma 3.11, the above left-hand side is equal to

$$\frac{q'}{2}\gamma_n \int_0^\infty \tilde{\Phi}^c\left(\frac{2^n e^{-q' 2^{n-1} t}}{\gamma_n}\right) dt,$$

i.e. $2^n \| \exp^{-q' 2^{n-1}} \|_{\mathrm{L}^{\tilde{\Phi}^c}} \leq \gamma_n$ by the definition of the Orlicz norm. □

Proof of Theorem 3.10. Since $\mathcal{L}\colon \mathrm{L}^\infty(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded, it holds that $\sum_n \mathcal{C}_q[\mu_n] < \infty$ by Theorem 3.5. There exists a positive sequence $(\gamma_n)_n$ such that $\gamma_n \to \infty$ sufficiently slowly as $|n| \to \infty$, and $\sum_n \gamma_n^{q-1}\mathcal{C}_q[\mu_n] < \infty$. It's no restriction to assume that $\gamma_n \geq 1$ for every n. Let $\tilde{\Phi}^c$ be as in Lemma 3.12, i.e.

$$2^n \| \exp^{-q' 2^{n-1}} \|_{\mathrm{L}^{\tilde{\Phi}^c}} \leq \gamma_n.$$

If $\Phi(t) = \tilde{\Phi}(t^{q'})$, then Theorem 3.6 implies that $\mathcal{L}\colon \mathrm{L}^\Phi(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded. □

3.3 Laplace–Carleson embeddings from $\mathrm{L}^\Phi(0,\tau_0)$

In this section we develop finite time analogues of the preceding results on Laplace–Carleson embeddings. More precisely, we consider Laplace transforms of functions supported on $(0,\tau_0)$ for some $\tau_0 > 0$. We begin with the case of $\mathrm{L}^\infty(0,\tau_0)$, and then progress to $\mathrm{L}^\Phi(0,\tau_0)$ for more general Young's functions Φ. We will find that the value of τ_0 is immaterial.

Theorem 3.13. *Let $q \geq 2$, and μ be a positive regular Borel measure supported on $\mathbb{C}_+$. Suppose that $\tau_0 \in [2^M, 2^{M+1}]$ for some integer M, and let μ^M denote the restriction of μ to the strip $\{0 \leq \operatorname{Re} z \leq 2^{-M}\}$. Then $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded if and only if*

$$\sum_{n=-M}^{\infty} \mathcal{C}_q[\mu_n] + \mathcal{C}_q[\mu^M] < \infty \tag{11}$$

with an associated equivalence of norms, where the equivalence constant depends only on q. Moreover, if $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded, then $\mathcal{L} : \mathrm{L}^\infty(0,\tau) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded whenever $\tau > 0$.

Proof. We start by noting that it is sufficient to consider $\tau_0 = 2^M$. Indeed, if $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded, then $\mathcal{L} : \mathrm{L}^\infty(0,2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded. The core of the proof is to prove that this is equivalent to (11). It is easy to see that if we replace M by $M+1$ in (11), then we obtain an equivalent condition. This in turn implies boundedness of $\mathcal{L} : \mathrm{L}^\infty(0,2^{M+1}) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$, and hence of $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$. This argument immediately implies that boundedness of $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ yields boundedness of $\mathcal{L} : \mathrm{L}^\infty(0,\tau) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ for all $\tau > 0$.
The proof that (11) is necessary is largely analogous to the proof of Theorem 3.5.
We fix $M \in \mathbb{Z}$. For $n \geq -M$, we define f_n, F_n, N as in the proof of Theorem 3.5. For $k = 0, \ldots, N-1$, define

$$g_k = \sum_{m\in\mathbb{Z}, mN+k \geq -M} f_{mN+k} \quad \text{and } G_k = \mathcal{L} g_k.$$

Note that $g_k \in \mathrm{L}^\infty(0, 2^M)$ for $k = 0, \dots, N-1$. As in the proof of Theorem 3.5, we obtain:
If $n \in \mathbb{Z}$, $nN + k \geq -M$, and $z \in T_{nN+k}$, then $|G_k(z)| \geq \frac{1}{2}|F_{nN+k}(z)| \geq c2^{-nN-k-1}$. This implies,

$$\sum_{n \geq -M} \mathcal{C}_q[\mu_n] \lesssim \sum_{k=1}^{N} \sum_{\substack{n\in\mathbb{Z};\\ nN+k\geq -M}} \int_{T_{nN+k}} |G_k|^q \, d\mu \leq \sum_{k=1}^{N} \int_{\mathbb{C}_+} |G_k|^q \, d\mu.$$

Assuming that $\mathcal{L}\colon \mathrm{L}^\infty(0, 2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded, the above left-hand side is finite.
We still have to check boundedness of the second term in Condition (11). For any interval I with $|I| = 2^{-M-1}$, let c be the center and define $f = \chi_{[0,2^M]}(t)e^{ict}$, $F = \mathcal{L}f$. Then

$$F(s) = \int_0^{2^M} e^{-(s-ic)t} dt = \int_0^{2^M} e^{-(\operatorname{Re} s)t} e^{i(c-\operatorname{Im} s)t} dt.$$

Note that $t \operatorname{Re} s \leq \frac{1}{2}$ and $|c - \operatorname{Im} s|t \leq \frac{1}{4}$ for $t \in [0, 2^M]$, $s \in Q_I$, thus

$$|F(s)| \gtrsim 2^{M+1} \text{ for } s \in Q_I,$$

and, using boundedness of $\mathcal{L} : \mathrm{L}^\infty(0, 2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$,

$$1 \gtrsim \int_{\mathbb{C}_+} |F(z)|^q \, d\mu(z) \geq \int_{Q_I} |F(s)|^q \, d\mu(s) \gtrsim \frac{\mu(Q_I)}{|I|^q}.$$

We now turn to sufficiency of (11). Boundedness of the embedding for the measure $\sum_{n=-M}^{\infty} \mu_n$ follows directly from Theorem 3.3. To finish the proof, it is sufficient to show that if μ is supported on $[0, 2^{-M}) \times \mathbb{R}$ and $\mu(Q_I) \lesssim |I|^q$ for all $|I| = 2^{-M+1}$, then $\mathcal{L} : \mathrm{L}^\infty(0, 2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded.
Note that $\mathcal{L} : \mathrm{L}^\infty(0, 2^M) \to \mathrm{H}^2(\mathbb{C}_{-2^{-M},+})$ is bounded with norm proportional to $2^{M/2}$, since the function $t \mapsto e^{-t \operatorname{Re} s} f(t)$ lies in $\mathrm{L}^2(0, 2^M)$ when $f \in \mathrm{L}^\infty(0, 2^M)$, with the corresponding norm estimate. Here $\mathrm{H}^2(\mathbb{C}_{-2^{-M},+})$ is the Hardy space on the larger half-plane $\{s : \operatorname{Re} s > -2^{-M}\}$.
We observe that for the norm of the embedding $\mathcal{E}$, we have

$$\|\mathcal{E}\|_{\mathrm{H}^2(\mathbb{C}_{-2^{-M},+}) \to \mathrm{L}^q(\mu)} = \|\mathcal{E}\|_{\mathrm{H}^2(\mathbb{C}_+) \to \mathrm{L}^q(\tilde{\mu}_{2^{-(M+1)}})},$$

where

$$\tilde{\mu}_{2^{-(M+1)}}(E) = \mu(E - 2^{-(M+1)}).$$

Now $\tilde{\mu}_{2^{-(M+1)}}$ is supported on the strip S_{-M-1}, and we may directly apply Theorem 3.9 in order to obtain that $\|\mathcal{E}\|_{\mathrm{H}^2(\mathbb{C}_{-2^{-M},+}) \to \mathrm{L}^q(\mu)} \lesssim 2^{-M/2}$. This finishes the proof. □

Theorem 3.14. *Assume $q \geq 2$, $\tau_0 > 0$, and that $\mathcal{L}\colon \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded. Then there exists a Young function $\Phi\colon [0,\infty) \to [0,\infty)$ for which $\mathcal{L}\colon \mathrm{L}^\Phi(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$ is bounded.*

Proof. By Theorem 3.13, we may assume that $\tau_0 = 2^M$ for some integer M. Let $\mu = \mu' + \mu^M$, where $\mu' = \sum_{n=-M}^\infty \mu_n$. Assuming boundedness of $\mathcal{L}\colon \mathrm{L}^\infty(0,2^M) \to \mathrm{L}^q(\mathbb{C}_+, d\mu)$, condition (11) together with Theorem 3.3 implies boundedness of

$$\mathcal{L}\colon \mathrm{L}^\infty(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, d\mu').$$

By Theorem 3.10, there exists a Young's function Φ such that $\mathcal{L}\colon \mathrm{L}^\Phi(0,\infty) \to \mathrm{L}^q(\mathbb{C}_+, \mu')$, and since $\mathrm{L}^\Phi(0,2^M) \hookrightarrow \mathrm{L}^\Phi(0,\infty)$ isometrically, $\mathcal{L}\colon \mathrm{L}^\Phi(0,2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu')$ is bounded.
The proof will be complete once we have established that $\mathcal{L}\colon \mathrm{L}^\Phi(0,2^M) \to \mathrm{L}^q(\mathbb{C}_+, \mu^M)$ is bounded. Note that the Young's function Φ obtained in the proof of Theorem 3.10 is of the form $\Phi(t) = \tilde{\Phi}(t^{q'})$ for some other Young's function $\tilde{\Phi}$. By Hölder's inequality for Orlicz spaces, it follows that $\mathrm{L}^\Phi(0,2^M) \hookrightarrow \mathrm{L}^{q'}(0,2^M)$. Repeating an argument from the proof of Theorem 3.13, $\mathcal{L}\colon \mathrm{L}^\Phi(0,2^M) \to \mathrm{H}^q(\mathbb{C}_{-2^{-M}})$, and $\mathrm{H}^q(\mathbb{C}_{-2^{-M}}) \hookrightarrow \mathrm{L}^q(\mathbb{C}_+, d\mu^M)$, again by condition (11). $\square$

Corollary 3.15. *Let $q \geq 2$, and μ be a positive regular Borel measure supported on $\mathbb{C}_+$. Suppose that $\mathcal{L} : \mathrm{L}^\infty(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+, \mu)$ is bounded for some $\tau_0 > 0$. Then*

$$\lim_{\tau \to 0} \|\mathcal{L}\|_{\mathrm{L}^\infty(0,\tau) \to \mathrm{L}^q(\mathbb{C}_+,\mu)} = 0.$$

In fact, with Φ as in Theorem 3.13, it holds that

$$\|\mathcal{L}\|_{\mathrm{L}^\infty(0,\tau) \to \mathrm{L}^q(\mathbb{C}_+,\mu)} \leq \|\mathcal{L}\|_{\mathrm{L}^\Phi(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+,\mu)} \|\chi_{[0,\tau]}\|_{\mathrm{L}^\Phi(0,\infty)}$$

whenever $\tau \in (0,\tau_0]$.

Proof. Let $f \in \mathrm{L}^\infty(0,\infty)$ have unit norm, and support on $(0,\tau)$. With Φ as in Theorem 3.13,

$$\|\mathcal{L}f\|_{\mathrm{L}^q(\mathbb{C}_+,\mu)} \leq \|\mathcal{L}\|_{\mathrm{L}^\Phi(0,\tau_0) \to \mathrm{L}^q(\mathbb{C}_+,\mu)} \|f\|_{\mathrm{L}^\Phi(0,\tau_0)}.$$

The desired estimate now follows from $\|f\|_{\mathrm{L}^\Phi(0,\tau_0)} \leq \|\chi_{[0,\tau]}\|_{\mathrm{L}^\Phi(0,\infty)}$. $\square$

3.4 A Laplace–Carleson embedding for a specific class of Orlicz spaces

In this section, we want to present applications of the theory developed above to some concrete Orlicz spaces.

In the following let

$$\Phi_{\exp}(t) = \exp t - t - 1 \qquad \text{and} \qquad \tilde{\Phi}_{\exp}(t) = \exp\sqrt{t} - \sqrt{t} - 1,$$

so that $\Phi(t) = \tilde{\Phi}(t^2)$. We will show that for this specific Young function the boundedness of the Laplace–Carleson embedding from $\mathrm{L}^\Phi(0,1)$ to $\mathrm{L}^2(\mathbb{C}_+,\mu)$ can be characterized in terms of the capacity, in an analogous way as in Theorem 3.3 for L^∞.

Theorem 3.16. *Let μ be a positive regular Borel measure on $\mathbb{C}_+$. Then $\mathcal{L} : \mathrm{L}^\Phi(0,1) \to \mathrm{L}^2(\mathbb{C}_+,\mu)$ is bounded, if and only if*

$$\sum_{n=1}^{\infty} n^2 \mathcal{C}_2[\mu_n] + \sup_{I \textit{ interval}, \ |I|=2} \mu(Q_I) < \infty \tag{12}$$

with an associated equivalence of norms.

Proof. To prove the necessity we will reuse some notation and quantities from the proof of Theorem 3.5. In particular, for each integer $n \geq 2$ we let T_n denote the right half of a Carleson square Q_{I_n} with side length 2^{n+1} and $\mathcal{C}_2[\mu_n] \leq 2^{3-2n}\mu_n(T_n)$. Moreover, the functions $f_m = \chi_{(2^{-m-1},2^{-m}]}(t)e^{ic_m t}$, with ic_m being the midpoint of I_m, are L^∞-normalized functions with disjoint supports such that $F_m = \mathcal{L}f_m$ is essentially localized to T_m: There exists $c, C > 0$ for which

$$z \in T_m \implies |F_m(z)| \geq c2^{-m} \quad \text{and} \quad |F_n(z)| \leq C2^{-m-|m-n|}. \tag{13}$$

For a given $\epsilon > 0$, we may choose N such that

$$\sum_{m=1}^{n-1} m2^{mN} \leq \epsilon n 2^{nN} \quad \text{and} \quad \sum_{m=n+1}^{\infty} m2^{-mN} \leq \epsilon n 2^{-nN}$$

uniformly in n. This can be seen by comparison with a Riemann integral. For such an N and $k \in \{0, \dots, N-1\}$, let

$$g_k = (\log 2) \sum_{m=0}^{\infty} m f_{k+mN}. \tag{14}$$

and write $G_k = \mathcal{L}g_k$. Note that

$$\int_0^1 \Phi(|g_k(t)|)dt \leq \int_0^1 e^{|g_k(t)|}dt = \sum_{m=0}^{\infty} 2^{-(k+mN+1)}2^m \leq 1,$$

whence $\|g_k\|_\Phi \leq 1$. Moreover, for $z \in T_{k+nN}$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}, m\geq 0, n\neq m} m|F_{k+mN}(z)| &\leq C \sum_{m\in\mathbb{Z}, m\geq 0, n\neq m} m2^{-(k+nN+|m-n|N)} \\
&= C\sum_{m=0}^{n-1} m2^{-(k+2nN-mN)} \\
&\quad + C\sum_{m=n+1}^{\infty} m2^{-(k+mN)} \\
&\leq C\epsilon n2^{1-k-nN} \\
&\leq \frac{2C\epsilon n}{c}|F_{k+nN}(z)|,
\end{aligned}$$

and hence

$$\begin{aligned}
|G_k(z)| &\geq (\log 2)\left(n|F_{k+nN}(z)| - \sum_{m\in\mathbb{Z}, m\geq 0, n\neq m} m|F_{k+mN}(z)|\right) \\
&\gtrsim n|F_{k+nN}(z)| \geq cn2^{-k-nN},
\end{aligned}$$

provided that ϵ is sufficiently small. A possible choice is $\epsilon = \frac{1}{8}\frac{c}{C}$, where c, C are the constants from (13).
Hence for $k \in \{0, \ldots, N-1\}$,

$$\sum_{n=1}^{\infty} n^2 \mathcal{C}_2[\mu_{k+nN}] \lesssim \sum_{n=1}^{\infty} n^2 2^{-2(k+nN)} \mu(T_{nN+k}) \lesssim_N \int_{\mathbb{C}_+} |G_k(z)|^2 d\mu.$$

Adding over $k = 0, \ldots, N-1$, we obtain the required norm bound of the first term in (12), with a constant only depending on N (therefore on ϵ, and hence only on c, C). To control the second term in (12), just consider $f = e^{itc_I}\chi_{(0,1)}$, where c_I is the midpoint of the interval I.

To prove the sufficiency of Condition (12), note first that boundedness of

$$\mathcal{L} : \mathrm{L}^\Phi(0,1) \to \mathrm{L}^2(\mathbb{C}_+, \mu^{-1})$$

follows immediate from the continuous embedding $\mathrm{L}^\Phi(0,1) \subset L^2(0,1)$, together with the Carleson Embedding Theorem for Paley-Wiener spaces, see e.g. [11]. Here, as in the notation of Theorem 3.13, μ^{-1} denotes the restriction of μ to the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z \leq 2\}$.
For the remaining part of the measure μ, one may use a straightforward adaptation of Theorem 3.6,

$$\|\mathcal{L}\colon \mathrm{L}^\Phi(0,1) \to \mathrm{L}^2(\mathbb{C}_+, d\mu)\|^2 \lesssim \sum_{n=1} 2^n \| \exp^{-2^n} \|_{\mathrm{L}^{\tilde{\Phi}^c}(0,1)} \mathcal{C}_2[\mu_n].$$

Thus, in order to conclude that $\sum_n n^2 \mathcal{C}_2[\mu_n] < \infty$, it suffices to establish the estimate

$$2^n \| \exp^{-2^n} \|_{\mathrm{L}^{\tilde{\Phi}^c}(0,1)} \lesssim n^2 \qquad \forall n \in \mathbb{N}.$$

We then need to show that for sufficiently large B, it holds that

$$\int_0^1 \tilde{\Phi}^c \left(\frac{2^n \exp(-2^n t)}{Bn^2} \right) dt \leq 1.$$

This is indeed possible but requires a somewhat arduous explicit computation. It suffices to do this for large n, since the above integral is always finite and converges to 0 as $B \to \infty$. It is straightforward to see that

$$\tilde{\phi}(t) := \tilde{\Phi}'(t) = \frac{\exp \sqrt{t} - 1}{2\sqrt{t}},$$

and by a comparison of power series,

$$\frac{\exp\left(\sqrt{t}/2\right)}{2} \leq \tilde{\phi}(t) \leq \frac{\exp\left(\sqrt{t}\right)}{2}.$$

It follows that $\tilde{\phi}^c$, the left-continuous inverse of $\tilde{\phi}$, vanishes on $[0, 1/2]$, and satisfies

$$(\log(2t))^2 \leq \tilde{\phi}^c(t) \leq 4 (\log(2t))^2$$

for $t > 1/2$. If one defines

$$\Psi(t) = \begin{cases} 0, & t \in [0, 1/2], \\ 4 \int_{1/2}^t (\log(2s))^2 \, ds, & t > 1/2, \end{cases}$$

then $\tilde{\Phi}^c(t) \leq \Psi(t)$. Assuming n is sufficiently large for $2^n \exp(-2^n)/Bn^2 < 1/2$, we apply Fubini's theorem to obtain

$$\begin{aligned}
\int_0^1 \tilde{\Phi}^c \left(\frac{2^n \exp(-2^n t)}{Bn^2} \right) dt &\leq \int_0^{\frac{1}{2^n} \log\left(\frac{2^{n+1}}{Bn^2}\right)} \Psi \left(\frac{2^n \exp(-2^n t)}{Bn^2} \right) dt \\
&= 4 \int_{t=0}^{\frac{1}{2^n} \log\left(\frac{2^{n+1}}{Bn^2}\right)} \int_{s=1/2}^{\frac{2^n \exp(-2^n t)}{Bn^2}} (\log(2s))^2 \, ds dt \\
&= 4 \int_{s=1/2}^{2^n/Bn^2} (\log(2s))^2 \int_{t=0}^{\frac{1}{2^n} \log\left(\frac{2^{n+1}}{Bn^2 s}\right)} dt ds \\
&= 4 \int_{s=1/2}^{2^n/Bn^2} (\log(2s))^2 \frac{1}{2^n} \log\left(\frac{2^{n+1}}{Bn^2 s}\right) ds.
\end{aligned}$$

Through a rather arduous calculation, one finds the limit of the above integral as $n \to \infty$ to be $4 (\log 2)^2 / B$. □

As an alternative to the concrete calculations in the proof above, we can take a slightly different path and observe that

Lemma 3.17. *Let $N \in \mathbb{Z}$, $N \geq 0$. Then $\mathcal{L} : \mathrm{L}^\Phi(0,1) \to H^2(\mathbb{C}_{+,2^N})$ is bounded with norm*

$$\|\mathcal{L}\|_{\mathrm{L}^\Phi(0,1) \ \to H^2(\mathbb{C}_{+,2^N})} \lesssim N \frac{1}{2^{N/2}}.$$

Proof. Let $\|f\|_{\mathrm{L}^\Phi} = 1$. Note that by the Paley-Wiener Theorem, it is enough to prove that

$$\|f \exp^{-2^N}\|_2 \lesssim N \frac{1}{2^{N/2}}. \tag{15}$$

Let $p, q > 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then by Hölder's inequality,

$$\|f \exp^{-2^N}\|_2 \lesssim \|f\|_p \| \exp^{-2^N}\|_q \lesssim p \frac{1}{2^{N/q}},$$

with constants independent of p, N, where we use the weak exponential integrability of f,

$$|\{t \in (0,1) : |f(t)| > \alpha\}| \lesssim e^{-\alpha},$$

and standard estimates of the Γ-function. Choosing $p = N$, we find in case $N \geq 2$

$$\|f e^{-2^N t}\|_2 \lesssim N \frac{1}{2^{N/2}}.$$

In case $N = 0, 1$, the estimate follows trivially from $\mathrm{L}^\Phi(0,1) \subset L^2(0,1)$. □

To finish the proof of Theorem 3.16, note that the embedding

$$H^2_{\mathbb{C}_{+,2^n}} \to L^2(\mu_{n+1})$$

has norm equivalent to $(C_2[\mu_{n+1}])^{1/2}$ by the classical Carleson Embedding Theorem, applied to the shifted half-plane $\mathbb{C}_{+,2^n}$.
The remainder follows now from the decomposition of $\mathbb{C}_+$ into the strips S_n, $n \geq 1$, together with the strip $\{z \in \mathbb{C}_+ : 0 \leq \operatorname{Re} z \leq 2\}$, and the inclusion $\mathrm{L}^\Phi \subset L^2(0,1)$:

$$\begin{aligned} \|\mathcal{L}f\|^2_{\mathrm{L}^2(\mathbb{C}_+,\mu)} &\leq \|\mathcal{L}f\|^2_{\mathrm{L}^2(S,\mu)} + \sum_{N \geq -1} \|\mathcal{L}f\|^2_{\mathrm{L}^2(\mathbb{C}_+,\mu_{N+1})} \\ &\lesssim \|f\|_2^2 + \sum_{N \in \mathbb{Z}} 2^N C_2[\mu_{N+1}] \|\mathcal{L}f\|^2_{H^2_{\mathbb{C}_{+,2^N}}} \\ &\lesssim \left(1 + \sum_{N \geq 0} N^2 C_2[\mu_N]\right) \|f\|_{\mathrm{L}^\Phi}. \end{aligned}$$

This proof extends without difficulty to the case of the Young function $\Phi_\alpha(t) = \exp(t^\alpha) - t^\alpha - 1$ on $[0,1]$, where $\alpha \geq 1$.

Using analogous estimates and choosing $p = N\alpha$ in the application of Hölder's inequality, we obtain the correct analogue of (15):

$$\|f \exp^{-2^N}\|_2 \lesssim N^{1/\alpha} \frac{1}{2^{N/2}} \text{ for } \|f\|_{\mathrm{L}^{\Phi_\alpha}} \leq 1. \tag{16}$$

The rest of the sufficiency proof follows as above. The proof of necessity again follows along the same lines, replacing the test function g_k in (14) by

$$g_k = (\log 2)^{1/\alpha} \sum_{m=0}^{\infty} m^{1/\alpha} f_{k+mN}. \tag{17}$$

Altogether, we obtain

Theorem 3.18. *Let μ be a positive regular Borel measure supported on $\mathbb{C}_+$ and let $\alpha > 1$. Then $\mathcal{L} : \mathrm{L}^{\Phi_\alpha}(0,1) \to \mathrm{L}^2(\mathbb{C}_+, \mu)$ is bounded, if and only if*

$$\sum_{n=1}^{\infty} n^{2/\alpha} \mathcal{C}_2[\mu_n] + \sup_{I \text{ interval}, \ |I|=2} \mu(Q_I), < \infty \tag{18}$$

with an associated equivalence of norms.

Remark 3.19. *An inspection of the proof above reveals that the implied constants can be chosen independent of α. Hence Theorem 3.3 may (in case $q = 2$) be obtained as a limiting case of Theorem 3.18, in the limit $\alpha \to \infty$.*

Acknowledgements

Part of this research was carried out within the project *MOTADA*, jointly funded by the University of Hamburg and Lund University. ER, SP and FS are grateful for this support. ER was also supported by the Knut and Alice Wallenberg foundation, KAW 2016.0442. SP also gratefully acknowledges support by VR grant 2015-05552.

References

[1] C. Bennett and R. Sharpley, *Interpolation of operators.* Academic Press, Inc., Boston, MA, 1988.

[2] J.B. Garnett, *Bounded Analytic Functions.* Graduate Texts in Mathematics 236, Springer 2007.

[3] B. Haak, *Kontrolltheorie in Banachräumen und quadratische Abschätzungen.* PhD thesis, Universität Karlsruhe, 2005.

[4] B. Haak and C. Le Merdy, α-admissibility of observation and control operators. *Houston J. Math.* 31 (2005), no. 4, 1153–1167.

[5] S. Hansen and G. Weiss. The operator Carleson measure criterion for admissibility of control operators for diagonal semigroups on l^2. *Systems Control Lett.*, 16 (1991), no. 3, 219–227.

[6] B. Jacob and J.R. Partington, Admissibility of control and observation operators for semigroups: a survey. *Current trends in operator theory and its applications*, 199–221, Oper. Theory Adv. Appl., 149, Birkhäuser, Basel, 2004.

[7] B. Jacob, R. Nabiullin, J.R. Partington, and F. Schwenninger, Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM J. Control Optim.*, 56 (2018), no. 2, 868–889.

[8] B. Jacob, J.R. Partington and S. Pott, On Laplace–Carleson embedding theorems, *J. Functional Analysis* 264 (2013), 738–814.

[9] B. Jacob, J.R. Partington and S. Pott, Applications of Laplace–Carleson embeddings to admissibility and controllability. *SIAM J. Control. Optim.* 52 (2014), 1299–1313.

[10] B. Jacob, F. L. Schwenninger, and H. Zwart, On continuity of solutions for parabolic control systems and input-to-state stability. *J. Differential Equations*, 266 (2019), no. 10, 6284–6306.

[11] J.R. Partington, S. Pott and R. Zawiski, Laplace-Carleson embeddings on model spaces and boundedness of truncated Hankel and Toeplitz operators. *Integral Equations Operator Theory* 92 (2020), no. 4, Paper No. 37, 15 pp.

[12] E. Rydhe, On Laplace–Carleson embeddings, and L^p-mapping properties of the Fourier transform. *Ark. Mat* 58 (2020), no. 2, 437–457.

[13] D. Salamon, Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.* 300 (1987), no. 2, 383–431.

[14] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

[15] G. Weiss. Admissible observation operators for linear semigroups, *Israel J. Math.*, 65 (1989), 17–43.

[16] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27 (1989), no. 3, 527–545.

[17] G. Weiss. Two conjectures on the admissibility of control operators. In *Estimation and control of distributed parameter systems (Vorau, 1990)*,

volume 100 of *Internat. Ser. Numer. Math.*, pages 367–378. Birkhäuser, Basel, 1991.

[18] H. Zwart, Sufficient conditions for admissibility. *Systems Control Lett.*, 54 (2005), 973–979.