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Abstract

In this paper we present how nonlinear stochastic Itô differential equations arising in the modelling of perturbed rigid bodies can be solved numerically in such a way that the solution evolves on the correct manifold. To this end, we formulate an approach based on Runge-Kutta–Munthe-Kaas (RKMK) schemes for ordinary differential equations on manifolds.

Moreover, we provide a proof of the strong convergence of this stochastic version of the RKMK schemes applied to the rigid body problem and illustrate the effectiveness of our proposed schemes by demonstrating the structure preservation of the stochastic RKMK schemes in contrast to the stochastic Runge-Kutta methods.

Keywords: stochastic Runge-Kutta method, Runge-Kutta–Munthe-Kaas scheme, nonlinear Itô SDEs, rigid body problem
2000 MSC: 60H10, 70G65, 91G80

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1. Introduction

We consider the nonlinear Itô stochastic differential equation (SDE) of the form

$$dy_t = F_0(y_t)\,dt + \sum_{i=1}^{m} F_i(y_t)\,dW_t^i, \quad y_0 \in \mathcal{M}, \tag{1}$$

where the solution y_t, $t \geq 0$, evolves on a n-dimensional, homogeneous submanifold $\mathcal{M}$ of $\mathbb{R}^N$, $F_i\colon \mathcal{M} \to T\mathcal{M}$ for $i = 0, \ldots, m$ and $W_t^1, \ldots, W_t^m$ are independent Wiener processes. A solution can be defined via $y_t = \Lambda(\exp(\Omega_t), y_0)$, where $\Lambda\colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is a Lie group action on $\mathcal{M}$, i.e. for two elements $y_1, y_2 \in \mathcal{M}$ we can find a matrix G, an element of the Lie group $\mathcal{G}$, such that $\Lambda(G, y_1) = y_2$.

The variable Ω_t is an element of the corresponding Lie algebra $\mathfrak{g}$, which is the tangent space at the identity e of $\mathcal{G}$, i.e. $\mathfrak{g} = T\mathcal{G}|_e$. It satisfies

$$d\Omega_t = A_t\,dt + \sum_{i=1}^{m} \Gamma_t^{(i)}\,dW_t^i, \quad \Omega_0 = 0, \tag{2}$$

where the coefficients $A_t, \Gamma_t^{(i)} \in \mathfrak{g}$ depend on the coefficients of (1), $F_i\colon \mathcal{M} \to T\mathcal{M}$, $i = 0, \ldots, m$. We refer to [6] for more details on a general representation of these coefficients and these SDEs. A specific representation for the case $\mathcal{M} = S^2$ can be found in Section 2.

Our aim is to exploit the Euclidean-like geometry of the Lie algebra by applying *stochastic Runge-Kutta (sRK)* schemes to (2) and projecting the numerical solution back onto the manifold $\mathcal{M}$ to express an approximation of the solution of the SDE (1) since a direct application of sRK schemes to (1) would result in a *drift-off*. This approach is based on the *Runge-Kutta–Munthe-Kaas (RKMK)* schemes for ordinary differential equations (ODEs) on manifolds [11]. Their application to rigid body equations has been analyzed in [2].

Stochastic extensions of RKMK methods and their proof of convergence have already been considered in [6, 1, 12, 10]. The authors of [6] focus on the convergence of the *exponential Lie series*, while the authors of [1] consider only weak convergence. The proof of convergence in [12] applies only to the Euler-Maruyama scheme on matrix Lie groups and the proof of strong convergence in [10] is restricted to linear SDEs on matrix Lie groups which occur for example in the approximation of correlation matrices [9].

In this paper we extend the idea of Munthe-Kaas to SDEs on homogeneous manifolds and give a proof of the strong convergence of stochastic Runge-Kutta–Munthe-Kaas (sRKMK) schemes for nonlinear Itô SDEs of the form (1) occurring in the modelling of perturbed rigid bodies. We will show that the strong order of convergence γ depends on the order of convergence of the applied sRK method in the Lie algebra and the truncation index in the series representation of the drift and diffusion coefficients of (2).

The structure of the paper is as follows. In Section 2 we formulate based on the deterministic case the SDE that describes the motion of a rigid body that is perturbed by stochastic processes. Then, in Section 3 we present the schemes to solve this SDE numerically such that the numerical solution evolves on the correct manifold. The results of simulating the rigid body problem are provided in Section 4. At last, a conclusion of our findings and an outlook are given in Section 5.

2. The stochastic rigid body problem

Let $\mathcal{M}$ be the n-sphere $S^n = \{y \in \mathbb{R}^{n+1} : y^\top y = 1\}$. Then the Lie group action Λ, i.e. the transport across this manifold, can be described via the matrix-vector product $\Lambda(G, y) = Gy$ with a rotation matrix G in the Lie group $\mathcal{G} := \mathrm{SO}(n+1)$. The corresponding Lie algebra $\mathfrak{so}(n+1)$ is the space of skew-symmetric $(n+1) \times (n+1)$-matrices.

For $n = 2$ this example can be illustrated by the rigid body problem [8]. Consider a free rigid body, whose centre of mass is at the origin. Let the vector $y = (y_1, y_2, y_3)^\top$ represent the angular momentum in the body frame. The motion of this free rigid body is described by the Euler equations

$$\dot{y} = V(y)y, \quad V(y) = \begin{pmatrix} 0 & y_3/I_3 & -y_2/I_2 \\ -y_3/I_3 & 0 & y_1/I_1 \\ y_2/I_2 & -y_1/I_1 & 0 \end{pmatrix},$$

where I_1, I_2 and I_3 denote the principal moments of inertia.

We suppose that the rigid body is perturbed by Wiener processes, i.e. that the motion is driven by an Itô SDE of the form (1) with $\mathcal{M} = S^2$. The diffusion coefficients $F_i\colon S^2 \to TS^2$ are given by $F_i(y_t) = V_i(y_t)y_t$, where $V_i\colon S^2 \to \mathfrak{so}(3)$ are defined as above,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & y_3/I_{i3} & -y_2/I_{i2} \\ -y_3/I_{i3} & 0 & y_1/I_{i1} \\ y_2/I_{i2} & -y_1/I_{i1} & 0 \end{pmatrix}, \tag{3}$$

for constants I_{i1}, I_{i2}, I_{i3}, $i = 1, \dots, m$. For the drift coefficient we have

$$F_0(y_t) = V_0(y_t)y_t + \frac{1}{2}\sum_{i=1}^{m}\left(V_i^2(y_t) + \left(\frac{d}{d\Omega}d\exp_{\Omega}(\Gamma_t^{(i)})\right)\Gamma_t^{(i)}\right)y_t$$

with

$$\left(\frac{d}{d\Omega}d\exp_{\Omega}(\Gamma_t^{(i)})\right)\Gamma_t^{(i)} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(k+j+2)}\frac{(-1)^{j+1}}{k!(j+1)!}\,\mathrm{ad}_{\Omega}^{k}\Big(\mathrm{ad}_{\Gamma_t^{(i)}}\big(\mathrm{ad}_{\Omega}^{j}(\Gamma_t^{(i)})\big)\Big),$$

$\Gamma_t^{(i)}$ being the diffusion coefficient in (2) and $V_0(y_t) \in \mathfrak{so}(3)$ having the same structure as described in (3).

Summarizing these notations the SDE we are considering for the motion of a perturbed rigid body reads

$$dy_t = \left(V_0(y_t) + \frac{1}{2}\sum_{i=1}^{m}K_i\big(y_t, \Gamma_t^{(i)}\big)\right)y_t\,dt + \sum_{i=1}^{m}V_i(y_t)y_t\,dW_t^i, \tag{4}$$

where $y_0 \in S^2$ and

$$K_i(y_t, \Gamma_t^{(i)}) = V_i^2(y_t) + \left(\frac{d}{d\Omega}d\exp_{\Omega}(\Gamma_t^{(i)})\right)\Gamma_t^{(i)}.$$

Note that this *stochastic rigid body* problem has been considered before in [6, 15] but modelled as the *Stratonovich SDE*

$$dy_t = V_0(y_t)y_t\,dt + \sum_{i=1}^{m}V_i(y_t)y_t \circ dW_t^i, \quad y_0 \in S^2, \tag{5}$$

with the corresponding SDE in the Lie algebra given by

$$d\Omega_t = A_t\,dt + \sum_{i=1}^{m}\Gamma_t^{(i)} \circ dW_t^i, \quad \Omega_0 = 0.$$

The coefficients of the Stratonovich and the Itô SDE (2) coincide and can be specified by

$$A_t = d\exp_{\Omega_t}^{-1}\big(V_0(y_t)\big), \quad \Gamma_t^{(i)} = d\exp_{\Omega_t}^{-1}\big(V_i(y_t)\big), \quad i = 1, \dots, m, \tag{6}$$

if the expression

$$d\exp_\Omega(H) = \sum_{k\geq 0} \frac{1}{(k+1)!}\,\mathrm{ad}_\Omega^k(H)$$

given in the derivative of the matrix exponential $\exp(\Omega) = \sum_{k\geq 0} \Omega^k/k!$,

$$\left(\frac{d}{d\Omega}\exp(\Omega)\right) H = d\exp_\Omega(H)\exp(\Omega),$$

is invertible.

By $\mathrm{ad}_\Omega(H) = [\Omega, H] = \Omega H - H\Omega$ we express the adjoint operator which is defined iteratively

$$\mathrm{ad}_\Omega^0(H) = H, \quad \mathrm{ad}_\Omega^k(H) = \left[\Omega, \mathrm{ad}_\Omega^{k-1}(H)\right] = \mathrm{ad}_\Omega\big(\mathrm{ad}_\Omega^{k-1}(H)\big), \quad k \geq 1.$$

According to the classical Lemma of Baker (1905, see e.g. [3, p. 84]) $d\exp_\Omega(H)$ is invertible, if the eigenvalues of ad_Ω are different from $2\ell\pi i$ with $\ell \in \{\pm 1, \pm 2, \dots\}$. Since B_k are the Bernoulli numbers the inverse reads

$$d\exp_\Omega^{-1}(H) = \sum_{k=0}^{\infty} \frac{B_k}{k!}\,\mathrm{ad}_\Omega^k(H), \tag{7}$$

which converges for $\|\Omega\| < \pi$.

The definition of these coefficients follows from the Itô formula and their derivation can be done by using right multiplication instead of left multiplication of the solution in the proof of Theorem 5 in [7].

3. Stochastic Runge-Kutta–Munthe-Kaas schemes

In the sequel we present a stochastic version of Runge-Kutta–Munthe-Kaas (RKMK) schemes for solving (4).

Algorithm 3.1. *Divide the time interval* $[0, T]$ *uniformly into* L *subintervals* $[t_\ell, t_{\ell+1}]$, $\ell = 0, 1, \dots, L-1$ *and define the time step* $\Delta = t_{\ell+1} - t_\ell$. *Starting with* $t_0 = 0$, $y(t_0) = y_0$ *and* $\Omega_0 = 0_{n\times n}$ *the following steps are repeated over successive intervals* $[t_\ell, t_{\ell+1}]$ *until* $t_{\ell+1} = T$.

1. ***Initialization step:*** *Let* y_ℓ *be the approximation of* y_t *at time* $t = t_\ell$.
2. ***Numerical method step:*** *Compute an approximation* $\Omega_1 \approx \Omega_\Delta$ *by applying a stochastic Runge-Kutta method to the matrix SDE* (2).

3. ***Projection step:*** *Set* $y_{\ell+1} = \exp(\Omega_1) y_\ell$.

Consider a truncated approximation for (7) denoted by

$$\operatorname{dexpinv}(\Omega, H, q) = \sum_{k=0}^{q} \frac{B_k}{k!} \operatorname{ad}_{\Omega}^{k}(H). \tag{8}$$

By adapting the notations of Rößler's explicit s-stage sRK scheme [13] with coefficients given in Table 1 we can specify the algorithm above for $m = 1$, see Algorithm 1.

c	A	B	
$\tilde{c}$	$\tilde{A}$	$\tilde{B}$	
	α	$\beta^{(1)}$	$\beta^{(2)}$
		$\beta^{(3)}$	$\beta^{(4)}$

Table 1: Butcher tableau

Algorithm 1 sRKMK

for $\ell = 0, 1, \ldots, L-1$ **do**
  for $i = 1, 2, \ldots, s$ **do**
    $\bar{\Omega}_i = \sum_{j=1}^{i-1} a_{ij} A(\bar{\Omega}_j)\Delta + \sum_{j=1}^{i-1} b_{ij} \Gamma(\tilde{\Omega}_j) \frac{I_{(1,0)}}{\Delta}$
    $\tilde{\Omega}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij} A(\bar{\Omega}_j)\Delta + \sum_{j=1}^{i-1} \tilde{b}_{ij} \Gamma(\tilde{\Omega}_j) \sqrt{\Delta}$
    $A(\bar{\Omega}_i) = \operatorname{dexpinv}\left(\bar{\Omega}_i, V_0(\exp(\bar{\Omega}_i) y_\ell), q\right)$
    $\Gamma(\tilde{\Omega}_i) = \operatorname{dexpinv}\left(\tilde{\Omega}_i, V_1(\exp(\tilde{\Omega}_i) y_\ell), q\right)$
  end for
  $\Omega_1 = \sum_{i=1}^{s} \alpha_i A(\bar{\Omega}_i)\Delta + \sum_{i=1}^{s} \beta_i^{(1)} \Gamma(\tilde{\Omega}_i) I_{(1)} + \sum_{i=1}^{s} \beta_i^{(2)} \Gamma(\tilde{\Omega}_i) \frac{I_{(1,1)}}{\Delta} + \sum_{i=1}^{s} \beta_i^{(3)} \Gamma(\tilde{\Omega}_i) \frac{I_{(1,0)}}{\Delta} + \sum_{i=1}^{s} \beta_i^{(4)} \Gamma(\tilde{\Omega}_i) \frac{I_{(1,1,1)}}{\Delta}$
  $y_{\ell+1} = \exp(\Omega_1) y_\ell$
end for

The question now is how to choose the truncation index q in (8) so that the sRKMK procedure inherits the strong convergence order of the underlying sRK scheme.

Theorem 3.2. *Let the applied stochastic Runge-Kutta method in the second step of Algorithm 3.1 be of strong order γ. Assume that $V_i \colon S^2 \to \mathfrak{so}(3)$, $i = 0, \dots, m$, are given as in (3). If the truncation index q in (8) satisfies $q \geq 2\gamma - 2$, then the method of Algorithm 3.1 is of strong order γ.*

Proof. We define Ω_Δ^q as the exact solution of the truncated version of (2) at $t = \Delta$, namely

$$\Omega_\Delta^q = \int_0^\Delta \sum_{k=0}^{q} \frac{B_k}{k!} \operatorname{ad}_{\Omega_s}^k \big(V_0(y_s)\big) ds + \sum_{i=1}^{m} \int_0^\Delta \sum_{k=0}^{q} \frac{B_k}{k!} \operatorname{ad}_{\Omega_s}^k \big(V_i(y_s)\big) dW_s^i.$$

Then, the absolute error considered in the Frobenius norm can be split into a modelling and a numerical error,

$$\mathbb{E}[\|\Omega_\Delta - \Omega_1\|_F] \leq \left(\mathbb{E}\left[\|\Omega_\Delta - \Omega_\Delta^q\|_F^2\right]\right)^{1/2} + \left(\mathbb{E}\left[\|\Omega_\Delta^q - \Omega_1\|_F^2\right]\right)^{1/2}.$$

Since the numerical error has the correct order by construction it remains to be shown that

$$\left(\mathbb{E}\left[\|\Omega_\Delta - \Omega_\Delta^q\|^2\right]\right)^{1/2} \leq C\Delta^{(q+2)/2}, \quad C < \infty.$$

Analyzing the left hand side of the inequality above and using the Itô isometry, we get

$$\begin{aligned}
&\left(\mathbb{E}\left[\|\Omega_\Delta - \Omega_\Delta^q\|_F^2\right]\right)^{1/2} \\
&\leq \left(\mathbb{E}\left[\Big\| \int_0^\Delta \sum_{k=q+1}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega_s}^k \big(V_0(y_s)\big) ds \Big\|_F^2\right]\right)^{1/2} \\
&\qquad + \sum_{i=1}^{m} \left(\mathbb{E}\left[\Big\| \int_0^\Delta \sum_{k=q+1}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega_s}^k \big(V_i(y_s)\big) dW_s^i \Big\|_F^2\right]\right)^{1/2} \\
&\leq \sum_{i=0}^{m} \left(\int_0^\Delta \mathbb{E}\left[\Big\| \sum_{k=q+1}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega_s}^k \big(V_i(y_s)\big) \Big\|_F^2\right] ds \right)^{1/2} \\
&\leq \sum_{i=0}^{m} \left(\int_0^\Delta \mathbb{E}\left[\Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!} \big\| \operatorname{ad}_{\Omega_s}^k \big(V_i(y_s)\big) \big\|_F \Big)^2\right] ds \right)^{1/2}.
\end{aligned}$$

By using the submultiplicativity of the Frobenius norm, one can show that

$$\big\| \operatorname{ad}_{\Omega_s}^k \big(V_i(y_s)\big) \big\|_F \leq 2^k \big\|\Omega_s\big\|_F^k \big\|V_i(y_s)\big\|_F$$

holds via induction. Due to the specific structure of $V_i(y)$ (3) we have

$$\|V_i(y_s)\|_F^2 = 2\left(\left(\frac{y_{s,1}}{I_{i1}}\right)^2 + \left(\frac{y_{s,2}}{I_{i2}}\right)^2 + \left(\frac{y_{s,3}}{I_{i3}}\right)^2\right) \leq \frac{2}{I_{i,\min}^2}\|y_s\|_2^2,$$

where we define $I_{i,\min} = \min\{|I_{i1}|, |I_{i2}|, |I_{i3}|\}$. Moreover, it holds

$$\|y_s\|_2^2 = \|\exp(\Omega_s)y_0\|_2^2 \leq \|\exp(\Omega_s)\|_2^2\|y_0\|_2^2 \leq \left(\exp\left(\mu_2(\Omega_s)\right)\right)^2 = 1,$$

where we have used that $y_0 \in S^2$ and that the logarithmic matrix norm $\mu_2(\Omega_s) = \lambda_{\max}((\Omega_s + \Omega_s^\top)/2) = 0$ for skew-symmetric matrices Ω_s.

Inserting these results in the expected value we get

$$\mathbb{E}\left[\Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!}\left\| \operatorname{ad}_{\Omega_s}^k\left(V_i(y_s)\right)\right\|_F\Big)^2\right] \leq \frac{2}{I_{i,\min}^2}\mathbb{E}\left[\Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!}2^k\|\Omega_s\|_F^k\Big)^2\right].$$

Let $f\colon I \to \mathbb{R}$, $x \mapsto \frac{x}{2}\left(1 + \cot\left(\frac{x}{2}\right)\right) + 2$ with $I = \{x \in \mathbb{R} : \frac{x}{2\pi} \notin \mathbb{Z}\}$. Then it is true that

$$\sum_{k=0}^{\infty} \frac{|B_k|}{k!}x^k = f(x),$$

where $f(x)$ can be expressed by

$$f(x) = \sum_{k=0}^{q} \frac{f^{(k)}(0)}{k!}x^k + R_q(x), \quad R_q(x) = \frac{f^{(q+1)}(\xi)}{(q+1)!}x^{q+1},$$

if Taylor's theorem is applied to f at the point 0. In doing so, we are using the Lagrange form of the remainder for some real number ξ between 0 and x. Next, we set $x = 2\|\Omega_s\|$ and consider the restriction $f_{\tilde{I}}$ with $\tilde{I} = \{x \in \mathbb{R} : |x| < 2\pi\}$ since (7) only converges for $\|\Omega\| < \pi$. The restriction $f_{\tilde{I}}$ is bounded, in particular there exists an upper bound M_q such that $|\, f|_{\tilde{I}}^{(q+1)}(\xi)| \leq M_q$ for all ξ between 0 and x and therefore

$$|R_q(x)| \leq \frac{M_q}{(q+1)!}(2\|\Omega_s\|)^{q+1}.$$

This leads us to

$$\mathbb{E}\left[\Big(\sum_{k=q+1}^{\infty} \frac{|B_k|}{k!}(2\|\Omega_s\|_F)^k\Big)^2\right] \leq \left(\frac{2^{q+1}M_q}{(q+1)!}\right)^2 \mathbb{E}\left[\|\Omega_s\|_F^{2q+2}\right].$$

Lastly, we insert an Itô-Taylor expansion according to Proposition 5.9.1 [5],

$$\Omega_s = \Omega_0 + R_s = R_s, \quad \mathbb{E}\left[\|R_s\|_F^2\right] \leq C_1 s$$

for some $C_1 < \infty$ such that

$$\mathbb{E}\left[\|\Omega_s\|_F^{2(q+1)}\right] = \mathbb{E}\left[\|R_s\|_F^{2(q+1)}\right] \leq C_1 s^{q+1}.$$

Summing up, we have

$$\sum_{i=0}^{m}\left(\int_0^\Delta \mathbb{E}\Big[\Big(\sum_{k=q+1}^{\infty}\frac{|B_k|}{k!}\Big\|\operatorname{ad}_{\Omega_s}^k\Big(V_i\big(\exp(\Omega_s)y_0\big)\Big)\Big\|_F\Big)^2\Big]ds\right)^{1/2}$$
$$\leq \frac{2^{q+1}M_q}{(q+1)!}\sum_{i=0}^{m}\left(\frac{2C_1}{I_{i,\min}^2}\int_0^\Delta s^{q+1}ds\right)^{1/2} = \mathcal{O}\left(\Delta^{(q+2)/2}\right).$$

□

4. Simulation

For the simulation of our theoretic results above we have implemented Algorithm 3.1 to solve (4) in the software package MATLAB. We have set $m = 1$, $y_0 = (\cos(0.9), 0, \sin(0.9))^\top$ as the initial value in S^2 and the moments of inertia as $I_0 = (3, 1, 2)$ and $I_1 = (1, 0.5, 1.5)$, where $I_i := (I_{i1}, I_{i2}, I_{i3})$ for $i = 0, 1$.

In the Numerical method step of Algorithm 3.1 we have used the Euler-Maruyama scheme and the sRK methods SRI1 [14] of strong order $\gamma = 1$ and SRI1W1 of strong order $\gamma = 1.5$ [13]. Since applying these sRK schemes together with a Projection step in Algorithm 3.1 preserves the geometric properties of the manifold in contrast to applying them directly to (4) we use the abbreviations *gEM*, *gSRI1* and *gSRI1W1*, resp., to emphasise the geometric aspect. The truncation index q in (8) was chosen according to Theorem 3.2, namely $q = 0$ for gEM and for gSRI1 and $q = 1$ for gSRI1W1. A log-log-plot of the simulation of the strong convergence order can be viewed in Figure 1.

For the estimation of the absolute error we computed

$$\frac{1}{M}\sum_{j=1}^{M}\left\|y_{T,j}^{\text{ref}} - \hat{y}_{T,j}\right\|_2$$

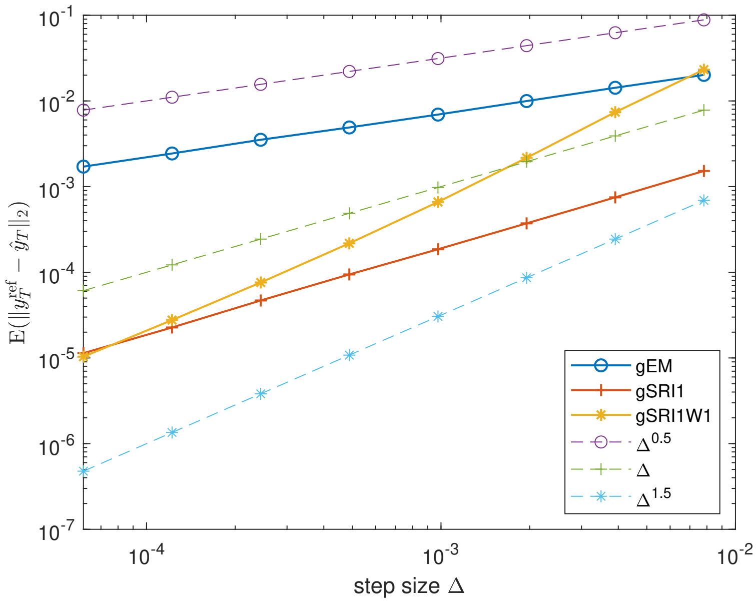

Figure 1: Simulation of the strong convergence order for $M = 1000$ paths.

where we have used the step sizes $\Delta = 2^{-14}, 2^{-13}, 2^{-12}, 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}$ to obtain the approximations $\hat{y}_T$. The reference solution y_T^{ref} was computed using gSRI1W1 with $\Delta = 2^{-16}$ and with the Cayley map $\mathrm{cay}(\Omega) = (I - \Omega)^{-1}(I + \Omega)$ instead of the matrix exponential in the Projection step of Algorithm 3.1 (see [3, 10]). As the Cayley map and the analogue expression to (7), namely

$$d\,\mathrm{cay}_{\Omega}^{-1}(H) = \frac{1}{2}(I - \Omega)H(I + \Omega),$$

are given by a finite product of matrices, there is no modelling error being made.

Note that we could also have used the closed-form expressions for the matrix exponential and (7) from [4, Appendix B] for the reference solution.

Figure 1 shows that the chosen truncation indices are sufficient for the sRKMK schemes to inherit the strong convergence order γ of the sRK scheme chosen in the second step of Algorithm 3.1.

The structure-preserving property of sRKMK schemes is visualised in Figure 2. It shows a sample path of gSRI1 of strong order $\gamma = 1$ applied to (4) with the same initial value y_0 and moments of inertia I_0 and I_1 as above

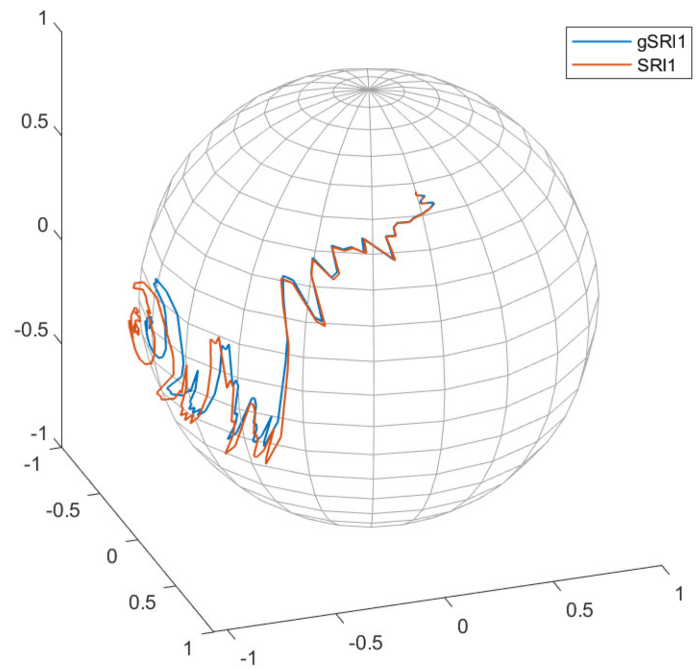

Figure 2: Sample path of the sRK method SRI1 and its structure-preserving counterpart gSRI1.

for 450 steps with a step size of $\Delta = 0.1$. In contrast to the sample path of SRI1 applied directly to (4), the sample path of gSRI1 remains on the manifold. The drift-off of the sRK scheme is also shown in Figure 3.

5. Conclusion

Since the analytical solution of the SDE considered in the stochastic rigid body problem lies on the unit sphere, a numerical approximation of the solution should also lie on the unit sphere. Based on the RKMK schemes for ODEs on manifolds, we have presented an extension to sRKMK schemes for nonlinear SDEs that arise in rigid body modelling under the assumption that there is a perturbation caused by stochastic processes.

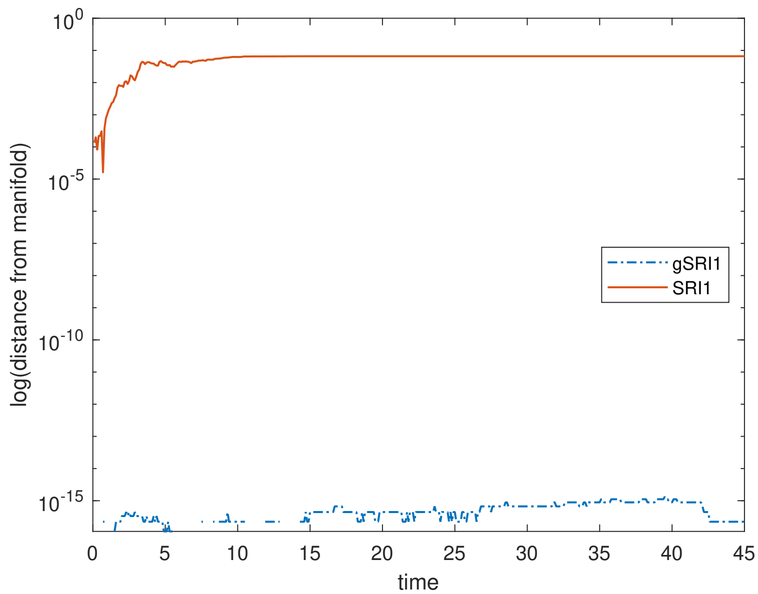

Figure 3: Log-distance of the numerical solution to the unit sphere

Moreover, we proved that the sRKMK schemes inherit the strong convergence order of the underlying sRK schemes when a condition on the truncation index q of (8) is satisfied.

Since the construction of sRKMK methods for nonlinear SDEs and their proof of convergence in this paper are limited to the modelling of perturbed rigid bodies, in future work we will generalise the results to SDEs on arbitrary homogeneous manifolds.

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