Bergische Universität Wuppertal

Fakultät für Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 21/15

Michelle Muniz, Matthias Ehrhardt and Michael Günther

Correlation Matrices driven by Stochastic Isospectral Flows

May 19, 2021

http://www.imacm.uni-wuppertal.de

Correlation Matrices driven by Stochastic Isospectral Flows

Michelle Muniz, Matthias Ehrhardt, and Michael Günther

Abstract In many important areas of finance and risk management, time-dependent correlation matrices must be specified. We create valid correlation matrices by extending the idea of correlation flows based on isospectral flows. To incorporate the stochastic behavior of correlations, we adapt this approach by modeling the isospectral flow as a stochastic differential equation (SDE) instead of an ordinary differential equation (ODE).
The solution of this SDE lies on the manifold of symmetric and positive semi-definite matrices, so structure-preserving schemes are needed for its numerical approximation. We apply stochastic Lie group methods based on Runge-Kutta–Munthe-Kaas schemes for ODEs to guarantee that the numerical solution evolves on the correct manifold. We also present an application example to illustrate our methodology.

1 Introduction

In this paper, we construct time-dependent correlation matrices that approximate the true correlation using real market data, reflect the stochastic nature of correlations, and satisfy the following properties of a valid correlation matrix:

1. All diagonal elements of a correlation matrix are equal to one and absolute values of all non-diagonal elements are less than or equal to one.
2. Correlation matrices are real symmetric and positive semi-definite, i.e. all eigenvalues are non-negative.

To ensure these properties, we take up the idea presented in [6, 3]. The authors constructed *covariance flows*, i.e., covariance matrices based on the isospectral flux

Michelle Muniz · Matthias Ehrhardt · Michael Günther
Bergische Universität Wuppertal, Wuppertal, Germany
e-mail: {muniz,ehrhardt,guenther}@uni-wuppertal.de

$$\dot{P}_t = [Y_t, P_t], \quad t \geq 0, \tag{1}$$

where P_0 is a given valid covariance matrix, i.e. symmetric and positive semi-definite, Y_t is a skew-symmetric matrix, $Y_t \in \mathfrak{so}(n)$, and $[A,B] = AB - BA$ is the matrix commutator. The solution P_t is a differential curve on the manifold

$$\widehat{\mathrm{Sym}}(n) = \{P_t = Q_t P_0 Q_t^\top : Q_t \in \mathrm{SO}(n),\, P_0 \text{ positive semi-definite}\}, \tag{2}$$

where SO(n) denotes the space of orthogonal matrices with determinant +1. Note that the matrices in $\widehat{\mathrm{Sym}}(n)$ are similar to P_0.

The corresponding *correlation flow* is obtained by the transformation $R_t = \Sigma_t^{-1} P_t \Sigma_t^{-1}$ with $\Sigma_t = \big(\mathrm{diag}(P_t)\big)^{1/2}$.

Our goal is to extend this approach by incorporating the stochastic behavior of correlations. To this end, we formulate an isospectral flow based on (1) driven by a stochastic differential equation (SDE) rather than an ordinary differential equation (ODE). Since the solution of this SDE evolves on the manifold $\widehat{\mathrm{Sym}}(n)$, we need a method for its numerical approximation that preserves the geometric properties of the manifold. Therefore, we will present a structure-preserving Euler-Maruyama scheme based on Runge-Kutta- Munthe-Kaas (RKMK) schemes for ODEs on manifolds [5]. Further details on stochastic RKMK schemes can be found in [2, 4].

The remainder of the paper is organized as follows. In Sect. 2 we construct covariance flows based on an isospectral flow driven by a SDE. Since correlation matrices play an important role in finance and risk management we provide an application example of our methodology from the viewpoint of a risk manager using real market data in Sect. 3. A conclusion of our results is given in Sect. 4.

2 Covariance flows based on stochastic isospectral flows

The space of covariance matrices $\widehat{\mathrm{Sym}}(n)$ is a homogeneous manifold, i.e. there exists an element Q in a corresponding Lie group such that $\Lambda(Q, P_1) = P_2$ for two arbitrary elements P_1 and P_2 of the manifold. The considered Lie group regarding isospectral flows is the space of rotation matrices SO(n) and the map $\Lambda: \mathrm{SO}(n) \times \widehat{\mathrm{Sym}}(n) \to \widehat{\mathrm{Sym}}(n)$, called the *Lie group action*, can be chosen as

$$\Lambda(Q, P) = QPQ^\top, \tag{3}$$

see [5]. Corresponding to this Lie group action there exists a *Lie algebra action* $\lambda: \mathfrak{so}(n) \times \widehat{\mathrm{Sym}}(n) \to \widehat{\mathrm{Sym}}(n)$ given by

$$\lambda(\Omega, P) = \exp(\Omega) P \exp(-\Omega), \tag{4}$$

where the Lie algebra $\mathfrak{so}(n)$ is the tangent space at the identity I of the Lie group SO(n), i.e. $\mathfrak{so}(n) = T_I\mathrm{SO}(n)$, which is the space of skew-symmetric matrices.

The matrix exponential $\exp\colon \mathfrak{so}(n) \to \mathrm{SO}(n)$, $\Omega \mapsto \sum_{k\geq 0} \Omega^k/k!$ acts as a map from the Lie algebra to the Lie group and its derivative is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\exp(\Omega)\right)H = (\mathrm{d}\exp_\Omega(H))\exp(\Omega), \quad \mathrm{d}\exp_\Omega(H) = \sum_{k\geq 0}\frac{1}{(k+1)!}\,\mathrm{ad}_\Omega^k(H)\;, \tag{5}$$

see [1, p. 83]. By $\mathrm{ad}_\Omega(H) = [\Omega, H] = \Omega H - H\Omega$ we express the adjoint operator

$$\mathrm{ad}_\Omega^0(H) = H, \quad \mathrm{ad}_\Omega^k(H) = \left[\Omega, \mathrm{ad}_\Omega^{k-1}(H)\right] = \mathrm{ad}_\Omega\left(\mathrm{ad}_\Omega^{k-1}(H)\right), \quad k \geq 1\;.$$

Theorem 1. *Assume that* $\mathrm{d}\exp_\Omega(H)$ *in* (5) *is invertible and let* $\Omega_t \in \mathfrak{so}(n)$ *be driven by*

$$\mathrm{d}\Omega_t = A_t\,\mathrm{d}t + \sum_{i=1}^m \Gamma_{i,t}\,\mathrm{d}W_{i,t}\;, \quad \Omega_0 = 0\;. \tag{6}$$

Then $P_t = \exp(\Omega_t)P_0\exp(-\Omega_t)$ *obeying*

$$\mathrm{d}P_t = \left([Y_{0,t}, P_t] + \frac{1}{2}\sum_{i=1}^m \left[Y_{i,t}, [Y_{i,t}, P_t]\right]\right)\mathrm{d}t + \sum_{i=1}^m [Y_{i,t}, P_t]\,\mathrm{d}W_{i,t} \tag{7}$$

is an isospectral flow in $\widehat{\mathrm{Sym}}(n)$*, where* $Y_{i,t} \in \mathfrak{so}(n)$ *for* $i = 0, \ldots, m$*.*

The coefficients in (6) *are given by*

$$A_t = \mathrm{d}\exp_{\Omega_t}^{-1}\left(Y_{0,t} - \frac{1}{2}\sum_{i=1}^m C_{i,t}\right)\;, \qquad \Gamma_{i,t} = \mathrm{d}\exp_{\Omega_t}^{-1}(Y_{i,t})\;,$$

where

$$C_{i,t} = \left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathrm{d}\exp_{\Omega_t}(\Gamma_{i,t})\right)\Gamma_{i,t} = \sum_{k=0}^\infty\sum_{j=0}^\infty \frac{1}{(k+j+2)}\frac{(-1)^{j+1}}{k!(j+1)!}\,\mathrm{ad}_{\Omega_t}^k\left(\mathrm{ad}_{\Gamma_{i,t}}\left(\mathrm{ad}_{\Omega_t}^j(\Gamma_{i,t})\right)\right)\;.$$

The SDE (7) and the coefficients in (6) can be derived by applying Itô's lemma to $P_t = \exp(\Omega_t)P_0\exp(-\Omega_t)$ and assuming an additive perturbation by independent Wiener processes $W_{1,t}, \ldots, W_{m,t}$ to the ODE (1). Since $\exp(\Omega_t)P_0\exp(-\Omega_t)$ corresponds to the Lie algebra action (4) with $P \equiv P_0$, the solution P_t will evolve in $\widehat{\mathrm{Sym}}(n)$ by construction.

The expression $\mathrm{d}\exp_\Omega(H)$ in (5) is invertible if the eigenvalues of ad_Ω are different from $2\ell\pi i$ with $\ell \in \{\pm 1, \pm 2, \ldots\}$. The inverse converges for $\|\Omega\| < \pi$ and is given by

$$d\exp_\Omega^{-1}(H) = \sum_{k=0}^\infty \frac{B_k}{k!}\,\mathrm{ad}_\Omega^k(H)\;, \tag{8}$$

where B_k denotes the Bernoulli numbers (see Lemma III.4.2 (Baker, 1905) in [1]).

Note that the assumption of a SDE in the Lie algebra gives the benefit of applying actions in a linear space whereas applying linear actions to (7) on the manifold $\widehat{\mathrm{Sym}}(n)$ would result in a *drift-off*.

3 Simulation of correlation flows

We assume the following scenario: A risk manager retrieves from the middle office's reporting system the initial correlation matrix

$$R_0^{\text{hist}} = \begin{pmatrix} 1 & -0.0159 \\ -0.0159 & 1 \end{pmatrix}, \tag{9}$$

of the moving correlations between the S&P 500 index and the Euro/US-Dollar exchange rate on a daily basis computed with a window size of 30 days from January 3, 2005 to January 6, 2006 seen in Fig. 1. Furthermore, we assume that the risk manager is aware of the density function of the considered correlation as the path shown in Fig. 1 is only one of many possible realizations. Therefore, we estimate a density function from the historical data using kernel smoothing functions (see Fig. 2). Now, the risk manager's task is to create valid time-dependent correlation matrices that reflect the stochastic nature of correlations while trying to match the density function of the historical data.

Our proposed methodology for the risk manager is given by the following steps:

1. Compute a covariance matrix P_0 based on the historical correlation matrix R_0^{hist} and consider the covariance flow $P_t = \exp(\Omega_t) P_0 \exp(-\Omega_t)$ obeying (7) where the skew-symmetric matrices $Y_{0,t}, \ldots, Y_{m,t}$ are set such that they contain parameters as degrees of freedom.
2. Solve the SDE (6) in the Lie algebra numerically and define a solution of (7) according to the Lie algebra action (4). Transform the obtained covariance matrices to corresponding correlation matrices.
3. Estimate the density function from the so-obtained correlation flow and calibrate the involved parameters such that the density function of the correlation flow matches the density function of the historical correlation.

These steps are now specified for $n = 2$ and $m = 2$.

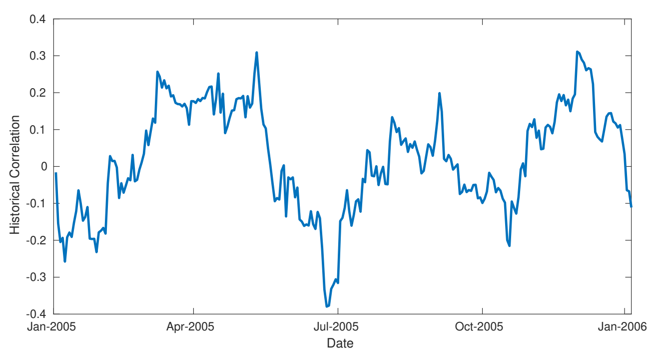

Fig. 1 The 30-day historical correlations between S&P 500 and Euro/US-Dollar exchange rate, source of data: www.yahoo.com.

Setting P_0 and $Y_{0,t}, Y_{1,t}, Y_{2,t}$

For the construction of P_0 we set D as the diagonal matrix containing the eigenvalues of the estimated covariance matrix of the whole historical data and we tried to find an orthogonal matrix H such that $P_0 = H^\top DH$ and $\|R_0 - R_0^{\text{hist}}\|_F \to \min$, where $R_0 = \Sigma_0^{-1} P_0 \Sigma_0^{-1}$ with $\Sigma_0 = \big(\text{diag}(P_0)\big)^{1/2}$ (see [3]). We report the so-found covariance matrix as

$$P_0 = \begin{pmatrix} 0.0233 & -0.0005 \\ -0.0005 & 0.0427 \end{pmatrix} . \tag{10}$$

Time-dependent, skew-symmetric matrices can be obtained by multiplying an arbitrary time-dependent function $g_i(t)$ with the generator G of $\mathfrak{so}(2)$. Experimenting with different functions we chose

$$g_0(t) = x_1 t \sin(x_2 t), \quad g_1(t) = x_3 + x_4 t, \quad g_2(t) = x_5 + x_6 t, \tag{11}$$

as they worked regarding the given historical data and we set $Y_i(t) = g_i(t)G$ for $i = 0,1,2$. The parameters $x_1, \ldots, x_6 \in \mathbb{R}$ can be associated with possible degrees of freedom.

Structure-preserving Euler-Maruyama scheme

We solve (7) with the initial value and coefficients specified in the previous step by applying the following algorithm which is based on RKMK schemes for ODEs [5].

Algorithm 2 (Structure-preserving Euler-Maruyama scheme) *Divide the time interval $[0,T]$ uniformly into J subintervals $[t_j, t_{j+1}]$, $j = 0,1,\ldots,J-1$ and define $\Delta = t_{j+1} - t_j$ and $\Delta W_i \sim \mathcal{N}(0,\Delta)$. Starting with $t_0 = 0$ and $\Omega_0 = 0$ these steps are repeated until $t_{j+1} = T$:*

1. *Let P_j be the approximation of P_t at time $t = t_j$.*
2. *Compute Ω_1 by applying the Euler-Maruyama scheme to the SDE* (6)*.*
3. *Define a numerical solution of* (7) *as $P_{j+1} = \exp(\Omega_1) P_j \exp(-\Omega_1)$.*

The computation of the correlation flow can be listed as an additional step:

4. Set $R_{j+1} = \Sigma_{j+1}^{-1} P_{j+1} \Sigma_{j+1}^{-1}$ with $\Sigma_{j+1} = \big(\text{diag}(P_{j+1})\big)^{1/2}$.

Calibration

We calibrate the parameters $x_1, \ldots, x_6$ in (11) such that the mean squared error, $\frac{1}{N}\sum_{j=1}^{N} \big(f^{\text{hist}}(z_j) - f^{\text{flow}}(z_j)\big)^2$ is minimized, where $f^{\text{hist}}(z)$ and $f^{\text{flow}}(z)$ are the empirical density function of the historical data and the correlation flow, resp., estimated with the MATLAB function `ksdensity` at $N = 100$ equally spaced points.

Choosing $(x_1, x_2, x_3, x_4, x_5, x_6) = (6.22, -5.22, 9.88, -5.19, -0.62, -16.63)$ we computed a mean squared error of $9.57 \cdot 10^{-4}$. A corresponding plot that shows how well the density function of our correlation flow approximates the historical data can be found in Fig. 2.

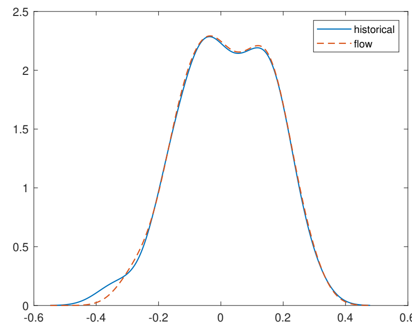

Fig. 2 Empirical density function of the historical correlation and the correlation flow between S&P 500 and Euro/US-Dollar exchange rate, computed with the MATLAB function `ksdensity`.

4 Conclusion

We have presented an approach that shows that the correlation model of [6] can be extended such that the stochastic behaviour of correlations is included by modelling the isospectral flow as a SDE instead of an ODE. Moreover, we introduced a structure-preserving scheme that keeps the numerical solution of this stochastic isospectral flow on the correct manifold $\widehat{\mathrm{Sym}}(n)$. Lastly, we have seen that our methodology for the approximation of correlation matrices based on the stochastic isospectral flow works quite well. In future work one could extend our model such that more correlations ($n > 2$) are approximated. For this purpose, one could adjust the number of diffusion coefficients $Y_{i,t}$ and the time-dependent functions $g_i(t)$ or apply higher order methods.

References

1. Hairer, E., Lubich, C., Wanner, G.: Geometric Numerical Integration. Springer (2006)
2. Malham, S.J.A., Wiese, A.: Stochastic Lie group Integrators. SIAM J. Sci. Comput. **30**(2), 597–617 (2008)
3. Muniz, M., Ehrhardt, M., Günther, M.: Approximating Correlation Matrices using Stochastic Lie Group Methods Mathematics **9**(1):94 (2021)
4. Muniz, M., Ehrhardt, M., Günther, M., Winkler, R.: Higher Strong Order Methods for Itô SDEs on Matrix Lie Groups. IMACM Preprint 21/03, January 2021.
5. Munthe-Kaas, H.: High order Runge-Kutta Methods on Manifolds. Appl. Numer. Math. **29**, 115–127 (1999)
6. Teng, L., Wu, X., Günther, M., Ehrhardt, M.: A new methodology to create valid time-dependent correlation matrices via isospectral flows. ESAIM: Math. Model. Numer. Anal. **54**(2), 361–371 (2020)