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Observability for port-Hamiltonian systems

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Abstract—The class of port-Hamiltonian systems incorporates many physical models, such as mechanical systems in the finite-dimensional case and wave and beam equations in the infinite-dimensional case. In this paper we study a subclass of linear first order port-Hamiltonian systems. In [3], it is shown that these systems are exactly observable when the energy is not dissipated internally and when sufficient observations are made at the boundary. In this article we study the observability properties for these systems when internal dissipation of energy is possible. We cannot show the exact observability, but we do show that the Hautus test is satisfied. In general, the Hautus test is weaker than exact observability, but stronger than approximate observability. Hence we conclude that these systems are approximately observable.

I. INTRODUCTION

In this paper we investigate the exact observability of port-Hamiltonian systems of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + G_0 \right) (\mathcal{H}(\zeta)x(\zeta, t)), \\ x(\zeta, 0) &= x_0(\zeta), \\ 0 &= \begin{bmatrix} W_{B,1} & W_{B,0} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \\ y(t) &= \begin{bmatrix} W_{C,1} & W_{C,0} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \end{aligned}$$

where $\zeta \in [0, 1]$ and $t \geq 0$. We refer to Section III for the precise assumptions on P_1 , G_0 , $\begin{bmatrix} W_{B,1} & W_{B,0} \end{bmatrix}$, $\begin{bmatrix} W_{C,1} & W_{C,0} \end{bmatrix}$ and \mathcal{H} .

This class of systems describes a special class of port-Hamiltonian systems, which however is rich enough to cover in particular the wave equation, the transport equation and the Timoshenko beam equation, and also coupled beam and wave equations. For more information on this class of port-Hamiltonian systems we refer to the first paper on this topic, [15], the monograph [11] and the survey [12].

Provided the port-Hamiltonian system is well-posed, that is, for every initial condition x_0 the port-Hamiltonian system possesses a unique (mild) solution, we aim to characterize *observability* and to study the related Hautus test. On one hand, exact observability implies the Hautus test and on the other hand, the Hautus test implies approximate observability, see Section II for the definitions of these notions.

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In [3] it is shown, that if additionally to the general assumptions of Section III the matrix G_0 is skew-adjoint and the norm of the solution x is non-increasing in time, then the port-Hamiltonian system is exactly observable. Actually, in [3] the dual statement concerning controllability is shown. In this article, we provide a simple proof if the port-Hamiltonian system is strongly stable and G_0 is skew-adjoint. Moreover, we show that the Hautus test is satisfied for exponentially stable port-Hamiltonian systems. Thus these systems are in particular approximately observable.

II. BACKGROUND ON EXACT OBSERVABILITY

In this section we assume that A is the generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on a Hilbert space X and C is a linear bounded operator from $\mathbf{D}(A)$, equipped with the graph norm, to another Hilbert space Y . We now consider the observation system

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \quad x(0) = x_0, \quad (1)$$

$$y(t) = Cx(t), \quad t \geq 0. \quad (2)$$

If the initial condition x_0 is an element of $\mathbf{D}(A)$, then the unique (classical) solution of (1)-(2) is given by

$$x(t) = T(t)x_0, \quad y(t) = CT(t)x_0, \quad t \geq 0,$$

thanks to the fact that $\mathbf{D}(A)$ is a $(T(t))_{t \geq 0}$ -invariant subspace of X . In order to define a (mild) solution of (1)-(2) for arbitrary initial conditions $x_0 \in X$ an extra condition is required.

Definition 2.1: The operator C is called an *admissible observation operator* for $(T(t))_{t \geq 0}$, if there is a time $t_0 > 0$ and a constant $m > 0$ such that

$$\int_0^{t_0} \|CT(t)x_0\|^2 dt \leq m \|x_0\|^2, \quad \text{for all } x_0 \in \mathbf{D}(A). \quad (3)$$

There is a rich literature on admissible observation operators, see for example [4]. If C is an admissible observation operator for $(T(t))_{t \geq 0}$, then (3) holds for every $t_0 > 0$, where the constant m may depend on t_0 . If the constant m can be chosen independent of t_0 , then C is called an *infinite-time admissible observation operator* for $(T(t))_{t \geq 0}$. If the C_0 -semigroup $(T(t))_{t \geq 0}$ is exponentially stable, then the notion of admissibility and infinite-time admissibility are equivalent. Moreover, if C is an admissible observation operator for $(T(t))_{t \geq 0}$ and $\alpha \in \mathbb{R}$, then C is also an admissible observation operator for $(e^{\alpha t}T(t))_{t \geq 0}$. Admissibility of the observation operator C guarantees that the mapping

$$\Psi_{t_0} : \mathbf{D}(A) \rightarrow L^2(0, t_0; Y), \quad x_0 \mapsto CT(\cdot)x_0$$

has a continuous extension to a linear bounded operator from X to $L^2(0, t_0; Y)$, again denoted by Ψ_{t_0} . Thus, in this case for every $x \in X$ the unique (mild) solution of (1)-(2) is given by

$$x(t) = T(t)x_0, \quad y = \Psi_{t_0}x_0, \quad t \in [0, t_0].$$

In this article we are in particular interested in observability, which we define next.

Definition 2.2: The observation system (1)-(2) is called *exact observable in finite time*, if there exists a time $t_0 > 0$ and a constant $\delta > 0$ such that

$$\int_0^{t_0} \|CT(t)x_0\|^2 dt \geq \delta \|x_0\|^2, \quad \text{for all } x_0 \in \mathbf{D}(A), \quad (4)$$

the observation system (1)-(2) is called *exact observable in infinite time*, if there exists a constant $\delta > 0$ such that

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq \delta \|x_0\|^2, \quad \text{for all } x_0 \in \mathbf{D}(A). \quad (5)$$

Further, the observation system (1)-(2) is called *approximate observable in infinite time*, if

$$\int_0^\infty \|CT(t)x_0\|^2 dt > 0, \quad \text{for all } x_0 \in \mathbf{D}(A), x_0 \neq 0.$$

If C is an admissible observation operator for $(T(t))_{t \geq 0}$, then the observation system (1)-(2) is exact observable in finite-time if and only if there is a $t_0 > 0$ such that the mapping Ψ_{t_0} has a linear bounded left-inverse. Further, if the C_0 -semigroup $(T(t))_{t \geq 0}$ is exponentially stable, then the notion of observability in finite time and infinite time are equivalent. We recall, that a C_0 -semigroup $(T(t))_{t \geq 0}$ is *exponentially stable*, if there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}, \quad t \geq 0.$$

A C_0 -semigroup $(T(t))_{t \geq 0}$ on a Hilbert space X is *strongly stable*, if

$$\lim_{t \rightarrow \infty} \|T(t)x\| = 0, \quad x \in X.$$

Next we provide an equivalent characterization of exact observability in term of Lyapunov (in)equalities.

Proposition 2.1: Let A be the generator of a strongly stable C_0 -semigroup $(T(t))_{t \geq 0}$ on a Hilbert space X . Then the following statements are equivalent

- 1) The observation system (1)–(2) is exact observable in infinite time.
- 2) There exists a linear bounded self-adjoint operator L on X satisfying

$$\langle x, Lx \rangle \geq \delta \|x\|^2 \quad (6)$$

for all $x \in X$ and some $\delta > 0$, and

$$\langle Ax, Lx \rangle + \langle Lx, Ax \rangle = -\|Cx\|^2 \quad (7)$$

for every $x \in \mathbf{D}(A)$.

- 3) There exists a linear bounded self-adjoint operator L on X satisfying (6) for all $x \in X$ and some $\delta > 0$, and

$$\langle Ax, Lx \rangle + \langle Lx, Ax \rangle \geq -\|Cx\|^2 \quad (8)$$

for every $x \in \mathbf{D}(A)$.

Inequality (8) is a Lyapunov inequality.

Proof: The implication 1) \Rightarrow 2) is well-know. For bounded operators $C \in L(X, Y)$ the proof can be found in [1, Exercise 6.10]. However, the general proof hardly differs. Furthermore, it is clear that 2) \Rightarrow 3). So we concentrate on the implication 3) \Rightarrow 1).

For $x_0 \in \mathbf{D}(A)$ we calculate

$$\begin{aligned} \frac{d}{dt} \langle T(t)x_0, LT(t)x_0 \rangle &= \langle AT(t)x_0, LT(t)x_0 \rangle + \\ &\quad \langle T(t)x_0, LAT(t)x_0 \rangle \\ &\geq -\|CT(t)x_0\|^2. \end{aligned}$$

In the last step we used (8) with $x = T(t)x_0$. Integrating both sides from $t = 0$ to $t = t_0$, gives

$$\langle T(t_0)x_0, LT(t_0)x_0 \rangle - \langle x_0, Lx_0 \rangle \geq -\int_0^{t_0} \|CT(t)x_0\|^2 dt.$$

Since the semigroup $(T(t))_{t \geq 0}$ is strongly stable and using (6), we conclude that

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq \delta \|x_0\|^2.$$

This concludes the proof. \blacksquare

In Russell and Weiss [14] it is shown that a necessary condition for exact observability of exponentially stable systems is the following version of the Hautus test:

There exists a constant $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathbf{D}(A)$:

$$\|(sI - A)x\|^2 + |\operatorname{Re} s| \|Cx\|^2 \geq m |\operatorname{Re} s|^2 \|x\|^2, \quad (\text{HT})$$

Here \mathbb{C}_- denotes the open left half plane. The Hautus test (HT) is sufficient for approximate observability of exponentially stable systems [14] and for polynomially stable systems [5]. Further, the Hautus test (HT) is sufficient for exact observability of strongly stable Riesz-spectral systems with finite-dimensional output spaces [6], for exponentially stable systems with A is bounded on H [14], and for exponentially stable systems if the constant m in (HT) equals one [2]. Further, the Hautus test (HT) is sufficient for exponentially stable systems with a normal C_0 -group [10]. However, in general the Hautus test (HT) is not sufficient for exponentially stable systems [9]. We refer the reader to Russell and Weiss [14], and Jacob and Zwart [6], [8] for more information on this Hautus test.

III. PROBLEM STATEMENT AND MAIN RESULTS

We consider the port-Hamiltonian system in the following form on a one-dimensional spatial domain of the form

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + G_0 \right) (\mathcal{H}(\zeta)x(\zeta, t)), \quad (9)$$

$$x(\zeta, 0) = x_0(\zeta), \quad (10)$$

$$0 = [W_{B,1} \quad W_{B,0}] \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad (11)$$

$$y(t) = [W_{C,1} \quad W_{C,0}] \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad (12)$$

where $\zeta \in [0, 1]$ and $t \geq 0$, the $n \times n$ Hermitian matrix P_1 is invertible, G_0 is a $n \times n$ matrix, $[W_{B,1} \ W_{B,0}]$ and $[W_{C,1} \ W_{C,0}]$ are full row rank $n \times 2n$ -matrices, and $\mathcal{H}(\zeta)$ is a positive $n \times n$ Hermitian matrix for almost all $\zeta \in [0, 1]$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C}^{n \times n})$. Thus, the matrix $P_1 \mathcal{H}(\zeta)$ can be diagonalized as $P_1 \mathcal{H}(\zeta) = S^{-1}(\zeta) \Delta(\zeta) S(\zeta)$, where $\Delta(\zeta)$ is a diagonal matrix and $S(\zeta)$ is an invertible matrix for almost all $\zeta \in [0, 1]$. We suppose the technical assumption that $S^{-1}, S, \Delta : [0, 1] \rightarrow \mathbb{C}^{n \times n}$ are continuously differentiable.

The function x denotes the state of the system, u the input function and y the output of the system. For more information we refer to [11], [12].

We equip the space $X := L^2(0, 1; \mathbb{C}^n)$ with the energy norm $\langle \cdot, \cdot \rangle_X := \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{H}}}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2(0, 1; \mathbb{C}^n)$. Note, that the energy norm is equivalent to the standard norm on $L^2(0, 1; \mathbb{C}^n)$.

Our standing assumption is that the operator $A : \mathbf{D}(A) \subset X \rightarrow X$ defined by

$$Ax := \left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x), \quad x \in \mathbf{D}(A), \quad (13)$$

$$\mathbf{D}(A) := \{x \in X \mid \mathcal{H}x \in H^1(0, 1; \mathbb{C}^n)\} \quad (14)$$

$$\text{and } [W_{B,1} \ W_{B,0}] \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0 \Big\},$$

generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X . This assumption guarantees that the port-Hamiltonian system (9)-(12) has for every initial condition $x_0 \in \mathbf{D}(A)$ a unique classical solution.

In [3] the dual version of the following proposition is shown.

Proposition 3.1: If A generates a contraction semigroup and $G_0^* = -G_0$, then the port-Hamiltonian system (9)-(12) is exactly observable in finite time, that is, there exist a time $t_0 > 0$ and constant $\delta > 0$ such that for every $x_0 \in \mathbf{D}(A)$ we have

$$\int_0^{t_0} \|y(t)\|^2 dt \geq \delta \|x_0\|^2.$$

Proof: In [12, Section 8] it is shown that the dual system is again a port-Hamiltonian system. As exact controllability is the dual notion of exact observability, the statement follows from the results in [3]. ■

Our first main result provides a direct proof for strongly stable systems.

Theorem 3.1: Suppose that the matrix

$$\mathcal{W}_{BC} := \begin{bmatrix} W_{B,1} & W_{B,0} \\ W_{C,1} & W_{C,0} \end{bmatrix} \quad (15)$$

is invertible, the C_0 -semigroup generated by A is strongly stable and $\text{Re } G_0 = 0$. Then the port-Hamiltonian system (9)-(12) is exactly observable in finite time, that is, there exist a time $t_0 > 0$ and constant $\delta > 0$ such that for every $x_0 \in \mathbf{D}(A)$ we have

$$\int_0^{t_0} \|y(t)\|^2 dt \geq \delta \|x_0\|^2. \quad (16)$$

The proof of the main result is given in Section IV. In applications, the assumption that the matrix \mathcal{W}_{BC} is invertible is not a restriction as we do not want to measure quantities that we already have set to zero.

The term $G_0 + G_0^*$ relates to the internal dissipation of energy, see also Lemma 4.1. So an interesting question is what happens when $\text{Re } G_0 := \frac{1}{2}(G_0 + G_0^*) \neq 0$. In the following finite-dimensional example we show that observability can be lost.

Example 3.1: We take as state space \mathbb{C}^2 and choose

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = [\sqrt{2} \ 0].$$

It is easy to see that

$$A + A^* = -C^* C$$

and so (C, A) is observable and A is Hurwitz. However, with

$$G_0 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

we have that $(C, A + G_0)$ is not observable, whereas $A + G_0$ is Hurwitz. □

Note that if in this example $\text{Re}(G_0)$ would be zero, then $(C, A + G_0)$ would be observable. So we see that in general observability can get lost when using general perturbations. However, for port-Hamiltonian system the Hautus test still holds.

Theorem 3.2: Suppose that the matrix \mathcal{W}_{BC} , see (15), is invertible and the C_0 -semigroup generated by A is exponentially stable. Further, we suppose that the C_0 -semigroup generated by $A - (\text{Re } G_0)\mathcal{H}$ is strongly stable. Then the port-Hamiltonian system (9)-(12) satisfies the Hautus test (HT), and thus the system is approximately observable.

The proof of Theorem 3.2 is given in Section V.

IV. PROOF OF THEOREM 3.1

This section is devoted to the proof of Theorem 3.1 and we denote by A the operator given by (13)-(14). Again we consider the port-Hamiltonian system (9)-(12). Recall that for every initial condition $x_0 \in \mathbf{D}(A)$ the port-Hamiltonian system (9)-(12) has a unique classical solution. It is useful to define the so called *boundary effort* and *boundary flow*, which are given by

$$e_\partial = \frac{1}{\sqrt{2}} ((\mathcal{H}x)(1) + (\mathcal{H}x)(0)),$$

$$f_\partial = \frac{1}{\sqrt{2}} (P_1(\mathcal{H}x)(1) - P_1(\mathcal{H}x)(0)),$$

respectively. We write this as a matrix vector product, i.e.,

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix}, \quad (17)$$

with $R_0 \in \mathbb{C}^{2n \times 2n}$ defined as

$$R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}. \quad (18)$$

Note that R_0 is invertible. Further, we define

$$P := \mathcal{W}_{BC}^{-*} R_0^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} R_0 \mathcal{W}_{BC}^{-1}.$$

Lemma 4.1: For $x_0 \in \mathbf{D}(A)$ and $t \geq 0$ the classical solution of (13)-(14) satisfies

$$\begin{aligned} & \langle Ax(\cdot, t), x(\cdot, t) \rangle_X + \langle x(\cdot, t), Ax(\cdot, t) \rangle_X \\ &= \frac{1}{2} \begin{bmatrix} 0 & y^*(t) \end{bmatrix} P \begin{bmatrix} 0 \\ y(t) \end{bmatrix} \\ &+ \operatorname{Re} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* (G_0 \mathcal{H}(\zeta)x(\zeta, t)) d\zeta. \end{aligned}$$

Proof: We calculate

$$\begin{aligned} & \langle Ax(\cdot, t), x(\cdot, t) \rangle_X + \langle x(\cdot, t), Ax(\cdot, t) \rangle_X \\ &= \frac{1}{2} \int_0^1 x(\zeta, t)^* \mathcal{H}(\zeta) \left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x)(\zeta, t) d\zeta \\ &+ \frac{1}{2} \int_0^1 \left(\left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x)(\zeta, t) \right)^* \mathcal{H}(\zeta)x(\zeta, t) d\zeta. \end{aligned}$$

Using the fact that P_1 is self-adjoint we get

$$\begin{aligned} & \langle Ax(\cdot, t), x(\cdot, t) \rangle_X + \langle x(\cdot, t), Ax(\cdot, t) \rangle_X \\ &= \frac{1}{2} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* \left(P_1 \frac{d}{d\zeta} (\mathcal{H}x)(\zeta, t) \right) d\zeta \\ &+ \frac{1}{2} \int_0^1 \left(\frac{d}{d\zeta} (\mathcal{H}x)(\zeta, t) \right)^* P_1 \mathcal{H}(\zeta)x(\zeta, t) d\zeta \\ &+ \frac{1}{2} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* G_0 \mathcal{H}(\zeta)x(\zeta, t) d\zeta \\ &+ \frac{1}{2} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* G_0^* \mathcal{H}(\zeta)x(\zeta, t) d\zeta \\ &= \frac{1}{2} \int_0^1 \frac{d}{d\zeta} ((\mathcal{H}x)^*(\zeta, t) P_1 (\mathcal{H}x)(\zeta, t)) d\zeta \\ &+ \operatorname{Re} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* (G_0 \mathcal{H}(\zeta)x(\zeta, t)) d\zeta \\ &= \frac{1}{2} ((\mathcal{H}x)^*(1, t) P_1 (\mathcal{H}x)(1, t) - (\mathcal{H}x)^*(0, t) P_1 (\mathcal{H}x)(0, t)) \\ &+ \operatorname{Re} \int_0^1 (\mathcal{H}(\zeta)x(\zeta, t))^* (G_0 \mathcal{H}(\zeta)x(\zeta, t)) d\zeta. \end{aligned}$$

Combining this equality with (17) and the definitions of R_0 and P , we obtain the statement of the lemma. ■

Proposition 4.1: For $x_0 \in \mathbf{D}(A)$ we have

$$\begin{aligned} & \langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle_X \\ & \geq -\|P\| \|Cx_0\|^2 \\ & + \operatorname{Re} \int_0^1 (\mathcal{H}(\zeta)x_0(\zeta))^* (G_0 \mathcal{H}(\zeta)x_0(\zeta)) d\zeta, \end{aligned}$$

where $Cx_0 := [W_{C,1} \quad W_{C,0}] \begin{bmatrix} (\mathcal{H}x_0)(1) \\ (\mathcal{H}x_0)(0) \end{bmatrix}$.

Proof: of Theorem 3.1: Follows directly from Proposition 4.1 together with Proposition 2.1. ■

V. PROOF OF THEOREM 3.2

We start with a general lemma.

Lemma 5.1: Consider the system (1)–(2) with A the infinitesimal generator of a strongly stable semigroup and assume that this system is exactly observable in finite time. Let G be a bounded operator. Then there exists a $m > 0$ and $\alpha > 0$ such that for all $\operatorname{Re}(s) < -\alpha$ and $x \in \mathbf{D}(A)$ there holds

$$\|(sI - A - G)x\|^2 + |\operatorname{Re}(s)| \|Cx\|^2 \geq m \operatorname{Re}(s)^2 \|x\|^2.$$

So we can interpret this as that the Hautus test holds on a smaller left half plane.

Proof: By Proposition 2.1 we know that there exists a self-adjoint bounded operator L such that $L \geq \delta I$ and (7) holds. As $L \geq \delta I$, $\langle \cdot, L \cdot \rangle$ defines a norm on X and so for $x \in \mathbf{D}(A)$

$$\begin{aligned} \langle (sI - A)x, L(sI - A)x \rangle &\leq 2\langle Gx, LGx \rangle + \quad (19) \\ &2\langle (sI - A - G)x, L(sI - A - G)x \rangle. \end{aligned}$$

Using the Lyapunov equation we have for $x \in \mathbf{D}(A)$ and $s = r + i\omega$ that

$$\begin{aligned} & \langle (sI - A)x, L(sI - A)x \rangle \\ &= \langle ((r + i\omega)I - A)x, L((r + i\omega)I - A)x \rangle \\ &= r^2 \langle x, Lx \rangle + \\ & \quad r \langle x, L(i\omega I - A)x \rangle + r \langle (i\omega I - A)x, Lx \rangle + \\ & \quad \langle (i\omega I - A)x, L(i\omega I - A)x \rangle \\ &= r^2 \langle x, Lx \rangle + r \|Cx\|^2 + \langle (i\omega I - A)x, L(i\omega I - A)x \rangle \\ &\geq r^2 \langle x, Lx \rangle + r \|Cx\|^2. \quad (20) \end{aligned}$$

So from (19) and (20) we find that

$$\begin{aligned} & 2\langle (sI - A - G)x, L(sI - A - G)x \rangle \\ & \geq \langle (sI - A)x, L(sI - A)x \rangle - 2\langle Gx, LGx \rangle \\ & \geq r^2 \langle x, Lx \rangle + r \|Cx\|^2 - 2\langle Gx, LGx \rangle \\ & \geq r^2 \delta \|x\|^2 - 2\|L\| \|G\|^2 \|x\|^2 + r \|Cx\|^2 \\ & \geq r^2 \frac{\delta}{2} \|x\|^2 + r \|Cx\|^2, \end{aligned}$$

where the last inequality for $r < -\sqrt{\frac{2\|L\|}{\delta}} \|G\|$. This implies the statement of the lemma. ■

Lemma 5.2: Let $s \in \mathbb{C}$ and P_1, G_0 and \mathcal{H} as in Section III. Then the solution of the system of ordinary differential equations

$$sx(\zeta) = \left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x)(\zeta) + f(\zeta), \quad \zeta \in [0, 1], \quad (21)$$

is given by

$$x(\zeta) = \Psi^s(\zeta, 0)x(0) + \int_0^\zeta \Psi^s(\zeta, \tau) \mathcal{H}^{-1}(\tau) P_1^{-1} f(\tau) d\tau,$$

where $\Psi^s(\zeta, \tau)$ satisfy for $v \in \mathbb{C}^n$, and $\zeta, \tau \in [0, 1]$

$$\begin{aligned} & \tilde{M} e^{-|\operatorname{Re}(s)| \tilde{c}_0(\zeta - \tau)} \|v\| \\ & \leq \|\Psi^s(\zeta, \tau)v\| \leq M e^{|\operatorname{Re}(s)| c_0(\zeta - \tau)} \|v\|, \end{aligned}$$

with constants $M, \tilde{M} > 0$, and $\tilde{c}_0, c_0 \geq 0$ independent of s , v , and ζ .

Proof: Equation (21) is a linear non-homogeneous ordinary differential equation in $\mathcal{H}x$ with non-homogeneity $\mathcal{H}^{-1}P_1^{-1}f$. This shows the representation of the solution. It remains to prove the estimate of $\Psi^s(\cdot, \cdot)$. Thus in the following we study the corresponding homogeneous ordinary differential equation

$$sx(\zeta) = \left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x)(\zeta), \quad \zeta \in [0, 1]. \quad (22)$$

Writing $\tilde{x} = \mathcal{H}x$, (22) can be equivalently written as

$$\mathcal{H}(\zeta)P_1\tilde{x}'(\zeta) = s\tilde{x}(\zeta) - \mathcal{H}(\zeta)G_0\tilde{x}(\zeta),$$

where the prime denotes the (spatial) derivative.

We write $s = i\omega + r$ with $\omega, \tau \in \mathbb{R}$ and diagonalize $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, see Section III. Thus, $\mathcal{H}(\zeta)P_1 = S^*(\zeta)\Delta(\zeta)S^{-*}(\zeta)$ and we get

$$\begin{aligned} \Delta(\zeta)S^{-*}(\zeta)\tilde{x}'(\zeta) &= i\omega S^{-*}(\zeta)\tilde{x}(\zeta) + \\ & (rI - S^{-*}(\zeta)\mathcal{H}(\zeta)G_0S^*(\zeta))S^{-*}(\zeta)\tilde{x}(\zeta). \end{aligned}$$

Using the substitution $z = S^{-*}\tilde{x}$, gives the equivalent differential equation

$$z'(\zeta) = i\omega\Delta^{-1}(\zeta)z(\zeta) + (r\Delta^{-1}(\zeta) + Q(\zeta))z(\zeta), \quad (23)$$

where

$$Q(\zeta) := -\Delta^{-1}(\zeta)S^{-*}(\zeta)\mathcal{H}(\zeta)G_0S^*(\zeta) + (S^{-*})'(\zeta)S^*(\zeta).$$

Thus, equation (22) is equivalent to equation (23). Due to the fact that $P_1\mathcal{H}(\zeta)$ has real eigenvalues, $\Delta(\zeta)$ is a diagonal, real matrix and $i\omega\Delta^{-1}(\zeta)$ is a diagonal, purely imaginary matrix. We write $\Delta^{-1}(\zeta) = \text{diag}_{k=1, \dots, n}(\alpha_k(\zeta))$ with $\alpha_k(\zeta) : [0, 1] \rightarrow \mathbb{R}$ and define $\Phi_\omega(\zeta) = \text{diag}(\exp(-i\omega \int_0^\zeta \alpha_k(\tau) d\tau))$ which satisfies

$$\|\Phi_\omega(\zeta)\|_{\mathcal{L}(\mathbb{C}^n)} = 1, \quad \zeta \in [0, 1].$$

Multiplying (23) with $\Phi_\omega(\zeta)$, we get

$$\begin{aligned} \Phi_\omega(\zeta)z'(\zeta) - i\omega\Delta^{-1}(\zeta)\Phi_\omega(\zeta)z(\zeta) &= \\ \Phi_\omega(\zeta)(r\Delta^{-1}(\zeta) + Q(\zeta))z(\zeta) \end{aligned}$$

or equivalently

$$\begin{aligned} (\Phi_\omega(\zeta)z(\zeta))' &= \\ = (r\Delta^{-1}(\zeta) + \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta))\Phi_\omega(\zeta)z(\zeta). \end{aligned}$$

Using the substitution $y = \Phi_\omega z$, this ordinary differential equation becomes

$$y'(\zeta) = (r\Delta^{-1}(\zeta) + Q_\omega(\zeta))y(\zeta), \quad (24)$$

where $Q_\omega(\zeta) := \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta)$.

There exist constants $c_0, c_1 \geq 0$, independent of ω , such that

$$2 \max_{\zeta \in [0, 1]} \|r\Delta^{-1}(\zeta) + Q_\omega(\zeta)\| \leq |r|c_0 + c_1. \quad (25)$$

The solution y of (24) satisfies

$$\begin{aligned} \frac{d}{d\zeta} \|y(\zeta)\|^2 &= y(\zeta)^* [(r\Delta^{-1}(\zeta) + Q_\omega(\zeta))y(\zeta)] + \\ & [(r\Delta^{-1}(\zeta) + Q_\omega(\zeta))y(\zeta)]^* y(\zeta). \end{aligned} \quad (26)$$

This together with (25) implies

$$-(|r|c_0 + c_1)\|y(\zeta)\|^2 \leq \frac{d}{d\zeta} \|y(\zeta)\|^2 \leq (|r|c_0 + c_1)\|y(\zeta)\|^2,$$

and thus

$$\begin{aligned} e^{-(|r|c_0 + c_1)(\zeta - \tau)} \|y(\tau)\|^2 \\ \leq \|y(\zeta)\|^2 \leq e^{(|r|c_0 + c_1)(\zeta - \tau)} \|y(\tau)\|^2. \end{aligned}$$

As the mapping $x \mapsto y$ is boundedly invertible on $L^2(0, 1, \mathbb{C}^n)$, with norm independent on ω , the statement follows. \blacksquare

We are now in the position to prove Theorem 3.2.

Proof: of Theorem 3.2: By Theorem 3.1 and Lemma 5.1 the Hautus Test (HT) is satisfied on some left half plane $\{s \in \mathbb{C} \mid \text{Re } s < \alpha\}$. Thus, if the Hautus test (HT) is not satisfied, then there would exist a sequence of complex numbers $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{C}^-$ with $\sup_n |\text{Re } s_n| < \infty$ and a sequence of elements in the domain of A , $\{x_n\}_{n \in \mathbb{N}}$, with norm one such that

$$\lim_{n \rightarrow \infty} \|(s_n I - A)x_n\|^2 + |\text{Re}(s_n)| \|Cx_n\|^2 = 0. \quad (27)$$

Since A generates an exponentially stable semigroup, the real part of s_n cannot converges to zero. We write $s_n = r_n + i\omega_n$, and define $f_n = (s_n I - A)x_n$. Using the definition of A , we see that $(s_n I - A)x_n = f_n$ can be rewritten as the ode

$$s_n x_n(\zeta) = \left(P_1 \frac{d}{d\zeta} + G_0 \right) (\mathcal{H}x_n)(\zeta) + f_n(\zeta). \quad (28)$$

By Lemma 5.2 we know that the solution of (28) is given by

$$\begin{aligned} x_n(\zeta) &= \Psi^{s_n}(\zeta, 0)x_n(0) + \\ & \int_0^\zeta \Psi^{s_n}(\zeta, \tau)\mathcal{H}^{-1}(\tau)P_1^{-1}f_n(\tau)d\tau. \end{aligned}$$

Furthermore, since the real parts of s_n are bounded, we find by the same lemma that $\Psi^{s_n}(\cdot, \cdot)$ are uniformly bounded on $[0, 1] \times [0, 1]$. Moreover, since r_n lies in an interval (α, δ) with $\delta < 0$ we obtain from (27) that $Cx_n \rightarrow 0$, and thus $x_n(0) \rightarrow 0$. By assumption $f_n \rightarrow 0$, so we conclude that $x_n \rightarrow 0$. However, this is in contradiction to the fact that x_n all have norm one. \blacksquare

VI. CONCLUSIONS AND REMARKS

In the previous section we have shown that port-Hamiltonian systems of the form (9)–(12) are exactly observable provided $G_0^* + G_0 = 0$, and satisfy the Hautus test for any bounded G_0 . Since the Hautus test implies approximate observability we have that this weaker form of observability does hold. Whether or not port-Hamiltonian systems of the form (9)–(12) are always exactly observable is

an open problem. We assert that the answer to that question is yes.

Since controllability and observability are dual properties, similar controllability results hold if we have n -dimensional control at the boundary. Thus the (stable) system (9)–(10) with control

$$u(t) = \begin{bmatrix} W_{B,1} & W_{B,0} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}$$

is exactly controllable when $G_0^* + G_0 = 0$, and satisfies the (dual) Hautus test always.

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