Bergische Universität Wuppertal

Fakultät für Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 21/28

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September 10, 2021

http://www.imacm.uni-wuppertal.de

Mathematics of Control, Signals, and Systems manuscript No.
(will be inserted by the editor)

A Two-Dimensional Port-Hamiltonian Model for Coupled Heat Transfer

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the date of receipt and acceptance should be inserted later

Abstract The problem of conjugate heat transfer in gas turbine blades and their cooling ducts is studied by constructing a highly simplified mathematical model that focuses on the relevant coupling structures while aiming to reduce the unrelated complexity as much as possible. The Port-Hamiltonian formalism is then applied to the model and its subsystems, and the interconnections are examined. Finally, a simple spatial discretization is applied to the system to investigate the properties of the resulting finite-dimensional Port-Hamiltonian system and whether the order of coupling and discretization has any effect on the resulting semi-discrete system.

Keywords Port-Hamiltonian System · Conjugate Heat Transfer · Coupled System · Thermodynamics

Mathematics Subject Classification (2020) 80A19 · 93A99

1 Introduction

The role of gas turbines in the power grid will most likely change as the share of renewable energy sources continues to rise. Their short start-up times and high efficiency make them well suited as backup power plants, and leading manufacturers are working on new technologies to make them suitable for use in an energy storage system in which they would run on hydrogen or synthetic methane. This brings new requirements and places new demands on the design process. While there is an almost overwhelming amount of engineering research and development, surprisingly little has been done on the mathematics for gas turbines. A mathematical approach to

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turbine blade design is being attempted as part of the GivEn project [1], which aims to combine multiphysics simulations with multicriteria shape optimization.

One of the physical processes that must be considered to obtain useful results from shape optimization is heat transfer within the turbine blade. Since gas turbines operate at extreme temperatures for efficiency reasons – often close to or even above the nominal melting point of the alloy used for the turbine blades – measures must be taken to protect the blade from the 1200 °C to 1500 °C of the combustion gas surrounding it. One method is to insert small cooling channels into the turbine blade, filled with a continuous stream of relatively cool air, to cool the blade by convection cooling from the inside. The shape, arrangement, and wall structure of these channels are themselves the subject of extensive engineering research, as described, for example, in [10, 11]. Because the flow in these cooling channels is intentionally kept highly turbulent to optimize heat transfer, it is difficult to simulate the flow explicitly. While it is possible, it is usually too sensitive and costly to do so as part of a multiphysics simulation. Instead, in most cases, a parametric one-dimensional model is used. Although quite dated, [19, 16] gives a reasonable overview of the basics of such a one-dimensional model.

Combining these cooling channels with the heat transfer inside the turbine blade, we obtain the so-called *conjugate heat transfer (CHT)* problem, i.e., strong thermal interactions between solids and fluids. Although [12] focuses on the coupling with the hot fluid surrounding the blade and not with the internal cooling fluid, both problems belong to the same large group. Alternatively, [20] considers both the external and internal fluids, but places little emphasis on coupling.

In this paper, we present a highly simplified model of conjugate heat transfer involving the turbine blade and a cooling channel. While this model is too simplified to be useful for actual engineering purposes, it is intended to represent the coupling structure between the turbine blade and a cooling channel and to allow us to study this coupling without having to deal with other engineering difficulties that might cloud the results. We extend and improve on the work done in [15, 14], in which we had considered a one-dimensional model for heat conduction within the blade metal, which led to strange and undesirable behaviors and properties of the system. Instead, we will consider a two-dimensional heat equation and investigate whether this eliminates the problems of the one-dimensional model. We then formulate the model system as an infinite-dimensional *port-Hamiltonian system (PHS)* and apply a spatial discretization to obtain a finite-dimensional PHS. Port-Hamiltonian systems are closely related to the Hamiltonian formalism, which was originally developed in theoretical physics, and are therefore well suited for modeling physical systems. The formalism makes conservation laws, a property central to virtually all physical systems, explicit and allows the construction of new port-Hamiltonian systems by connecting two PHSs with a suitable coupling. It also allows time discretization schemes that preserve the conservation laws of the continuous system [18].

The present paper is organized as follows. In Section 2, we introduce and motivate the mathematical model of the coupled system we study. In Section 3, we present a port-Hamiltonian formulation for each of the subsystems and study the coupling structure of their connection to determine whether the connection of these two subsystems forms a PHS for the overall system. Section 4 will contain a spatial discretization of the PHS formulated in Section3. Here we will investigate whether the resulting semi-discrete systems form finite-dimensional port Hamiltonian systems and whether there is a difference between the coupling of the discretized systems and the discretization of the coupled system. Finally, we will summarize the results in Section 5, outline open questions, and make some concluding remarks.

2 The model system

Here we introduce the mathematical model of the coupled system under investigation. Let $\Omega_m = (0,1) \times (0,1) \subset \mathbb{R}^2$ denote the spatial domain of the blade metal. The heat equation on Ω_m is given by

$$\frac{\partial T}{\partial t}(x,y,t) = \frac{1}{c_m \rho_m} \operatorname{div}\left(\lambda \operatorname{grad} T(x,y,t)\right) \quad \text{for } (x,y) \in \Omega_m. \tag{1}$$

In Figure 1 we give a rough sketch of the model setting.

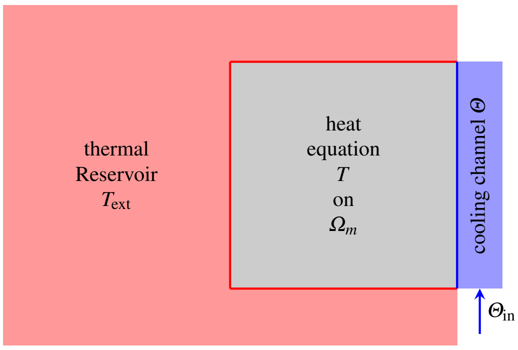

Fig. 1 Schematic of the 2D model system with $\partial\Omega_{ext}$ marked as a red line and $\partial\Omega_c$ as a blue line.

The left, upper and lower boundary ($x = 0$, $y = 0$ and $y = 1$) should be in contact with a thermal reservoir with a given temperature T_{ext}, leading to a Robin-BC:

$$-\lambda \frac{\partial T}{\partial x}(x,y,t) = h_0\left(T_{\text{ext}}(t) - T(x,y,t)\right) \quad \text{for } x = 0,\ y \in [0,1] \tag{2a}$$

$$-\lambda \frac{\partial T}{\partial y}(x,y,t) = h_0\left(T_{\text{ext}}(t) - T(x,y,t)\right) \quad \text{for } x \in (0,1),\ y = 0 \tag{2b}$$

$$\lambda \frac{\partial T}{\partial y}(x,y,t) = h_0\left(T_{\text{ext}}(t) - T(x,y,t)\right) \quad \text{for } x \in (0,1),\ y = 1 \tag{2c}$$

In the following, we denote these parts of the boundary with $\partial\Omega_{ext}$. The right boundary ($x = 1$), which is in contact with the cooling channel will be denoted by $\partial\Omega_c$, so that $\partial\Omega_m = \partial\Omega_{ext} \cup \partial\Omega_c$.

For $\partial\Omega_c$, we then have the boundary condition

$$-\lambda \frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big) \quad \text{for } (1,y) \in \partial\Omega, \tag{3}$$

where Θ denotes the temperature of the cooling channel and is governed by a transport equation with an additional source term describing the heat flux into the channel:

$$\frac{\partial \Theta}{\partial t}(y,t) = -v\frac{\partial \Theta}{\partial y}(y,t) + \frac{h_1}{c_c\rho_c}\big(T(1,y,t) - \Theta(y,t)\big), \tag{4}$$

$$\Theta(0,t) = \Theta_{\text{in}}(t). \tag{5}$$

3 Port-Hamiltonian Formulation

In this section, we formulate port-Hamiltonian systems for each of the two subsystems. To this end, we will use quadratic Hamiltonians (referred to as the *Lyapunov formulation* in [21]) rather than physical (thermodynamic) energy for two reasons. First, the resulting boundary ports of the heat equation will involve measurable quantities relevant in practice. Second, and more importantly, the transport equation causes problems with a non-quadratic Hamiltonian.

3.1 Heat Equation

For the heat equation in the metal rod we choose the Hamiltonian

$$H(t) = \frac{1}{2}\int_{\Omega_m} \rho(\vec{x})c_m(\vec{x})\,T(t,\vec{x})^2\,\mathrm{d}\vec{x} \tag{6}$$

with $T(t,\vec{x})$ the temperature and c_m denotes the *isochoric specific heat capacity* of the metal, i.e. the specific heat capacity at constant volume. We assume that c_m does not depend on the temperature, similar to the Dulong-Petit model. For further thermodynamic details, we refer the reader to [3]. These assumtions and choice of Hamiltonian are similar to those made in [21]. However, we choose to take the temperature T as our state variable, which allows us to eliminate the internal energy from occurring in our system.

We now choose the usual flow and effort variables

$$e_T = \delta_T H = T, \qquad f_T = \partial_t T, \tag{7}$$

with δ_T denoting the variational derivative w.r.t. the temperature T. Note that, just like in [21], ρc_m vanishes because we take the variational derivative with respect to the weighted $L^2_{\rho c_m}(\Omega)$ space, i.e. with measure $\rho(\vec{x})c_m(\vec{x})\,\mathrm{d}\vec{x}$.

With the above mentioned assumptions, the first law of thermodynamics gives us

$$\rho(\vec{x})c_m(\vec{x})\,\partial_t T(t,\vec{x}) = -\operatorname{div}\vec{\Phi}_Q(t,\vec{x}), \tag{8}$$

with the heat flux Φ_Q. From the (isotropic) Fourier's law, we have

$$\vec{\Phi}_Q(t,\vec{x}) = -\lambda \operatorname{grad} T(t,\vec{x}). \tag{9}$$

Note that an anisotropic thermal conductivity would also be possible, as in [21], but would not add anything interesting to the model while complicating the coupling formulation. Therefore we introduce the additional flow and effort variables similar to [21]

$$\vec{e}_Q = \vec{\Phi}_Q, \quad \vec{f}_Q = -\operatorname{grad} T, \tag{10}$$

to obtain the system of equations

$$\begin{pmatrix} \rho c_m f_T \\ \vec{f}_Q \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} e_T \\ \vec{e}_Q \end{pmatrix}, \tag{11}$$

$$\vec{e}_Q = \lambda \vec{f}_Q. \tag{12}$$

We can now calculate the time derivative of the Hamiltonian:

$$\begin{aligned} d_t H &= \int_{\Omega_m} \rho(\vec{x}) c_m(\vec{x}) \partial_t T(t,\vec{x}) T(t,\vec{x}) \,\mathrm{d}x \\ &= -\int_{\Omega_m} \operatorname{div}(\vec{\Phi}_Q) T \,\mathrm{d}x \\ &= \int_{\Omega_m} \vec{\Phi}_Q \operatorname{grad}(T) \,\mathrm{d}x - \int_{\partial\Omega_m} T \vec{\Phi}_Q \vec{n} \,\mathrm{d}\gamma \\ &= -\int_{\Omega_m} \vec{e}_Q \vec{f}_Q \,\mathrm{d}x - \int_{\partial\Omega_m} e_T (\vec{e}_Q \vec{n}) \,\mathrm{d}\gamma, \end{aligned} \tag{13}$$

recovering the same boundary port variables as [21], i.e. the temperature T and the heat flux *into* the system $-\vec{\Phi}_Q \cdot \vec{n}$.

Since our model system from section 2 does not prescribe the heat flux $\vec{\Phi}_Q$ directly, but has Robin boundary conditions instead, we need to modify this boundary port. We first split the single boundary port into two ports, one each for $\partial\Omega_{ext}$ and $\partial\Omega_c$. The latter part will then be coupled to the cooling channel later, while the port for $\partial\Omega_{ext}$ will stay an external port for the coupled system. To replicate the boundary conditions (2), we set

$$\vec{e}_Q \vec{n} = \vec{\Phi}_Q \vec{n} = h_0 (T - T_{\text{ext}}) \text{ on } \partial\Omega_{ext} \tag{14}$$

$$\vec{e}_Q \vec{n} = \vec{\Phi}_Q \vec{n} = h_1 (T - \Theta) \text{ on } \partial\Omega_c \tag{15}$$

so equation (13) becomes

$$\begin{aligned} d_t H &= -\int_{\Omega_m} \vec{e}_Q \vec{f}_Q \,\mathrm{d}x - \int_{\partial\Omega_m} e_T (\vec{e}_Q \vec{n}) \,\mathrm{d}\gamma \\ &= -\int_{\Omega_m} \vec{e}_Q \vec{f}_Q \,\mathrm{d}x \\ &\quad - \int_{\partial\Omega_{ext}} h_0 e_T^2 \,\mathrm{d}\gamma + \int_{\partial\Omega_{ext}} h_0 e_T T_{\text{ext}} \,\mathrm{d}\gamma \\ &\quad - \int_{\partial\Omega_c} h_1 e_T^2 \,\mathrm{d}\gamma + \int_{\partial\Omega_c} h_1 e_T \Theta \,\mathrm{d}\gamma, \end{aligned} \tag{16}$$

turning the boundary port of equation (13) into two new boundary ports and additional dissipative terms on the boundary. These new boundary ports have the internal (T) and external (T_{ext}) temperatures as the port variables for the external port, as well as T and Θ for the coupling port.

3.2 Cooling Channel

We remind the reader that we try to formulate the following equation as a PHS:

$$\frac{\partial \Theta}{\partial t}(y,t) = -v\frac{\partial \Theta}{\partial y}(y,t) + \frac{h_1}{c_c\rho_c}\big(T(1,y,t) - \Theta(y,t)\big), \tag{17}$$

with the temperature of the cooling fluid $\Theta(y,t)$ in the cooling channel and assuming that ρ_c and c_c are constant. We consider the Hamiltonian

$$H = \frac{1}{2}\int_0^1 \rho_c c_c \Theta^2(y,t)\,\mathrm{d}y. \tag{18}$$

By taking again the variational derivative w.r.t. the measure $\rho_c c_c\,\mathrm{d}y$, we have

$$e_\Theta = \delta_\Theta H = \Theta, \qquad f_\Theta = \frac{\partial \Theta}{\partial t}. \tag{19}$$

We choose

$$\begin{aligned}
&J = -v\rho_c c_c \frac{\partial}{\partial y}, && R = 0,\\
&B = 1, && P = 0,\\
&f_d = h_1\big(T(1,y,t) - \Theta(y,t)\big) = \vec{e}_x\vec{\Phi}(1,y,t), && e_d = \Theta,\\
&e_\partial = \frac{1}{\sqrt{2}}\big(\Theta(1,t) + \Theta(0,t)\big), && f_\partial = -\frac{1}{\sqrt{2}}v\rho_c c_c\big(\Theta(1,t) - \Theta(0,t)\big),
\end{aligned}$$

resulting in the system, cf. [17]

$$\rho_c c_c f_\Theta = (J-R)e_\Theta + (B-P)f_d, \tag{20a}$$

$$e_d = (B+P)^\top e_\Theta, \tag{20b}$$

$$e_\partial = \frac{1}{\sqrt{2}}\big(\Theta(1,t) + \Theta(0,t)\big), \tag{20c}$$

$$f_\partial = -\frac{1}{\sqrt{2}}v\rho_c c_c\big(\Theta(1,t) - \Theta(0,t)\big). \tag{20d}$$

This gives us

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_0^1 e_\Theta f_\Theta\,\mathrm{d}y = \int_0^1 \rho_c c_c \Theta \frac{\partial \Theta}{\partial t}\,\mathrm{d}y\\
&= \int_0^1 \Theta\Big(-v\rho_c c_c \frac{\partial \Theta}{\partial y} + f_d\Big)\,\mathrm{d}y\\
&= \underbrace{-v\rho_c c_c \int_0^1 \Theta \frac{\partial \Theta}{\partial y}\,\mathrm{d}y}_{= -\frac{v}{2}\rho_c c_c\left[\Theta^2\right]_{y=0}^{y=1}} + \int_0^1 \Theta f_d\,\mathrm{d}y.
\end{aligned} \tag{21}$$

For the system to be dissipative, we need

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &\le e_\partial f_\partial + \int_0^1 e_d f_d \,\mathrm{d}y \\ &= -\frac{v}{2}\rho_c c_c [\Theta^2]_{y=0}^{y=1} + \int_0^1 \Theta f_d \,\mathrm{d}y.\end{aligned} \tag{22}$$

A comparison with equation (21) shows that the equality holds. We therefore have the distributed output $e_d = \Theta(y,t)$ and the distributed input $f_d = h\big(T(1,y,t) - \Theta(y,t)\big)$.

For the boundary input, we set $u(t) = W_B \binom{f_\partial}{e_\partial}$, and for the output $w(t) = W_C \binom{f_\partial}{e_\partial}$. Then, according to [13, 25], we have a port-Hamiltonian system if $W_B \Sigma W_B^\top$ is positive semi-definite and $\binom{W_B}{W_C}$ has full rank. If we want to use the inlet temperature as the input and the outlet temperature as output, this setting results in

$$W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{v\rho_c c_c} & 1 \end{pmatrix}, \tag{23}$$

$$W_C = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{v\rho_c c_c} & 1 \end{pmatrix}, \tag{24}$$

which satisfies the above criteria.

3.3 Coupling

To recover a port-Hamiltonian formulation of the model system discussed in Section 2 by coupling the port-Hamiltonian systems discussed in Sections 3.1 and 3.2, we need the following equality:

$$-\lambda \frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big). \tag{25}$$

Considering the relevant inputs and outputs of the two systems, we find that

$$e_1 = T(1,y,t), \qquad f_1 = -\lambda \frac{\partial T}{\partial x}(1,y,t), \tag{26}$$

$$e_2 = h\big(T(1,y,t) - \Theta(y,t)\big), \qquad f_2 = T(1,y,t). \tag{27}$$

The standard ‘gyrative’ interconnection, cf. [7]

$$e_1 = f_2, \quad e_2 = -f_1, \tag{28}$$

is a Dirac structure, and obviously satisfies (25). Therefore, the combined system is again a port-Hamiltonian system.

4 Finite Difference Discretization

In this section, we present a finite difference discretization of the port-Hamiltonian systems from the previous section. A finite difference discretization is certainly not the only possible discretization and others, such as the *Partitioned Finite Element Method (PFEM)* presented in [6, 23, 4] and applied to the heat equation in [22, 9] and [5, Section 3.4], might give better results from a numerical point of view. However, the simplicity of a finite difference scheme makes it more suitable for our purposes, since it is easier to explicitly write down and compare the matrices of the discretized system. For an alternative approach to the heat equation based on finite difference we refer the reader to [24]. Here, variables considered on the spatial grid are indicated by an underscore, e.g. $\underline{x}$.

4.1 Heat Equation

We assume that ρ and c_m are constant. We consider a uniform grid in x-direction with $N+1$ equidistant discretization points $\underline{x}_0 = 0, \underline{x}_1 = \Delta x, \ldots, \underline{x}_N = 1$ and similarly for y with $M+1$ points, leading to the grid variable $\underline{y}$ and a space step Δy. We define $\underline{T} \in \mathbb{R}^{N\cdot M}$, such that $\underline{T}$ is defined on an offset grid, i.e. $\underline{T}_{i+jN} \approx T(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_j + \frac{\Delta y}{2}), \quad i = 0,\ldots,N-1,\ j = 0,\ldots,M-1$.

We discretize the Hamiltonian (6) w.r.t. space using the midpoint rule

$$\underline{H} = \frac{1}{2}\rho c_m \Delta x \Delta y \underline{T}^\top \underline{T}, \tag{29}$$

with the time derivative

$$\frac{\mathrm{d}\underline{H}}{\mathrm{d}t} = \rho c_m \Delta x \Delta y \underline{T}^\top \frac{\partial \underline{T}}{\partial t}, \tag{30}$$

giving us the internal energy change as flow variable and the temperature as effort variable:

$$\underline{f}^{(T)} = \rho c_m \Delta x \Delta y \frac{\partial \underline{T}}{\partial t}, \quad \underline{e}^{(T)} = \underline{T}. \tag{31}$$

Using central differences (with half step sizes) to discretize equation (9), we obtain the following approximation for the heat fluxes in the interior

$$\begin{aligned}
\Phi_x(\underline{x}_i, \underline{y}_j + \tfrac{\Delta y}{2}) &= -\lambda \frac{1}{\Delta x}\left(\underline{T}_{i+Nj} - \underline{T}_{i-1+Nj}\right) + \mathcal{O}(\Delta x^2), \\
\Phi_y(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_j) &= -\lambda \frac{1}{\Delta y}\left(\underline{T}_{i+Nj} - \underline{T}_{i+N(j-1)}\right) + \mathcal{O}(\Delta y^2).
\end{aligned}$$

On the boundary, we use one-sided difference quotients to approximate the heat fluxes, since this accuracy is sufficient according to Gustafsson [8]. Together with the boundary conditions, we then have for the left boundary

$$\begin{aligned}
\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) &= h\left(T_{ext}(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) - T(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2})\right) \\
\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) &= -\lambda \frac{2}{\Delta x}\left(T(\underline{x}_0 + \frac{\Delta x}{2}, \underline{y}_j + \tfrac{\Delta y}{2}) - T(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2})\right) + \mathcal{O}(\Delta x)
\end{aligned}$$

Solving both for $T(\underline{x}_0, \underline{y}_j + \frac{\Delta y}{2})$ and combining them, we find

$$\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) \approx \frac{2h\lambda}{2\lambda + h\Delta x}\big(T_{ext}(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) - \underline{T}_{0+Nj}\big).$$

Similarly, for the right, lower and upper boundary, we find

$$\begin{aligned}
\Phi_x(\underline{x}_N, \underline{y}_j + \tfrac{\Delta y}{2}) &\approx \frac{2h\lambda}{2\lambda + h\Delta x}\big(\underline{T}_{N-1+Nj} - \Theta(\underline{y}_j + \tfrac{\Delta y}{2})\big)\\
\Phi_y(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_0) &\approx \frac{2h\lambda}{2\lambda + h\Delta y}\big(T_{ext}(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_0) - T_i\big)\\
\Phi_y(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_M) &\approx \frac{2h\lambda}{2\lambda + h\Delta y}\big(T_{i+(M-1)N} - T_{ext}(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_M)\big)
\end{aligned}$$

Let $\underline{\Phi_x} \in \mathbb{R}^{(N+1)M}$ and $\underline{\Phi_y} \in \mathbb{R}^{N(M+1)}$ with

$$\begin{aligned}
\underline{\Phi_x}_{i+(N)j} &\approx \Phi_x(\underline{x}_i, \underline{y}_j + \tfrac{\Delta y}{2}) \qquad \text{for } i = 0,\dots,N \qquad \text{and } j = 0,\dots,M-1\\
\underline{\Phi_y}_{i+Nj} &\approx \Phi_y(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_j) \qquad \text{for } i = 0,\dots,N-1 \qquad \text{and } j = 0,\dots,M.
\end{aligned}$$

Using central differences again to discretize (8), we get

$$\rho c_m \Delta x \Delta y \frac{\partial \underline{T}_{i+Nj}}{\partial t} = \Delta y\big(\underline{\Phi_x}_{i+(N+1)j} - \underline{\Phi_x}_{i+1+(N+1)j}\big) + \Delta x\big(\underline{\Phi_y}_{i+Nj} - \underline{\Phi_y}_{i+N(j+1)}\big)$$

Let $J_{x,1} \in \mathbb{R}^{N\times(N+1)}$, $J_x \in \mathbb{R}^{(N*M)\times((N+1)*M)}$, $J_y \in \mathbb{R}^{(N*M)\times(N*(M+1))}$ and $I_{N\times N}$ the $N \times N$ unit matrix, with

$$J_{x,1} = \begin{pmatrix} 1 & -1 & & \\ & \ddots & -1 & \\ & & 1 & -1 \end{pmatrix}, \qquad J_x = \Delta y \begin{pmatrix} J_{x,1} & & \\ & \ddots & \\ & & J_{x,1} \end{pmatrix}, \tag{32}$$

$$J_y = \Delta x \begin{pmatrix} I_{N\times N} & -I_{N\times N} & & \\ & \ddots & \ddots & \\ & & I_{N\times N} & -I_{N\times N} \end{pmatrix}. \tag{33}$$

Also, let $R_x \in \mathbb{R}^{(N+1)M\times(N+1)M}$, $R_{x,1} \in \mathbb{R}^{(N+1)\times(N+1)}$ and $R_y \in \mathbb{R}^{N(M+1)\times N(M+1)}$ with

$$R_{x,1} = \Delta y \begin{pmatrix} \frac{2\lambda+h\Delta x}{2h\lambda} & & & & \\ & \frac{\Delta x}{\lambda} & & & \\ & & \ddots & & \\ & & & \frac{\Delta x}{\lambda} & \\ & & & & \frac{2\lambda+h\Delta x}{2h\lambda} \end{pmatrix}, \qquad R_x = \begin{pmatrix} R_{x,1} & & \\ & \ddots & \\ & & R_{x,1} \end{pmatrix}, \tag{34}$$

$$R_y = \Delta x \begin{pmatrix} \frac{2\lambda + h\Delta y}{2h\lambda} I_{N\times N} & & & & \\ & \frac{\Delta y}{\lambda} I_{N\times N} & & & \\ & & \ddots & & \\ & & & \frac{\Delta y}{\lambda} I_{N\times N} & \\ & & & & \frac{2\lambda + h\Delta y}{2h\lambda} I_{N\times N} \end{pmatrix}. \tag{35}$$

Finally let

$$b_{x,0} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^\top \in \mathbb{R}^{N+1}, \qquad b_{x,N} = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^\top \in \mathbb{R}^{N+1},$$

$$B_{x,0} = \Delta y \begin{pmatrix} b_{x,0} & & \\ & \ddots & \\ & & b_{x,0} \end{pmatrix} \in \mathbb{R}^{(N+1)M\times M}, \qquad B_{y,0} = \Delta x \begin{pmatrix} I_{N\times N} \\ 0 \end{pmatrix} \in \mathbb{R}^{N*(M+1)\times N},$$

$$B_{x,N} = -\Delta y \begin{pmatrix} b_{x,N} & & \\ & \ddots & \\ & & b_{x,N} \end{pmatrix} \in \mathbb{R}^{(N+1)M\times M}, \quad B_{y,M} = -\Delta x \begin{pmatrix} 0 \\ I_{N\times N} \end{pmatrix} \in \mathbb{R}^{N*(M+1)\times N}.$$

We can now recover the discretized version of (11) in the form of a port-Hamiltonian descriptor system (pHDAE) [2]:

$$\begin{pmatrix} \underline{f}^{(T)} \\ 0 \\ 0 \end{pmatrix} = (J - R) \begin{pmatrix} \underline{e}^{(T)} \\ \underline{\Phi}_x \\ \underline{\Phi}_y \end{pmatrix} + B \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ \Theta(\underline{y} + \frac{\Delta y}{2}) \end{pmatrix}, \tag{36}$$

$$\underline{w} = B^\top \begin{pmatrix} \underline{e} \\ \underline{\Phi}_x \\ \underline{\Phi}_y \end{pmatrix} \approx \begin{pmatrix} \Delta y\, \Phi_x(x_0, \underline{y} + \frac{\Delta y}{2}) \\ \Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_0) \\ -\Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_M) \\ -\Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \end{pmatrix}, \tag{37}$$

with

$$J = \begin{pmatrix} 0 & J_x & J_y \\ -J_x^\top & 0 & 0 \\ -J_y^\top & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_x & 0 \\ 0 & 0 & R_y \end{pmatrix}, \tag{38}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & B_{x,N} \\ 0 & B_{y,0} & B_{y,M} & 0 \end{pmatrix}. \tag{39}$$

4.2 Transport Equation

We choose the $M+1$ equidistant discretization points $\underline{y}_0 = 0$, $\underline{y}_1 = \Delta y, \dots, \underline{y}_M = 1$ with $\Delta y = \frac{1}{M}$. Then we set $\underline{\Theta} = (\Theta(\underline{y}_0 + \frac{\Delta y}{2}), \dots, \Theta(\underline{y}_{M-1} + \frac{\Delta y}{2}))^\top \in \mathbb{R}^M$, matching the discretization scheme of the heat equation above. Discretizing equation (18) with the midpoint rule results in the semi-discrete Hamiltonian

$$\underline{H} = \frac{1}{2}\Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i^2, \tag{40}$$

with the time derivative

$$\frac{\mathrm{d}\underline{H}}{\mathrm{d}t} = \Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i \frac{\partial \underline{\Theta}_i}{\partial t}, \tag{41}$$

allowing us to set the flow and effort variables

$$\underline{f}_i = \Delta y \rho_c c_c \frac{\partial \underline{\Theta}_i}{\partial t}, \qquad \underline{e}_i = \underline{\Theta}_i, \qquad \forall i = 0, \ldots, M-1. \tag{42}$$

Using an upwind discretization for the spatial derivative, i.e.

$$\frac{\partial \Theta}{\partial y}(\underline{y}_i) = \frac{\underline{\Theta}_i - \underline{\Theta}_{i-1}}{\Delta y} + \mathscr{O}(\Delta y), \tag{43}$$

we obtain the following discretized version of equations (20)

$$\underline{f} = (J - R)\underline{e} + (B - P)\underline{u}, \tag{44}$$

$$\underline{w} = (B + P)^\top \underline{e} + (S - N)\underline{u}, \tag{45}$$

with

$$J = \frac{1}{2} v \rho c_c \begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}, \qquad R = \frac{1}{2} v \rho c_c \begin{pmatrix} 2 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 2 \end{pmatrix}, \tag{46}$$

$$B = \begin{pmatrix} 1 & & 1 \\ & \ddots & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{pmatrix}, \qquad P = 0, \tag{47}$$

$$S = 0, \qquad N = 0, \tag{48}$$

$$\tag{49}$$

as well as the input $\underline{u}$ and output $\underline{w}$

$$\underline{u} = \begin{pmatrix} \Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ v \rho c_c \Theta_{in} \end{pmatrix}, \qquad \underline{w} = \begin{pmatrix} \underline{\Theta} \\ \underline{\Theta}_0 \end{pmatrix}. \tag{50}$$

Note that we could also use $\underline{\Theta}_{in}$ as input by moving the preceding factors into B, which would make those factors also appear in the output. If an output at the end of the cooling channel is desired, this requires an artificial feed-through between input and output, as in [14]. With matrices chosen as above, we easily find that

$$W = \begin{pmatrix} R & P \\ P^\top & S \end{pmatrix} \tag{51}$$

is positive semi-definite according to the Gershgorin circle theorem.

4.3 Coupling the Discretized Systems

We consider two finite-dimensional port-Hamiltonian (descriptor) systems of the form

$$Ef = (J-R)e + (B-P)u, \tag{52}$$
$$w = (B+P)^\top e + (S-F)u, \tag{53}$$

and Hamiltonian H. According to [18], the system resulting from an interconnection of these two systems is again a PHDAE, if there are matrices M and N, so that

$$Mu + Nw = 0, \tag{54}$$

with $u = \binom{u_1}{u_2}$ the combined inputs of both systems and $w = \binom{w_1}{w_2}$ their combined outputs. If $Mu + Nw = 0$ defines a Dirac structure for (w,u), the system can usually be made smaller through index reduction and row operations. The coupled system takes the form

$$\begin{pmatrix} Ef \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} J-R & B-P & 0 & 0 \\ -(B+P)^\top & S-F & I & -M^\top \\ 0 & -I & 0 & -N^\top \\ 0 & M & N & 0 \end{pmatrix} \begin{pmatrix} e \\ \hat{u} \\ \hat{w} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix} u, \tag{55}$$

$$w = \hat{w}, \tag{56}$$

with I the identity, $E = \operatorname{diag}(E_1, E_2)$, $J = \operatorname{diag}(J_1, J_2)$ and R, B, P, S, F similar. The form given here is equivalent to the one given in [18], but without requiring the (unspecified) permutation matrices present in their formulation.

Setting the heat equation to be the first system and the transport equation to be the second system, we can choose

$$M = \begin{pmatrix} 0 & 0 & 0 & I_{M\times M} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{M\times M} & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 & -I_{M\times M} & 0 \\ 0 & 0 & 0 & I_{M\times M} & 0 & 0 \end{pmatrix}, \tag{57}$$

which defines a Dirac structure for those parts of the input and output, that are involved in the interconnection. It should therefore be possible to shrink the system.

Writing down the vectors and matrices for the coupled system,

$$J = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & 0 & 0 & 0 \\ -J_y^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & J_\Theta \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R_x & 0 & 0 \\ 0 & 0 & R_y & 0 \\ 0 & 0 & 0 & R_\Theta \end{pmatrix}, \tag{58}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & B_{x,N} & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_\Theta \end{pmatrix}, \tag{59}$$

$$u = \hat{u} = \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ \underline{\Theta} \\ \Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ v\rho c_c \Theta_{in} \end{pmatrix}, \qquad w = \hat{w} = \begin{pmatrix} \Delta y\, \Phi_x(x_0, \underline{y} + \frac{\Delta y}{2}) \\ \Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_0) \\ -\Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_M) \\ -\Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ \underline{\Theta} \\ \underline{\Theta}_0 \end{pmatrix}, \tag{60}$$

we immediately see that $\underline{\Theta}$ and $\Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2})$ appear in both, input and output, and from the previous sections, we also know that they occur in e as well. We can therefore eliminate them from the input and output and move the relevant terms into J, resulting in the – significantly more compact – condensed system

$$\begin{pmatrix} f^{(T)} \\ 0 \\ 0 \\ f^{(\Theta)} \end{pmatrix} = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & -R_x & 0 & B_{x,N} \\ -J_y^\top & 0 & -R_y & 0 \\ 0 & -B_{x,N}^\top & 0 & J_\Theta - R_\Theta \end{pmatrix} \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix} \tag{61}$$

$$+ \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 \\ 0 & 0 & 0 & B_\Theta \end{pmatrix}}_{\widetilde{B}} \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ v\rho c_c \Theta_{in} \end{pmatrix}, \tag{62}$$

$$w = \widetilde{B}^\top \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix}, \tag{63}$$

with $B_\Theta = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^\top$.

4.4 Discretizing the coupled system

Coupling the two systems from Sections 3.1 and 3.2 results in a system with the Hamiltonian

$$H = \frac{1}{2} \int_\Omega \rho(\vec{x}) c_m(\vec{x})\, T(t, \vec{x})^2 \,\mathrm{d}\vec{x} + \frac{1}{2} \int_0^1 \rho_c c_c\, \Theta^2(y, t)\, \mathrm{d}y. \tag{64}$$

Discretizing T and Θ each with an appropriate midpoint rule as in subsections 4.1 and 4.2, results in

$$\underline{H} = \frac{1}{2}\rho c_m \Delta x \Delta y \underline{T}^\top \underline{T} + \frac{1}{2}\Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i^2. \tag{65}$$

Proceeding as in the previous sections, we then obtain the following system:

$$\begin{pmatrix} f^{(T)} \\ 0 \\ 0 \\ f^{(\Theta)} \end{pmatrix} = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & -R_x & 0 & B_{x,N} \\ -J_y^\top & 0 & -R_y & 0 \\ 0 & -B_{x,N}^\top & 0 & J_\Theta - R_\Theta \end{pmatrix} \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix} + B \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ v \rho c_c \Theta_{in} \end{pmatrix},$$

$$\underline{\widetilde{w}} = B^\top \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix},$$

with

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 \\ 0 & 0 & 0 & B_\Theta \end{pmatrix}, \qquad B_\Theta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{M \times 1},$$

all other matrices and vectors containing Θ as in Section 4.2 and the remaining quantities as in Section 4.1. As we can see, this is the same system we got by eliminating superfluous variables from the system in Section 4.3.

5 Conclusion and Outlook

In this work, we proposed a model consisting of two subsystems for a simplified conjugate heat transfer in a turbine blade. We were then able to show that each of these subsystems is a port-Hamiltonian system and their interconnection defines a Dirac structure. Therefore, the entire model is also a port-Hamiltonian system. While the one-dimensional model previously proposed in [15] had constraints on the physical parameters, this two-dimensional model no longer has constraints beyond those that are physically meaningful. However, the question of the existence and uniqueness of the solution is still open, since one of the subsystems involved has a spatial dimension larger than one.

In Section 4 it was then shown that the application of an appropriate but very simple spatial discretization leads to a finite-dimensional port-Hamiltonian system. The finite-dimensional port-Hamiltonian system resulting from the discretization of the subsystems and the coupling of the resulting finite-dimensional subsystems is equivalent to the system resulting from the coupling of the subsystems and the subsequent discretization of the complete coupled system (using the same discretization scheme). It might be worthwhile to investigate whether this is a peculiarity of the system and discretization method under consideration, or whether it is a general result that holds for all port-Hamiltonian systems.

While the intuitive choices for quadrature of the Hamiltonian and the spatial discretization worked quite well for this system, it remains an open question whether the same holds for the general case. It would also be interesting to investigate whether a particular choice of quadrature for the Hamiltonian uniquely determines the spatial discretization of the differential equations and vice versa.

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