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ABSTRACT
Heat transfer and cooling solutions play an important role in the design of gas turbine blades. However, the underlying mathematical coupling structures have not been thoroughly investigated. In this work, the port-Hamiltonian formalism is applied to the conjugate heat transfer problem in gas turbine blades. A mathematical model based on common engineering simplifications is constructed and further simplified to reduce complexity and focus on the coupling structures of interest. The model is then cast as a port-Hamiltonian system and examined for stability and well-posedness.

KEYWORDS
port-Hamiltonian system; heat transfer; cooling channel; coupled systems; well-posedness

1. Introduction

With the German government's plans to increase the share of renewable energies in the power grid to over 60% by 2050, gas turbines will most likely gain in importance and take on new roles and tasks in the power grid. Due to their short start-up times and high efficiency, they are particularly well suited as reserve power plants and can cushion power drops and demand peaks. They are also relevant in the 100% renewable energy sources scenario, when hydrogen and methane are used as energy storage. These changing application roles go hand in hand with changing design requirements, especially in terms of efficiency, reliability and flexibility of operation.

In order to accurately incorporate these requirements into the design process, the use of high-level multiphysics simulations is required, combining fluid dynamics, structural mechanics, heat conduction, convective heat transport and 1D flow networks, among others. The GivEn project [cf. 1] aims to integrate multiphysics simulation with a multicriteria shape optimisation process.

To achieve this goal and obtain useful results, other requirements must first be met. One of them is to ensure a suitable coupling structure between the different parts of the multiphysics simulation. In this work we want to model and investigate a special coupling structure - the heat transfer at the walls of the cooling channels of

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the turbine blade. It couples the heat conduction in the metal of the turbine blade with the transport of the cooling fluid flowing through the cooling channels.

To improve the thermal efficiency, modern gas turbines are operated at very high temperatures (1200 °C to 1500 °C). Since these temperatures greatly exceed the range in which the turbine blade's metal can be used safely, active cooling of the turbine blades is necessary for them to withstand these temperatures. One of the techniques used for this purpose is convection cooling, i.e. cooling channels are installed within the blade. These internal cooling channels are small ducts within the turbine blade that are filled with a stream of cooling fluid, usually (comparatively cool) air extracted from the compressor.

This approach leads to a so-called *conjugate heat transfer* (CHT) problem (strong thermal interactions between solids and fluids) [2], which is the heat transfer between the turbine blade, the internal flow in the cooling channel and the hot external flow. In this work we will focus on the role of the cooling channels, the heating of the blade by the external flow is discussed in [2]. In order to maximize heat transfer between the blade and the fluid, the flow within the cooling channels is kept deliberately turbulent, for example by use of so-called rib turbulators, periodic protrusions and recessions in the channel walls. For a more in-depth review of turbine blade cooling, see e.g. [3].

We will formulate a so-called *port-Hamiltonian system (pHs)* modeling this conjugate heat transfer and study in detail the resulting coupling structure. Since it is closely related to the Hamiltonian formalism originally developed in theoretical physics, this port-Hamiltonian framework is a natural fit for modeling physical systems and, in particular, their interconnections, since two port-Hamiltonian systems connected by a suitable coupling structure in turn form a port-Hamiltonian system. The formalism also makes conservation laws, a fundamental property of virtually any physical system, explicit. Moreover, a suitable port-Hamiltonian formulation makes the process of discretizing a continuous system for numerical simulation relatively simple, while ensuring that conservation laws still hold in the discretized system [cf. 4]. Moreover, several desirable properties such as stability or controllability are either inherent to port-Hamiltonian systems or can be guaranteed by some easily checked additional conditions.

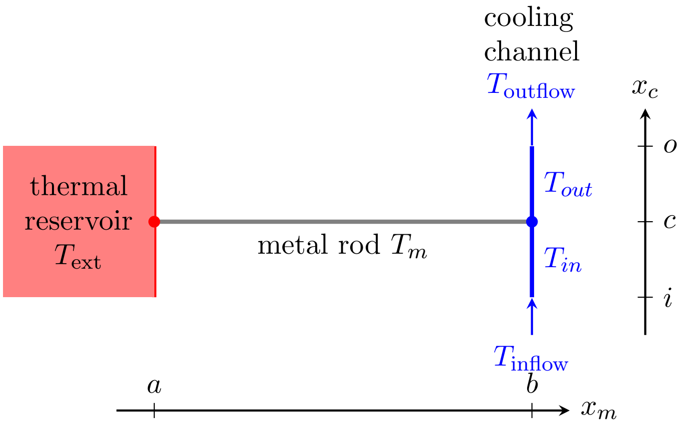
The paper is structured as follows. First, in Section 2 we motivate and introduce the mathematical model of the coupled system to be investigated including a rescaling of the variables. Next, in Section 3 we rewrite each subsystem of our model in the port-Hamiltonian framework, after which we combine these port-Hamiltonian systems and study the properties of the resulting coupled system. Finally, we summarize and interpret the results in Section 5 and give some concluding remarks.

2. The Model System

Our model system is based on a highly simplified model of heat transfer within the blade of a gas turbine, since we are mainly interested in the mathematical coupling structure.

In the design process of gas turbine blades, much work is done with two-dimensional slices, which are then stacked and interpolated to form the final three-dimensional blade. This is done because simulation and optimization is much easier and faster that way than with a full three-dimensional model and still gives "good enough" results. The cooling channel, which is more or less perpendicular to the slices, then has exactly one point of contact with each slice (neglecting 180° hairpin curves) where heat transfer can occur.

Since we are primarily interested in the thermal coupling structure between the turbine blade and the cooling channel, and a 2D-slice introduces additional complexity into the model, we decided to further simplify the system and use a one-dimensional rod instead. Thus, our model consists of a thermally conductive rod ($a \leq x_m \leq b$) and a cooling channel ($i \leq x_c \leq o$) through which a fluid is pumped. The left end of the rod at $x_m = a$ is in contact with a thermal reservoir with a given temperature $T_{\text{ext}}(t)$. The right end of the rod at $x_m = b$ is in contact with the wall of the cooling channel.



The temperature of the metal rod $T_m = T_m(x_m, t)$ is modeled by a heat equation supplied with Robin boundary conditions at $x_m = a$ and $x_m = b$. The temperatures $T_{in} = T_{in}(x_c, t)$ and $T_{out} = T_{out}(x_c, t)$ of the inflowing and outflowing parts of the cooling channel are described by simple transport equations. This is, again, a simplification, as it assumes that the convective heat transport dominates and we can neglect the diffusion in the cooling channel medium. For the usual flow rates and cooling fluids used in gas turbines, such as air or water vapor, this is a valid assumption, as they have a very low thermal conductivity compared to their heat capacity. The coupling at point $x_c = c$ is such that the outflowing temperature at point c is determined by the inflowing temperature at this point plus the heat flowing out of the metal rod due to the boundary condition at $x_m = b$.

2.1. The System of Equations

The above setting leads to the following equations for the temperatures in the metal rod and in the cooling channel:

$$\frac{\partial T_m}{\partial t} = \frac{k}{c_m}\frac{\partial^2 T_m}{\partial x_m^2}, \quad a < x_m < b, \quad t > 0, \tag{1a}$$

$$\frac{\partial T_{in}}{\partial t} = -v\frac{\partial T_{in}}{\partial x_c}, \quad i < x_c < c, \quad t > 0, \tag{1b}$$

$$\frac{\partial T_{out}}{\partial t} = -v\frac{\partial T_{out}}{\partial x_c}, \quad c < x_c < o, \quad t > 0, \tag{1c}$$

$$-k\,\frac{\partial T_m}{\partial x_m}(a,t) = h_a\big(T_{\text{ext}}(t) - T_m(a,t)\big), \quad t > 0, \tag{1d}$$

$$-k\,\frac{\partial T_m}{\partial x_m}(b,t) = h_b\big(T_m(b,t) - T_{in}(c,t)\big), \quad t > 0, \tag{1e}$$

$$T_{in}(i,t) = T_{\text{inflow}}(t), \quad t > 0, \tag{1f}$$

$$c_c v\big(T_{out}(c,t) - T_{in}(c,t)\big) = h_b\big(T_m(b,t) - T_{in}(c,t)\big), \quad t > 0. \tag{1g}$$

Hence, to summarize, the temperature field in the metal rod is described by the heat equation (1a) supplied with the two Robin boundary conditions (1d), (1e) modelling a Fourier-type heat transfer, the Newton's law of cooling. Next, in the inflow part of the cooling channel ($i < x_c < c$) a given temperature profile $T_{\text{inflow}}(t)$ at the left boundary $x_c = i$ is convected with the speed v, see (1b), (1f). Finally, in equation (1g) the coupling of the two systems at the point $x_m = b$ equals $x_c = c$ is described: the heat flux from the metal rod to the cooling channel, depending on the inflowing temperature T_{in} at $x_c = c$.

We remark that, because of equation (1e), the last boundary condition (1g) can alternatively be written as

$$c_c v\big(T_{out}(c) - T_{in}(c)\big) = -k\,\frac{\partial T_m}{\partial x_m}(b). \tag{1h}$$

2.2. The Rescaled System

A rescaling of system (1) such that the temperatures are defined on a unit interval $[0,1]$ allows us to write the system (1) in a more compact way and we see how geometric dimensions enter the system. For that purpose, we introduce the new space variables

$$\xi_m = \frac{x_m - a}{b - a} = \frac{x_m - a}{l_m}, \quad a \le x_m \le b,$$

$$\xi_{in} = \frac{x_c - i}{c - i} = \frac{x_c - i}{l_{in}}, \quad i \le x_c \le c,$$

$$\xi_{out} = \frac{x_c - c}{o - c} = \frac{x_c - c}{l_{out}}, \quad c \le x_c \le o,$$

and the rescaled temperature functions $\vartheta_j(\xi_j(x_{m/c}), t) = T_j(x_{m/c}, t)$ for each of the indices $j \in \{m, in, out\}$ respectively. Restating the system in the three scaled spatial variables ξ_j then results in

$$\frac{\partial \vartheta_m}{\partial t} = \frac{k}{c_m l_m^2} \frac{\partial^2 \vartheta_m}{\partial \xi_m^2}, \qquad 0 < \xi_m < 1, \quad t > 0, \tag{2a}$$

$$\frac{\partial \vartheta_{in}}{\partial t} = -\frac{v}{l_{in}} \frac{\partial \vartheta_{in}}{\partial \xi_{in}}, \qquad 0 < \xi_{in} < 1, \quad t > 0, \tag{2b}$$

$$\frac{\partial \vartheta_{out}}{\partial t} = -\frac{v}{l_{out}} \frac{\partial \vartheta_{out}}{\partial \xi_{out}}, \qquad 0 < \xi_{out} < 1, \quad t > 0, \tag{2c}$$

$$-\frac{k}{l_m} \frac{\partial \vartheta_m}{\partial \xi_m}(0,t) = h_a\big(T_{\text{ext}}(t) - \vartheta_m(0,t)\big), \quad t > 0, \tag{2d}$$

$$-\frac{k}{l_m} \frac{\partial \vartheta_m}{\partial \xi_m}(1,t) = h_b\big(\vartheta_m(1,t) - \vartheta_{in}(1,t)\big), \quad t > 0, \tag{2e}$$

$$\vartheta_{in}(0,t) = T_{\text{inflow}}(t), \quad t > 0, \tag{2f}$$

$$c_c v\big(\vartheta_{out}(0,t) - \vartheta_{in}(1,t)\big) = h_b\big(\vartheta_m(1,t) - \vartheta_{in}(1,t)\big), \quad t > 0. \tag{2g}$$

Now we are prepared to rewrite this system (2) as a port-Hamiltonian system in the next section.

3. The Port-Hamiltonian Formulation of infinite systems

As mentioned earlier, we now want to formulate our model system in the port-Hamiltonian framework, as this makes it easier to check for certain properties, especially stability.

Since the state variables ϑ_m, ϑ_{in}, ϑ_{out} in the model system (2) are continuous in space, we cannot apply the usual finite-dimensional port-Hamiltonian framework as described in, for example [5, 6]. Instead, we use a generalisation for distributed parameter systems, as presented in, for example, [7–9]. Since the system (2) only contains first-order derivatives w.r.t. time, we also restrict ourselves to linear first-order systems for simplicity.

Port-Hamiltonian systems (pHs) can be viewed as a combination of a Dirac structure (or Stokes-Dirac structure in the case of infinite dimensional systems) and a Hamiltonian. The Dirac structure defines the relation between the so-called *flow variables* f and *effort variables* e, while the Hamiltonian connects e and f to the state variables Θ and "contains the physics". We start with a definition of the underlying *Dirac structure*.

Definition 3.1 ((Stokes-)Dirac structure [7, 9, 10])**.**
Let $\mathcal{F}$ be a linear space, $\mathcal{E}$ its dual and $\langle e, f\rangle$ their dual product. Further let $\langle\langle \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \rangle\rangle = \langle e_1, f_2\rangle + \langle e_2, f_1\rangle, \quad \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \in \mathcal{E} \times \mathcal{F}$.

Then $\mathcal{D} \subset (\mathcal{E} \times \mathcal{F})$ is a *Stokes-Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$ with

$$\mathcal{D}^\perp = \{b \in \mathcal{E} \times \mathcal{F} | \langle\langle b, b_1 \rangle\rangle = 0 \quad \forall\, b_1 \in \mathcal{D}\}.$$

Remark 1.

$$\mathcal{D} = \{(e,f) \in \mathcal{E} \times \mathcal{F} | f = Je\}$$

is a (Stokes-)Dirac structure if J is a skew-adjoint operator.

This results in the following definition of a pHs, as given in [8, Definition 7.1.2]:

Definition 3.2 (port-Hamiltonian system [8])**.** Let $P_1 \in \mathbb{R}^{n\times n}$ invertible and self-adjoint, $P_0 \in \mathbb{R}^{n\times n}$ skew-adjoint, i.e. $P_0^\top = -P_0$ and $\mathcal{H} \in \mathbb{R}^{n\times n}$ symmetric such that $mI \leq \mathcal{H} \leq MI$ with constants $m, M > 0$. Further, let $X = L^2([a,b], \mathbb{R}^{n\times n})$ be a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{2}\int_a^b g(\xi)^* \mathcal{H}(\xi) f(\xi)\,\mathrm{d}\xi.$$

Then the differential equation

$$\frac{\partial \Theta}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\big(\mathcal{H}(\xi)\Theta(\xi,t)\big) + P_0\big(\mathcal{H}(\xi)\Theta(\xi,t)\big), \qquad a < \xi < b, \quad t > 0, \tag{3}$$

is a *linear first-order port-Hamiltonian system (pHs)* with the associated *Hamiltonian*

$$H(t) = \frac{1}{2}\int_a^b \Theta(\xi,t)^* \mathcal{H}(\xi)\Theta(\xi,t)\,\mathrm{d}\xi. \tag{4}$$

Remark 2. With the usual choice of $f = \frac{\partial \Theta}{\partial t}$ and $e = \mathcal{H}\Theta$, we can see the connection between the port-Hamiltonian system of Definition 3.2 and the Stokes-Dirac structure of Definition 3.1, since the operator $P_1 \frac{\partial}{\partial \xi} + P_0$ is skew-adjoint.

The most important difference between a Dirac structure and a Stokes-Dirac structure is the presence of a boundary port that governs the power flow across the boundary and takes the place of boundary conditions in 'regular' PDEs. For a port-Hamiltonian system as in Definition 3.2, the boundary port takes the following form [cf. 8, eqs. (7.26) and (7.27)]:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{R_0} \begin{pmatrix} \mathcal{H}\Theta(b) \\ \mathcal{H}\Theta(a) \end{pmatrix}. \tag{5}$$

While this is essentially a simple variable substitution, it is making the power flow across the boundary obvious, since now $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = e_\partial^\top f_\partial$ holds (in the absence of other external ports).

We equip the port-Hamiltonian system with boundary conditions of the form

$$u(t) = W_B \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \tag{6}$$

where $u(t)$ is a time-dependent input function, $W_B \in \mathbb{R}^{n\times 2n}$ has full rank, $W_B \Sigma W_B^\top \geq 0$ and $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. We note that the above property guarantees that the port-Hamiltonian system always has a unique (classical and mild) solution which is non-decreasing in the energy norm [cf. 8, Theorem 11.3.5]. Requiring $W_B \Sigma W_B^\top > 0$ would be sufficient for uniform exponential stability [8, Lemma 9.1.4], however, there are also other, less restrictive criteria [cf. 8, Theorem 9.1.3] for uniform exponential stability.

Sometimes it is more convenient to write the boundary conditions in a form directly dependent on the effort variables at the boundary, in the form of $u(t) = \widetilde{W}_B \begin{pmatrix} \mathcal{H}\Theta(b) \\ \mathcal{H}\Theta(a) \end{pmatrix}$. By comparison with equations (5) and (6), we see that

$$W_B = \widetilde{W}_B R_0^{-1}. \tag{7}$$

Not all port-Hamiltonian systems directly fit into the formalism above, especially if the order of space and time derivatives doesn't match. However, the formalism can be extended to dissipative systems [cf. 9, Chapter 6]:

$$\frac{\partial \vartheta_m}{\partial t}(\xi_m, t) = \big(\mathcal{J} - \mathcal{G}_R \mathcal{S} \mathcal{G}_R^*\big)\big(\mathcal{H}\vartheta_m\big)(\xi_m, t), \qquad 0 < \xi_m < 1, \quad t > 0 \tag{8}$$

$\mathcal{G}_R^*$ is the formal adjoint operator of $\mathcal{G}_R$ (i.e. the adjoint of $\mathcal{G}_R$ neglecting boundary conditions) and S is a coercive operator on $L^2([0,1], \mathbb{R})$. Since we are only considering linear first-order systems, the operators take the following form:

$$\mathcal{J} e = P_1 \frac{\partial e}{\partial \xi} + P_0 e, \quad \mathcal{G}_R f = G_1 \frac{\partial f}{\partial \xi} + G_0 f, \quad \mathcal{G}_R^* e = -G_1^\top \frac{\partial e}{\partial \xi} + G_0^\top e,$$

As shown by Villegas [9, Chapter 6.3], the operator $\mathcal{J} - \mathcal{G}_R \mathcal{S} \mathcal{G}_R^*$ in equation (8) is equivalent to the *expanded skew-symmetric operator* $\mathcal{J}_e$ together with the *closure relation* $e_r = \mathcal{S} f_r$ in the *expanded system*

$$\begin{pmatrix} f \\ f_r \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_r \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_r \end{pmatrix}, \tag{9a}$$

$$e_r = \mathcal{S} f_r. \tag{9b}$$

In this formulation, the Dirac structure induced by $\mathcal{J}_e$ has an additional port, called the *resistive port*. This port is then terminated with the resistive closure relation (9b). Obviously, this also means that the boundary conditions can also depend on e_r.

Since W_B only gives us n conditions, this leaves the other n open for use as outputs, which we can define as

$$y = W_C \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}. \tag{10}$$

To ensure that we do not use quantities as output that are already set by the input, we require that $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ is of full rank.

4. The PHS-Formulation of the Model System

In the following, we will first model each part of our model system as a port-Hamiltonian system, and then look at the combined system in Section 4.3.

4.1. The Heat Equation

The heat equation (2a) cannot be written as a port-Hamiltonian system of the form given in Definition 3.2. However, it can be written as a system with dissipation of the form given in equation (8).

With the usual choice of flow variable $f = \frac{\partial \vartheta_m}{\partial t}$ and effort variable $e = \mathcal{H}\vartheta_m$, as well as the choices of

$$P_1 = P_0 = G_0 = 0, \quad G_1 = \frac{1}{l_m}, \quad \mathcal{S} = k \quad \text{and} \quad \mathcal{H} = \frac{1}{c_m}$$

we have $\mathcal{J} = 0$, $\mathcal{G}_R = \frac{1}{l_m}\partial_\xi$, $\mathcal{G}_R^* = -\frac{1}{l_m}\partial_\xi$ and thus we recover the heat equation (2a) from equation (8) and obtain the associated quadratic Hamiltonian

$$H_m = \frac{1}{2}\int_0^1 \vartheta_m^*(\xi,t)\mathcal{H}\vartheta_m(\xi,t)\,\mathrm{d}\xi = \frac{1}{2c_m}\int_0^1 \big(\vartheta_m(\xi,t)\big)^2\,\mathrm{d}\xi. \tag{11}$$

Note that in this case the Hamiltonian (11) is not the physical energy, so the dissipation present in this system does not automatically violate the law of conservation of energy. If we wanted to explicitly use the physical energy as our Hamiltonian, we would also need to ensure that the second law of thermodynamics is satisfied, i.e., the system becomes irreversible. It is possible to extend port-Hamiltonian systems for irreversible cases, cf. [10], but this would add additional complexity and is not needed here.

Putting the heat equation into the form of (9), we obtain the following equations

$$\begin{pmatrix} f \\ f_r \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{l_m} \\ \frac{1}{l_m} & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} e \\ e_r \end{pmatrix}, \tag{12a}$$

$$e_r = k f_r. \tag{12b}$$

This formulation makes it clear how spatial derivatives of ϑ can occur in the boundary conditions. Remembering that $f = \frac{\partial \vartheta_m}{\partial t}$ and $e = \mathcal{H}\vartheta_m$, it follows that $e_r = \frac{k}{c_m l_m}\frac{\partial \vartheta_m}{\partial \xi}$.

The corresponding boundary conditions (2d), (2e) can now be rewritten in a formulation with inputs:

$$u(t) = \begin{pmatrix} \frac{h_a}{c_m} T_{\text{ext}}(t) \\ \frac{h_b}{c_m}\vartheta_{in}(1,t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & h_a & -1 \\ h_b & 1 & 0 & 0 \end{pmatrix}}_{=\widetilde{W}_B} \begin{pmatrix} e(1,t) \\ e_r(1,t) \\ e(0,t) \\ e_r(0,t) \end{pmatrix}. \tag{13}$$

With $R_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}$ we find

$$W_B = \widetilde{W}_B R_0^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} l_m & -h_a l_m & ha & -1 \\ l_m & h_b l_m & h_b & 1 \end{pmatrix}. \tag{14}$$

$$W_B \Sigma W_B^\top = \begin{pmatrix} 2h_a l_m & 0 \\ 0 & 2h_b l_m \end{pmatrix} \tag{15}$$

W_B is obviously of full rank. As h_a, h_b and l_m are all physical constants of the system and thus positive, we find that $W_B \Sigma W_B^\top > 0$ holds. Therefore, the heat equation with the chosen boundary conditions is has unique (classical and mild) solutions which are non-increasing in norm [cf. 9, Theorem 6.9].

As outputs, we choose the temperatures at the boundaries (scaled by some constants for convenience), so we get

$$y = \begin{pmatrix} \frac{h_a}{c_m} \vartheta_m(0) \\ \frac{h_b}{c_m} \vartheta_m(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & h_a & 0 \\ h_b & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e(1) \\ e_m(1) \\ e(0) \\ e_m(0) \end{pmatrix} \tag{16}$$

Then the combined matrix $\begin{pmatrix} \widetilde{W}_B \\ \widetilde{W}_C \end{pmatrix}$ has full rank, which means we are not measuring quantities we already set as input.

4.2. The Transport Equations

For the cooling channel, which we divide into an incoming and an outgoing channel (indices *in* and *out*), we do not have any dissipative terms, so we can write it directly as a linear first-order port-Hamiltonian system as defined in Definition 3.2

$$\begin{pmatrix} \frac{\partial \vartheta_{in}}{\partial t} \\ \frac{\partial \vartheta_{out}}{\partial t} \end{pmatrix} = \begin{pmatrix} -\frac{v}{l_{in}} & 0 \\ 0 & -\frac{v}{l_{out}} \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} \vartheta_{in} \\ \vartheta_{out} \end{pmatrix}, \tag{17}$$

i.e. with the choices

$$P_1 = \begin{pmatrix} -\frac{v}{l_{in}} & 0 \\ 0 & -\frac{v}{l_{out}} \end{pmatrix}, \quad P_0 = 0, \quad \mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, rewriting the boundary conditions (2f) and (2g) in a formulation with inputs, we obtain

$$u(t) = \begin{pmatrix} T_{\text{inflow}}(t) \\ h_b \vartheta_m(1,t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ (h_b - c_c v) & 0 & 0 & c_c v \end{pmatrix}}_{=\widetilde{W}_B} \begin{pmatrix} \vartheta_{in}(1,t) \\ \vartheta_{out}(1,t) \\ \vartheta_{in}(0,t) \\ \vartheta_{out}(0,t) \end{pmatrix}. \tag{18}$$

As outputs, we choose the temperature at the ends of each cooling channel part, i.e.

$$y(t) = \begin{pmatrix} \vartheta_{in}(1,t) \\ \vartheta_{out}(1,t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{\widetilde{W}_C} \begin{pmatrix} \vartheta_{in}(1,t) \\ \vartheta_{out}(1,t) \\ \vartheta_{in}(0,t) \\ \vartheta_{out}(0,t) \end{pmatrix}, \tag{19}$$

so we have again $\widetilde{W}_B$ as well as $\begin{pmatrix} \widetilde{W}_B \\ \widetilde{W}_C \end{pmatrix}$ of full rank.

Calculating W_B as in (7), we obtain

$$W_B = \widetilde{W}_B R_0^{-1} = \begin{pmatrix} \frac{l_{in}}{v} & 0 & 1 & 0 \\ l_{in}c_c - \frac{l_{in}h_b}{v} & c_c l_{out} & (h_b - c_c v) & c_c v \end{pmatrix}, \tag{20}$$

$$W_B \Sigma W_B^\top = \begin{pmatrix} \frac{l_{in}}{v} & 0 \\ 0 & c_c^2 l_{out} v - \frac{(c_c v - h_b)^2 l_{in}}{v} \end{pmatrix}. \tag{21}$$

From the last line we can see that this system is not stable for all variable choices, since the stability condition $W_B \Sigma W_B^\top \geq 0$ is not always satisfied. Positive-semidefinitness is only given for

$$\frac{l_{out}}{l_{in}} \geq \frac{(c_c v - h_b)^2}{c_c^2 v^2}$$

In the special case of $l_{in} = l_{out}$, this condition can be simplified to $2c_c v \geq h_b$.

4.3. The Combined system

When we combine the two Port-Hamiltonian systems discussed before, we get the following system of equations. As you can see in equation (24), not all boundary conditions are connected to an input anymore, but are instead set to zero. These are the coupling conditions between the two subsystems. Accordingly, we also have only two outputs. In detail, the system has the form given by the equations (9) resp. (8), with the choices of

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{v}{l_{in}} & 0 \\ 0 & 0 & -\frac{v}{l_{out}} \end{pmatrix}, \quad P_0 = G_0 = 0, \quad G_1 = \begin{pmatrix} \frac{1}{l_m} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{S} = k,$$

$$\mathcal{H} = \begin{pmatrix} \frac{1}{c_m} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inserting these choices into equation (9), we get the following form for the combined system:

$$\begin{pmatrix} \frac{\partial \vartheta_m}{\partial t} \\ \frac{\partial \vartheta_{in}}{\partial t} \\ \frac{\partial \vartheta_{out}}{\partial t} \\ f_r \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{l_m} \\ 0 & -\frac{v}{l_{in}} & 0 & 0 \\ 0 & 0 & -\frac{v}{l_{out}} & 0 \\ \frac{1}{l_m} & 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} \frac{1}{c_m}\vartheta_m \\ \vartheta_{in} \\ \vartheta_{out} \\ e_r \end{pmatrix}, \tag{22}$$

$$e_r = \frac{k}{f_r}, \tag{23}$$

$$\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{h_a}{c_m}T_{\text{ext}}(t) \\ T_{\text{inflow}}(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & h_a & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ h_b & -\frac{h_b}{c_m} & 0 & 1 & 0 & 0 & 0 & 0 \\ -h_b c_m & (h_b - c_c v) & 0 & 0 & 0 & 0 & c_c v & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{c_m}\vartheta_m(1) \\ \vartheta_{in}(1) \\ \vartheta_{out}(1) \\ e_r(1) \\ \frac{1}{c_m}\vartheta_m(0) \\ \vartheta_{in}(0) \\ \vartheta_{out}(0) \\ e_r(0) \end{pmatrix}, \tag{24}$$

$$y = \begin{pmatrix} \frac{1}{c_m}\vartheta_m(0) \\ \vartheta_{out}(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{c_m}\vartheta_m(1) \\ \vartheta_{in}(1) \\ \vartheta_{out}(1) \\ e_r(1) \\ \frac{1}{c_m}\vartheta_m(0) \\ \vartheta_{in}(0) \\ \vartheta_{out}(0) \\ e_r(0) \end{pmatrix}. \tag{25}$$

$\widetilde{W}_B$, $\widetilde{W}_C$ and $\begin{pmatrix} \widetilde{W}_B \\ \widetilde{W}_C \end{pmatrix}$ obviously have full rank. Note that we could also choose the heat flux at the left boundary $e_m(0)$ as an output, without changing the rank of the matrices.

It remains to check whether the stability condition $W_B \Sigma W_B^\top \geq 0$ holds:

$$W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} l_m & 0 & 0 & -l_m h_a & h_a & 0 & 0 & -1 \\ 0 & \frac{l_{in}}{v} & 0 & 0 & 0 & 1 & 0 & 0 \\ l_m & \frac{l_{in} h_b}{c_m v} & 0 & l_m h_b & h_b & -\frac{h_b}{c_m} & 0 & 1 \\ 0 & \frac{(c_c v - h_b) l_{in}}{v} & l_{out} c_c & -l_m h_b c_m & -h_b c_m & -c_c v + h_b & c_c v & 0 \end{pmatrix} \tag{26}$$

$$W_B \Sigma W_B^\top = \begin{pmatrix} 2\, l_m h_a & 0 & 0 & 0 \\ 0 & \frac{l_{in}}{v} & 0 & 0 \\ 0 & 0 & 2\, l_m h_b - \frac{l_{in} h_b^2}{c_m^2 v} & -l_m c_m h_b - \frac{(c_c v - h_b) l_{in} h_b}{c_m v} \\ 0 & 0 & -l_m c_m h_b - \frac{(c_c v - h_b) l_{in} h_b}{c_m v} & l_{out} c_c^2 v - \frac{(c_c v - h_b)^2 l_{in}}{v} \end{pmatrix} \tag{27}$$

We can see from equation (27) that both eigenvalues are non-negative, if the following two conditions hold:

$$a + c \geq 0,$$
$$ac \geq b^2$$

with

$$a = 2\,l_m h_b - \frac{l_{in} h_b^2}{c_m^2 v},$$
$$b = -l_m c_m h_b - \frac{(c_c v - h_b) l_{in} h_b}{c_m v},$$
$$c = l_{out} c_c^2 v - \frac{(c_c v - h_b)^2 l_{in}}{v}.$$

Since the boundary condition (1h) is equivalent to (1g) due to (1e), we can use that one to develop our coupling structure instead. If we do that, we find:

$$\widetilde{W_B} = \begin{pmatrix} 0 & 0 & 0 & 0 & h_a & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ h_b & -\frac{h_b}{c_m} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -c_c v & 0 & 1 & 0 & 0 & c_c v & 0 \end{pmatrix} \tag{28}$$

$$W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} l_m & 0 & 0 & -l_m h_a & h_a & 0 & 0 & -1 \\ 0 & \frac{l_{in}}{v} & 0 & 0 & 0 & 1 & 0 & 0 \\ l_m & \frac{l_{in} h_b}{c_m v} & 0 & l_m h_b & h_b & -\frac{h_b}{c_m} & 0 & 1 \\ l_m & c_c l_{in} & l_{out} c_c & 0 & 0 & -c_c v & c_c v & 1 \end{pmatrix} \tag{29}$$

$$W_B \Sigma W_B^\top = \begin{pmatrix} 2h_a l_m & 0 & 0 & 0 \\ 0 & \frac{l_{in}}{v} & 0 & 0 \\ 0 & 0 & 2h_b l_m - \frac{h_b^2 l_{in}}{c_m^2 v} & -\frac{c_c h_b l_{in}}{c_m} + l_m h_b \\ 0 & 0 & -\frac{c_c h_b l_{in}}{c_m} + l_m h_b & l_{out} c_c^2 v - l_{in} c_c^2 v \end{pmatrix} \tag{30}$$

Here it becomes immediately clear that the matrix for $l_{in} = l_{out}$ can never be positive definite, but only semi-definite, if $2h_b l_m - \frac{h_b^2 l_{in}}{c_m^2 v} \geq 0$ and $-\frac{c_c h_b l_{in}}{c_m} + l_m h_b = 0$ holds, which yields $c_c \geq \frac{h_b}{2c_m v}$. Thus it is more restrictive in this respect than the previous coupling.

Summarising, the combined system possesses unique (classical and mild) solutions on $[0, \infty)$ and these solution are non-increasing in the energy norm. We remark, that it is an open question whether exponential stability of the extended port-Hamiltonian system implies exponential stability of the combined system, that is, whether the condition $W_B \Sigma W_B^\top > 0$ guarantees exponential stability of the combined system.

Table 1. Variable naming conventions

Variable	Quantity	Unit
T	temperature	K
c	volumetric heat capacity	$\mathrm{J\,m^{-3}\,K^{-1}}$
h	heat transfer coefficient	$\mathrm{W\,m^{-2}\,K^{-1}}$
k	thermal conductivity	$\mathrm{W\,m^{-1}\,K^{-1}}$
l	length	m
v	flow speed	$\mathrm{m\,s^{-1}}$

5. Conclusion

While the heat equation as described in Section 4.1 is exponentially stable for all (physically meaningful, i.e. positive) values of the constants involved, this is not the case for the transport equations of the cooling channel described in Section 4.2, nor for the combined system of Section 4.3. For both systems, it is possible to find values of the constants that are physically reasonable but do not satisfy the stability criteria, regardless of which formulation is chosen for the coupling conditions.

It is also noteworthy that the two formulations of the coupling conditions studied, while technically equivalent, imply different regions of stability. Although some of the stability conditions have a clear, physical motivation, others – particularly those related to the coupling condition – are seemingly nonsensical. The most obvious example of the latter would be that the ratio between the lengths of the cooling channel parts can determine the stability of the system.

A likely explanation or interpretation for this is that our model system is oversimplified and does not properly capture the properties of the real system. The fact that all heat transfer occurs in a single point, as opposed to an extended region with a non-zero physical dimension, causes a discontinuity in the temperature distribution within the cooling channel, making it a likely candidate for the source of the above problems. In future work we will investigate whether the observed stability problems persist if we consider instead the coupling of a two-dimensional heat equation with heat transfer along the entire length of the cooling channel.

In any case, the port-Hamiltonian formalism has proved to be a very useful tool to study the properties of a model system describing several interconnected physical processes and to find problems and limitations of the model. With its help, it has been shown that simplifications – even those widely used in industry – are not always useful from a mathematical point of view and must be carefully evaluated before use.

Our future work will focus on investigating new discretisation techniques based on this coupling strategies and studying a more realistic two dimensional setting.

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Nomenclature/Notation

Th naming of variables denoting physical quantities follows the conventions set in the Table 1. Variables that are not directly referencing physical quantities are generally defined when introduced.

The index m is used for variables that refer to the metal subsystem, i.e. the heat conducting rod. The indices *in* and *out* are used for the inflowing and outflowing parts of the cooling channel, respectively. Finally, the index c is used for variables that refer to the entirety of the cooling channel.

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