Bergische Universität Wuppertal

Fakultät für Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 21/01

Torben Schäfers and Long Teng

Asymmetry in stochastic volatility models with threshold and time-dependent correlation

http://www.math.uni-wuppertal.de

Asymmetry in stochastic volatility models with threshold and time-dependent correlation

TORBEN SCHÄFERS, LONG TENG

Lehrstuhl für Angewandte Mathematik und Numerische Analysis,
Fakultät für Mathematik und Naturwissenschaften,
Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany

Abstract

In this work we study the effects by including threshold, constant and time-dependent correlation in stochastic volatility (SV) models to capture the asymmetry relationship between stock returns and volatility. We develop SV models which include only time-dependent correlated innovations and both threshold and time-dependent correlation, respectively. It has been shown in literature that the SV model with only constant correlation does a better job of capturing asymmetry than than threshold stochastic volatility (TSV) model. We show here that the SV model with time-dependent correlation performs better than the model with constant correlation on capturing asymmetry, and the comprehensive model with both threshold and time-dependently correlated innovations dominates models with pure threshold, constant and time-dependent correlation, and both threshold and constant correlation as well. In our comprehensive model, volatility and returns are time-dependently correlated, where the time-varying correlation is negative, and the volatility is more persistent, less volatile and higher following negative returns as expected. An empirical study is provided to illustrate our findings.

Keywords *Asymmetry stochastic volatility model, Threshold, Time-dependent correlation*

1 Introduction

Many existing empirical results, e.g., [Glosten et al., 1993, Harvey and Shephard, 1996, Jacquier et al., 2004, Nelson, 1991, Omori et al., 2007, Yu, 2005] indicate that the relationship between stock returns and their volatility is asymmetric. The asymmetric volatility is an important feature of stock returns, and especially noticeable when the stock market crashes. There are two widespread explanations for this relationship: the leverage effect [Black, 1976, Christie, 1982] and the volatility feedback effect [Campbell and Hentschel, 1992, French et al., 1987]. The leverage effect states that after negative returns, the financial leverage of a company increases and this leads to increased volatility [Christie, 1982, Schwert, 1989]. However, it was concluded that the leverage effect cannot solely be responsible for this asymmetric relationship. Nonetheless, this explanation has been strongly linked to the asymmetric relationship of return and volatility, that the leverage effect is often used as a synonym for asymmetric volatility. According to Campel and Hentschel's definition of the volatility feedback effect, volatility feedback strengthens large negative returns and large positive returns are weakened by volatility feedback. There are other explanations, for example the prospect theory [Kahneman and Tversky, 1979]. In the context of that theory, one is exposed to loss aversion. Because of that people prefer to avoid losses rather than make corresponding profits. This also explains the tendency of investors to sell assets that have increased in value while retaining assets that have decreased in value. Although there is no consensus about the causes of asymmetric volatility, whose existence plays definitely a major role in financial market.

Many various models have been developed to capture this asymmetric relationship. Asymmetric GARCH-models which are able to capture this asymmetric relationship clearly outperform GARCH-models that do not capture this asymmetric relationship, see [Glosten et al., 1993, Nelson, 1991]. In these asymmetric GARCH-models, volatility increases more strongly after large negative returns than after large positive returns. The SV model proposed in [Taylor, 1986] can be used to model the conditional volatility of a variety of financial time series. An established method to capture the asymmetric volatility is to correlate the shocks of the returns and the future volatility. Such correlated SV models, which were first introduced in [Wiggins, 1989, Chesney and Scott, 1989], generally provide estimates of a negative correlation between the stock returns and volatility. Another possibilty to capture this asymmetry is the threshold stochastic volatility (TSV) model[So et al., 2003]. In this model, the parameters that influence the volatility dynamics change with the sign of the previous stock returns and no correlation between return and volatility is allowed. In [Smith, 2009] a model is introduced which contains the threshold dynamic and also correlated innovations, and it is shown that modelling with only constant correlation does a better job of capturing the asymmetric relationship than the TSV model.

Indeed, the included correlation can not only improve the SV models, it plays generally a crucial role in modern finance, for example it is significant for pricing financial products. However, many studies have shown that correlation is hardly constant over time, e.g., [Buraschi et al., 2010, Goetzmann et al., 2005, Longin and Solnik, 1995, Teng et al., 2015b, Teng et al., 2015a, Teng et al., 2016a, Teng et al., 2016b, Teng et al., 2016c, Teng et al., 2016d, Teng et al., 2018b, Teng et al., 2018a, Teng and Clevenhaus, 2019, Teng et al., 2020, Tse, 2000]. In [Longin and Solnik, 1995] it was shown that market volatility has changed somewhat in the past indicating that there is no constant conditional correlation for stock returns. A high market volatility leads to an increase of the conditional correlation, and economic variables such as dividend returns and interest rates likely contain information about future volatility and correlation that is not contained in past returns alone.

To further extend the model in [Smith, 2009] in the context of non-constant correlation, X. Wu and H. Zhou [Wu and Zhou, 2014] proposed a triple-threshold leverage SV model, in which the asymmetric correlations between the return and volatility can be captured in the two regimes. This is to say that two different correlation values are allowed in the model. In this work we propose a novel SV model which includes both the threshold and time-dependent correlation instead of constant correlation. Furthermore, by an empirical study on the S&P500 index we show that the SV model with time-dependent correlation performs better than the model with constant correlation on capturing asymmetry, and the comprehensive model with both threshold and time-dependent correlated innovations dominates the pure threshold, constant and time-dependent correlation, and both threshold and constant correlation as well. Due to the broad applicability we use the algorithm proposed in [Smith, 2009] to estimate the parameters, i.e., the maximum likelihood estimation based on the extended Kalman filter and numerical integration over the volatility process.

In the next section we introduce several variants of SV models. Section 3 is devoted to the generalization of SV models by including a time-dependent correlation. In Section 4, we show the estimation of the extended SV models based on the extended Kalman filter and numerical integration. Finally, we present our empirical results in Section 6 and conclude in Section 7.

2 Asymmetric Stochastic Volatility Models

The volatility is not observable, depends on the random shocks and can not be measured with observable information and is presented as a separate random process. The estimation and forecasts are made considerably more difficult by the fact that volatility is latent. It is well-known that the volatility of an asset is not constant. In the following we introduce the different variants of SV models by starting with the standard stochastic volatility model.

2.1 Standard Stochastic Volatility (SV) Model

The standard SV model for the return series r_t, as a latent AR(1) process, can be used to model the log-volatility $x_t = \log \sigma_t^2$:

$$r_t = \mu + \phi r_{t-1} + \exp(x_t/2) z_t, \tag{1}$$

$$x_{t+1} = \omega + \beta x_t + \sigma_v \nu_t, \tag{2}$$

where ϕ describes the autocorrelation, the innovations z_t and ν_t are independent standard normally distributed random variables, β and σ describe the persistence and variance of volatility, respectively.

2.2 Correlated Stochastic Volatility (SVA) Model

The correlated stochastic volatility model, denoted SVA, were introduced in [Wiggins, 1989, Chesney and Scott, 1989] and reads

$$r_t = \mu + \phi r_{t-1} + \exp(x_t/2) z_t, \tag{3}$$

$$x_{t+1} = \omega + \beta x_t + \sigma_v \nu_t, \tag{4}$$

with

$$\rho = E(v_t z_t).$$

This asymmetric relationship between return and volatility is recorded at $\rho < 0$, as in this case the conditional volatility tends to be higher after negative returns than after positive returns.

2.3 Threshold Stochastic Volatility (TSV) Model

In [So et al., 2003] a new way to capture this asymmetric relationship is shown by including a threshold, where the parameters affecting the volatility dynamics alternate with the sign of the previous stock returns:

$$r_t = \mu_{s_{t-1}} + \phi_{s_{t-1}} r_{t-1} + \exp(x_t/2) z_t, \tag{5}$$

$$x_{t+1} = \omega_{s_t} + \beta_{s_t} x_t + \sigma_{v,s_t} v_t, \tag{6}$$

where z_t and v_t are independent standard normally distributed random variables, and s_t is the following indicator function

$$s_t = \begin{cases} 1, & r_t < 0 \\ 0, & r_t \geq 0 \end{cases} \tag{7}$$

and the time-varying coefficients satisfy

$$\begin{aligned} \mu_{s_{t-1}} &= \mu_0 + s_{t-1}\mu_1, \\ \phi_{s_{t-1}} &= \phi_0 + s_{t-1}\phi_1, \\ \omega_{s_t} &= \omega_0 + s_t\omega_1, \\ \beta_{s_t} &= \beta_0 + s_t\beta_1, \\ \sigma^2_{v,s_t} &= \sigma^2_{v,0} + s_t\sigma^2_{v,1}. \end{aligned}$$

The asymmetric relationship can be captured with the parameter ω. If $\omega_1 > 0$, the volatility will tend to be higher after negative returns than after positive returns. However, this increase is independent with size of the negative return in this model.

2.4 Correlated Threshold Stochastic Volatility (TSVA) Model

The TSV model in Sec. 2.3 can be extended by including both the correlation and threshold dynamic [Smith, 2009], i.e., by allowing a correlation between v_t and z_t, denoted by TSVA:

$$r_t = \mu_{s_{t-1}} + \phi_{s_{t-1}} r_{t-1} + \exp(x_t/2) z_t, \tag{8}$$

$$x_{t+1} = \omega_{s_t} + \beta_{s_t} x_t + \sigma_{v,s_t} v_t \tag{9}$$

with $\rho = E(v_t z_t)$ and

$$s_t = \begin{cases} 1, & r_t < 0 \\ 0, & r_t \geq 0 \end{cases},$$

$$\begin{aligned} \mu_{s_{t-1}} &= \mu_0 + s_{t-1}\mu_1, \\ \phi_{s_{t-1}} &= \phi_0 + s_{t-1}\phi_1, \\ \omega_{s_t} &= \omega_0 + s_t \omega_1, \\ \beta_{s_t} &= \beta_0 + s_t \beta_1, \\ \sigma^2_{v,s_t} &= \sigma^2_{v,0} + s_t \sigma^2_{v,1}. \end{aligned}$$

3 Time-dependent correlated Stochastic Volatility Model

In this section we generalize the SVA and TSVA models by using time-dependent correlation model instead of constant correlation coefficient.

3.1 Time-dependent Correlation Model

This section deals with the time-dependent correlation function proposed in [Teng et al., 2015b, Teng et al., 2016a], which reads

$$\rho(t) = E\left[\tanh(X_t)\right], \quad t \geq 0, \tag{10}$$

where X_t is any mean-reverting process with positive and negative values. For the known parameters of X_t, the correlation function $\rho(t)$ varies only with t, i.e., $t \to (-1, 1)$. The time-dependent correlation model (10) fulfills the properties: only takes values in $(-1, 1) \quad \forall t \geq 0$; converges with time due to the mean reversion of X_t. X_t in (10) could be any mean-reverting process that allows positive and negative outcomes. Let X_t be the Ornstein-Uhlenbeck (OU) process [Uhlenbeck and Ornstein, 1930]:

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t, \quad t \geq 0,$$

the time-dependent correlation function defined in (10) can be obtained by

$$\begin{aligned} \rho(t) = E\left[\tanh(X_t)\right] &= E\left[1 - e^{-X_t}\frac{2}{e^{-X_t} + e^{X_t}}\right] = 1 - E\left[e^{-X_t}\frac{1}{\cosh(X_t)}\right] \\ &= 1 - \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh(\frac{\pi u}{2})} CF(t, i + u | X_0, \kappa, \mu, \sigma) du, \end{aligned}$$

where $CF(t, u | X_0, \kappa, \mu, \sigma)$ denotes the characteristic function of the OU process X_t, which can be obtained analytically, e.g. by using the framework of the affine process [Duffie et al., 2003]. This is to say that one can calculate $CF(t, i + u | X_0, \kappa, \mu, \sigma)$ out as

$$CF(t, i + u | X_0, \kappa, \mu, \sigma) = e^{-A(t) - \frac{B(t)}{2} + iu(A(t) + B(t)) + u^2 \frac{B(t)}{2}}$$

with

$$A(t) = e^{-\kappa t} X_0 + \mu(1 - e^{-\kappa t}) \text{ and } B(t) = -\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{11}$$

Finally, the dynamic correlation function can be calculated as

$$\rho(t) = 1 - \frac{e^{-A(t)-\frac{B(t)}{2}}}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh(\frac{\pi u}{2})} e^{iu(A(t)+B(t))+u^2\frac{B(t)}{2}} du, \tag{12}$$

where $A(t)$ and $B(t)$ are defined in (11). In fact, X_0 in $A(t)$ is equal to $\text{artanh}(\rho(0))$.

3.2 Time-dependent correlated (Threshold) Stochastic Volatility Models

The SVA and TSVA models can be directly generalized to capture time-dependent correlated stochastic volatility (SVAD) and time-dependent correlated threshold stochastic volatility (TSVAD) models, respectively, by replacing $E[\nu_t z_t] = \rho$ by

$$E[\nu_t z_t] = \rho(t),$$

where $\rho(t)$ is given in (12).

4 Parameter Estimation Algorithm

Due to the fact that the algorithm introduced in [Smith, 2009] has good extensibility, we will adopt it for the SVAD and TSVAD models. Basically, the model parameters are estimated using maximum likelihood with a nonlinear numerical integration-based filter. The likelihood function is defined as the following T-dimensional integral:

$$L(\theta; r) = \int f(r|x, \theta) f(x|\theta) dx,$$

where r and x denote the T-dimensional sample paths of returns and volatility. Generally, the parameters of stochastic volatility models is complicated to be estimated because the volatility is latent. For example, the standard filters can not be used since the observable variables r_{t+1} and the log-volatility are not jointly normally distributed. In [Smith, 2009], the parameters are estimated with an algorithm that implements numerical integration with brute force over the unobserved latent volatility using the conditional probability formulas in [Kitagawa, 1987, Fridman and Harris, 1998]. Unlike the standard Kalman filter, this algorithm tracks the conditional density of the log-volatility at a finite set of N points, and the T-dimensional integral gets reduced to T different 1-dimensional integrals, thus allowing maximum likelihood estimation.

In the following we introduce the estimation algorithms for the SVAD and TSVAD models. We start with the SVAD model, let I_t be the information available to the econometrician and $f(x_t|I_t)$ be the known conditional density of log-volatility at t. We construct the forecast density of x_{t+1} as

$$f(x_{t+1}|I_t) = \int f(x_{t+1}, x_t|I_t) dx_t = \int f(x_{t+1}|x_t) f(x_t|I_t) dx_t,$$

where $f(x_{t+1}|x_t) = \phi(x_{t+1}; \omega + \beta x_t, \sigma_v^2)$ and $\phi(.; a, b)$ is the probability density function of a normal random variable with expected value a and variance b. For the log-likelihood function one needs the conditional density

$$f(r_{t+1}|I_t) = \int\int f(r_{t+1}, x_{t+1}, x_{t+2}|I_t) dx_{t+1} dx_{t+2}, \tag{13}$$

i.e., integration over both current and future return volatility, and note that the return depends on the change in volatility. To compute (13), one needs to calculate the joint density of $(r_{t+1}, x_{t+1}, x_{t+2})$ which depends on the two correlated normally distributed innovations z_{t+1} and v_{t+1}, which can be expressed as

$$z_{t+1} = u_{1,t+1}$$

and

$$v_{t+1} = \rho(t+1)u_{1,t+1} + \sqrt{(1-\rho(t+1)^2)}u_{2,t+1},$$

where $u_{1,t}$ and $u_{2,t}$ are two independent standard normally distributed random variables, and $\rho(t)$ is given in (12). The joint density can thus be calculated as

$$f(r_{t+1}, x_{t+1}, x_{t+2}|I_t) = \sigma_v^{-1} \exp\left(\frac{-x_{t+1}}{2}\right) \phi(u_{1,t+1})\phi(u_{2,t+1})f(x_{t+1}|I_t)$$

with

$$u_{1,t+1} = (r_{t+1} - \mu - \phi r_t)\exp\left(\frac{-x_{t+1}}{2}\right)$$

and

$$u_{2,t+1} = (x_{t+2} - \omega - \beta x_{t+1} - \rho(t)\sigma_v u_{1,t+1})\sigma_v^{-1}.$$

Finally, the log-likelihood function can be calculated as

$$L(r;\theta) = \sum_{t=0}^{T-1} \log f(r_{t+1}|I_t).$$

Now the inference about the distribution x_{t+1} can be updated by conditioning on the new information r_{t+1}

$$f(x_{t+1}|I_{t+1}) = \frac{\int f(r_{t+1}, x_{t+1}, x_{t+2}|I_t)dx_{t+2}}{f(r_{t+1}|I_t)},$$

and the input for the next conditional density is

$$f(x_{t+2}|I_{t+1}) = \frac{\int f(r_{t+1}, x_{t+1}, x_{t+2}|I_t)dx_{t+1}}{f(r_{t+1}|I_t)}.$$

As mentioned before, a numerical integration scheme will be employed to calculate the general integral

$$\int f(x)dx \approx \sum_{i=1}^{N} \omega_i f(x_i),$$

where w_i is a set of weights and $f(x_i)$ denotes the value of the function f, which is evaluated at N different points x_i for $i = 1, \ldots, N$. For example, the Gauss-Legendre integration is used in [Fridman and Harris, 1998, Smith, 2009]. This is to say that the estimation is implemented by numerical integration over the common grid of N different volatility points $\{x_i\}_{i=1}^N$ for both x_{t+1} and x_{t+2}. In the following we present the algorithm for calculating the log-likelihood function. The time $t+1$ volatility is indexed by $i = 1, \ldots, N$ and the time $t+2$ volatility is indexed by $j = 1, \ldots, N$ for simplicity. First we truncate the integral in (12) at $[a_\rho, b_\rho]$ to compute time-dependent correlations, namely

$$\tilde{\rho}(t) \approx 1 - \frac{e^{-A(t)-\frac{B(t)}{2}}}{2} \int_{a_\rho}^{b_\rho} \frac{1}{\cosh(\frac{\pi u}{2})} e^{iu(A(t)+B(t))+u^2\frac{B(t)}{2}} du, \tag{14}$$

in our experiment we set $[a_\rho, b_\rho] = [-20, 20]$. This integral gets solved numerically using global adaptive quadrature. The algorithm is initialized with $f(x_1 = x_i|I_0) = \phi(x_i|\omega(1-\beta)^{-1}, \sigma_\eta^2(1-\beta^2)^{-1})$ and $\tilde{\rho}(1)$ and computes the joint density function. Secondly, we generate the marginal density of the return $f(r_{t+1}|I_t)$, which is used to calculate the log-likelihood. Note that, for the $(t+1)$-th iteration, the forecast density of the volatility $f(x_{t+1} = x_i|I_t)$ together with $\tilde{\rho}(t+1)$ are used as the input. The inference of volatility can thus be updated as

$$f(x_{t+1}|I_{t+1}) = \frac{\int f(r_{t+1}, x_{t+1}, x_{t+2}|I_t)dx_2}{f(r_{t+1}|I_t)}$$

and the forecast of future volatility

$$f(x_{t+2}|I_{t+1}) = \frac{\int f(r_{t+1}, x_{t+1}, x_{t+2}|I_t)dx_1}{f(r_{t+1}|I_t)}$$

together with $\tilde{\rho}(t+2)$ are used as input for the $(t+2)$-th iteration.

SVAD-algorithm

1. Calculate the joint density for the $N \times N$ possible values of x_{t+1} and x_{t+2}:

$$f(r_{t+1}, x_{t+1} = x_i, x_{t+2} = x_j|I_t) = \sigma_v^{-1} \exp\left(-\frac{x_i}{2}\right) \phi(u_{1,t+1}^{(i)})\phi(u_{2,t+1}^{(i,j)}) f(x_{t+1} = x_i|I_t)$$

   for $i, j = 1, \ldots, N$ with

$$u_{1,t+1}^{(i)} = (r_{t+1} - \mu - \phi r_t) \exp\left(-\frac{x_i}{2}\right) \tag{15}$$

   and

$$u_{2,t+1}^{(i,j)} = (x_j - \omega - \beta x_i - \tilde{\rho}(t+1)\sigma_v u_{1,t+1}^{(i)})\sigma_v^{-1}, \tag{16}$$

   where $\tilde{\rho}(t+1)$ is computed using (14).

2. Compute the marginal density of the returns:

$$f(r_{t+1}|I_t) \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j f(r_{t+1}, x_{t+1} = x_i, x_{t+2} = x_j|I_t).$$

3. Update the log-likelihood:

$$L(r;\theta) = L(r;\theta) + \log f(r_{t+1}|I_t).$$

4. Update the inference about the latent volatility:

$$f(x_{t+1} = x_i|I_{t+1}) \approx \frac{\sum_{j=1}^{N} w_j f(r_{t+1}, x_{t+1} = x_i, x_{t+2} = x_j|I_t)}{f(r_{t+1}|I_t)}.$$

5. Compute inference about the future volatility

$$f(x_{t+2} = x_j|I_{t+1}) \approx \frac{\sum_{i=1}^{N} w_i f(r_{t+1}, x_{t+1} = x_i, x_{t+2} = x_j|I_t)}{f(r_{t+1}|I_t)}$$

   and $\tilde{\rho}(t+2)$, which are used as input for the next iteration.

6. Back to Step 1.

For the TSVAD model we need to use alternative definitions of the uncorrelated normal innovations in the algorithm above to allow the threshold model parameters. (15)-(16) need to be modified as

$$u_{1,t+1}^{(i)} = (r_{t+1} - \mu_{s_t} - \phi_{s_t} r_t) \exp\left(-\frac{x_i}{2}\right)$$

and

$$u_{2,t+1}^{(i,j)} = (x_j - \omega_{s_t} - \beta_{s_t} x_i - \tilde{\rho}(t+1)\sigma_{v,s_{t+1}} u_{1,t+1}^{(i)})\sigma_{v,s_{t+1}}^{-1}.$$

All other steps are identical to those for the SVAD model. We follow the way proposed in [Smith, 2009] to construct the unconditional density for initializing. First one defines the two $N \times N$ matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ with the element (j, i)

$$\mathcal{A}_k^{(j,i)} = w_i \phi(x_i; \omega_k + \beta_k x_j, \sigma_{\eta,k}^2), \quad k = 0, 1,$$

which are used to generate

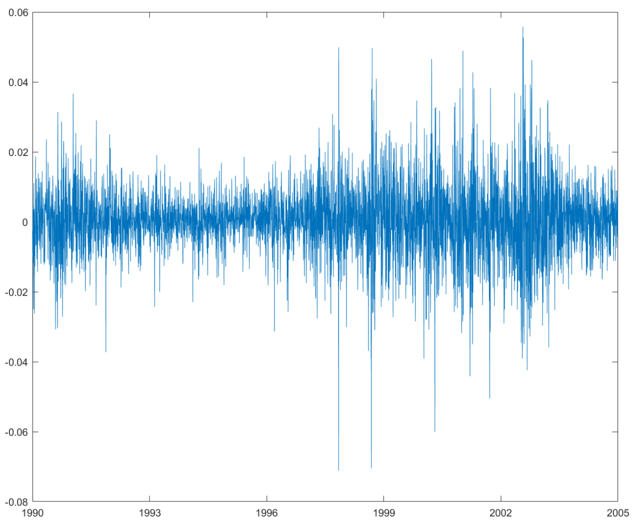
$$A = \lim_{n \to \infty} (0.5\mathcal{A}_0 + 0.5\mathcal{A}_0)^n \times A. \tag{17}$$

Note that one needs to find a suitable starting value and set a large value for n.

5 Empirical results

The empirical analysis uses the logarithmic returns of the value-weighted S&P 500 portfolio from January 1990 to December 2004. For the analysis the adjusted closing price of the returns was used, which was adjusted for dividends and splits. The development of the 3782 daily returns is shown in Figure 1.

Figure 1: The daily S&P 500 returns: 1990-2004.



In our empirical analysis we consider a range of SV models with various specifications for asymmetric volatility, which are all special cases of

$$r_t = \mu + \phi r_{t-1} + \exp(x_t/2) u_{1,t}, \tag{18}$$

$$x_{t+1} = \omega_0 + s_t\omega_1 + (\beta_0 + s_t\beta_1)x_t + \sqrt{\sigma_{0,\nu}^2 + s_t\sigma_{1,\nu}^2}\left(\rho(t)u_{1,t} + \sqrt{1-\rho^2(t)}u_{2,t}\right). \tag{19}$$

From (18) and (19) one can see that we consider the conditional mean as an AR(1) process, and only volatility dynamics change with the sign of lagged returns.

5.1 Results of Parameter Estimations

We report the results of parameter estimation and standard error (in parenthesis) in Table 1, the standard errors are computed using a numerically evaluated Hessian matrix. A comprehensive analysis and comparison between effects of constant correlation and threshold has already been done in [Smith, 2009], we will focus on analyzing the effect of time-dependent correlation. Furthermore, it has been indicated in [Smith, 2009] that the conclusions about the correlation and threshold effect are not driven by the choice of the numerical integration-based estimation method, by comparing to the Bayesian Markov chain Monte Carlo estimation [Meyer and Yu, 2000, Yu, 2005]. A comprehensive comparison to other estimation strategies, e.g., moment matching [Taylor, 1986, Wiggins, 1989], quasi-maximum likelihood [Harvey et al., 1994, Harvey and Shephard, 1996] etc. can be found in [Smith, 2009]. We do not repeat those comparisons in this paper, since our aim is to investigate the effect of the included time-dependent correlation. As in [Smith, 2009], we set both the parameters μ_1 and ϕ_1 to be 0 in the threshold models, since the impact on the quality of the models is quite small but the running time of parameter estimation increases sharply.

The returns are moderately autocorrelated for all the models, with values for ϕ ranging from 0.01213 to 0.02185, which are small but still of great statistical significance. The included constant or time-dependent correlation, or threshold dynamic leads to an increase in ϕ. When comparing the constant correlation with the time-dependent correlation it becomes clear that the included time-dependent correlation leads to a further increase in the autocorrelation of the return. However, when comparing the correlated threshold models (TSVA, TSVAD), the autocorrelation in the TSVAD model is slightly lower.

We see that the volatility is extremely persistent. For the models without the threshold dynamic (SV, SVA, SVAD), the values of β are between 0.98362 and 0.98965. Although both the constant or time-dependent correlations lead to a slight reduction in the persistence of the volatility, the persistence is higher in the SVAD model than it in the SVA model, which implies that the time-dependent correlation leads to an increase in the persistence compared to the constant correlation. For the threshold models (TSV, TSVA, TSVAD) there are two parameters for the persistence of the volatility β_0 and β_1. In the following, a distinction is made between $\beta_{s_t=0} = \beta_0$ and $\beta_{s_t=1} = \beta_0 + \beta_1$, since the persistence in the threshold models at time t depends on the sign of the return at time t-1. For the TSV model we have $\beta_{s_t=0}$=0.92002 and $\beta_{s_t=1}$=1.0581, for the TSVA $\beta_{s_t=0}$=0.93619 and $\beta_{s_t=1}$=1.03495 and for the TSVAD $\beta_{s_t=0}$= 0.93887 and $\beta_{s_t=1}$= 1.02709. It can be seen that the volatility is more persistent after negative returns in all those models, since $\beta_1 > 1$.

The parameter ω is the constant term of the AR(1) process of the log-volatility. Thus, ω_1 is used for the threshold models to capture an asymmetric relationship between the returns and volatility. If $\omega_1 > 0$, the volatility tends to be higher after negative returns than after positive returns. For the TSV model ω_1 is about 1.47273, for the TSVA model 0.96085 and for the TSVAD model 0.88053. This shows that for the given data the relationship between return and volatility is clearly asymmetric. A comparison of the pure correlation models (SVA, SVAD) shows that the time-dependent correlation leads to a slight increase of the value of the parameter ω. Again, a close examination of $\omega_{s_t=0} = \omega_0$ and $\omega_{s_t=1} = \omega_0 + \omega_1$ is useful for the threshold models. In the TSV model we have $\omega_{s_t=0} = -0.83631$ and $\omega_{s_t=1} = 0.63641$, in the TSVA model $\omega_{s_t=0} = -0.58986$ and $\omega_{s_t=1} = 0.37099$ and in the TSVAD model $\omega_{s_t=0} = -0.54878$ and $\omega_{s_t=1} = 0.33175$. The consideration of the constant or time-dependent correlation has thus a great influence on the parameters ω_0 and ω_1 of the threshold models. By taking any correlations into account, $\omega_{s_t=0}$ becomes larger and $\omega_{s_t=1}$ becomes smaller, i.e., the effect of threshold dynamics is slightly reduced. This indicate that threshold effects and constant or time-dependent correlation are complementary.

Table 1: Parameter estimates and standard errors for the various SV models: S&P 500, 1990-2004.

	SV	SVA	TSV	TSVA	SVAD	TSVAD
μ	0.00058	0.00028	0.00044	0.0003	0.00033	0.00028
	(0.00012)	(0.00013)	(0.00012)	(0.00013)	(0.00013)	(0.00012)
ϕ	0.01213	0.0216	0.01506	0.0211	0.02185	0.02053
	(0.0157)	(0.01564)	(0.01544)	(0.01565)	(0.01566)	(0.01534)
ω_0	-0.09852	-0.15618	-0.83631	-0.58986	-0.14888	-0.54878
	(0.03613)	(0.04404)	(0.02038)	(0.14354)	(0.039)	(0.02333)
β_0	0.98965	0.98362	0.92002	0.93619	0.98437	0.93887
	(0.00382)	(0.00468)	(0.00198)	(0.01555)	(0.00417)	(0.00165)
$\sigma^2_{0,\nu}$	0.01443	0.02379	0.02511	0.04025	0.02323	0.05044
	(0.00481)	(0.00665)	(0.00002)	(0.01781)	(0.0062)	(0.00005)
ρ		-0.66539		-0.66284		
		(0.04866)		(0.06157)		
ω_1			1.47273	0.96085		0.88053
			(0.04391)	(0.27325)		(0.01837)
β_1			0.13808	0.09876		0.08822
			(0.00361)	(0.02791)		(0.00145)
$\sigma^2_{1,\nu}$			-0.0248	-0.03244		-0.04959
			(0.00002)	(0.02147)		(0.00005)
κ					0.00798	0.10775
					(0.00219)	(0.10625)
σ_ρ					0.00323	0.0109
					(0.00211)	(0.0067)
μ_ρ					-9.96561	-0.76642
					(1.60657)	(0.20593)
$\rho(0)$					-0.28401	-0.76266
					(0.16154)	(0.05553)
LL	12470.79	12515.01	12505.74	12524.26	12521.46	12526.84
Joint	8.56876	3.87625	2.14291	3.48463	3.63333	3.84495
p-value	[0.01378]	[0.14397]	[0.34251]	[0.17511]	[0.16257]	[0.14624]
Sign	8.55167	3.90394	2.15771	3.52521	3.66229	3.87712
p-value	[0.00345]	[0.04817]	[0.14186]	[0.06044]	[0.05566]	[0.04895]
Neg. Size	5.16399	0.4785	0.30172	0.05241	0.38397	0.3187
p-value	[0.02306]	[0.4891]	[0.58281]	[0.81892]	[0.53549]	[0.57239]

The parameter $\sigma^2_{0,\nu}$ describes the variance of the volatility. For the SV model, the corresponding value is 0.01443. By taking the correlation into account, this value increases to 0.02379 for the SVA model and 0.02323 for the SVAD model. Both the constant and the time-dependent correlations have thus a great influence on the parameter $\sigma^2_{0,\nu}$ as well. For the threshold models again the parameters $\sigma^2_{s_t=0,\nu}= \sigma^2_{0,\nu}$ and $\sigma^2_{s_t=1,\nu}= \sigma^2_{0,\nu} + \sigma^2_{1,\nu}$ are interesting, which describe the variance of the volatility after positive and negative returns, respectively. $\sigma^2_{s_t=0,\nu}$ is 0.02511 for the TSV model, 0.04025 for the TSVA model and 0.05044 for the TSVAD model. $\sigma^2_{s_t=1,\nu}$ is 0.00004 for the TSV model, 0.0168 for the TSVA model and 0.00085 for the TSVAD model. The variance of volatility is extremely low after negative returns. The comparison of the constant and the time-dependent correlations in the threshold models show that capturing the time-dependent correlation decreases the variance of the volatility after negative returns even further and increase the variance after positive returns, thus amplifying the threshold dynamic in this context.
Firtsly, we see that the volatility is more persistent, less volatile and higher following negative returns after counting for the constant correlation. This is consistent with findings in [Smith, 2009]. And the phenomenon is further enhanced by using time-dependent correlation instead of constant correlation.

5.2 Comparison of stochastic volatility models

To check whether models are not too complex and do not contain unnecessarily many parameters, both the Akaike information criterion (AIC) and the Bayes information criterion (BIC) can be used:

$$AIC = 2k - 2\log L(\theta; y),$$
$$BIC = k\log(T) - 2\log L(\theta; y),$$

where k is the number of parameters in the model and T is the number of observations (in this case, the number of given returns). A model is preferred if it has the lower AIC/BIC value. Both criteria penalize models when they have many parameters, with the BIC penalizing more than the AIC.

Table 2: AIC und BIC

	LL	AIC	BIC
SV	12470.79	-24931.58	-24923.69
SVA	12515.01	-25018.02	-25008.55
TSV	12505.74	-24995.48	-24982.86
TSVA	12524.26	-25030.52	-25016.32
SVAD	12521.46	-25024.92	-25010.72
TSVAD	12526.84	-25029.67	-25010.74

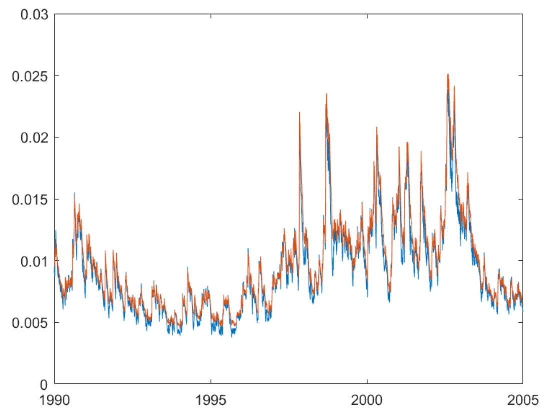
The results in Table 2 show that the SV model performs worst in all categories. This is to be expected since the SV model is the only model that cannot well capture the asymmetric relationship between the return and volatility. It can be concluded that for the given returns, capturing the asymmetric relationship by including correlation and threshold leads to a significant improvement in the modeling. It is also hardly surprising that the SVA model is clearly superior to the TSV model for both AIC and BIC, since the SVA model already achieves a better log-likelihood value even with less parameters. The TSV model is clearly superior to the SV model and the TSVA model performs better than the SVA model for both the log-likelihood value and both information criteria. The information criteria also suggests that including both the correlation and threshold dynamics is useful, as they appear to complement each other. The included time-dependent correlation instead of constant correlation leads to an increase in the log-likelihood values of the models. The log-likelihood value of the SVAD model is about 6.45 higher than the log-likelihood value of the SVA model. The difference between the TSVAD and TSVA models is about 2.58. The SVAD model also achieves better AIC and BIC values despite having more parameters than the SVA model. The TSVAD model has the best log-likelihood value of all models, but the TSVAD model performs slightly worse than the TSVA model for the AIC and BIC due to the higher number of parameters.

5.3 Volatility dynamics of the models

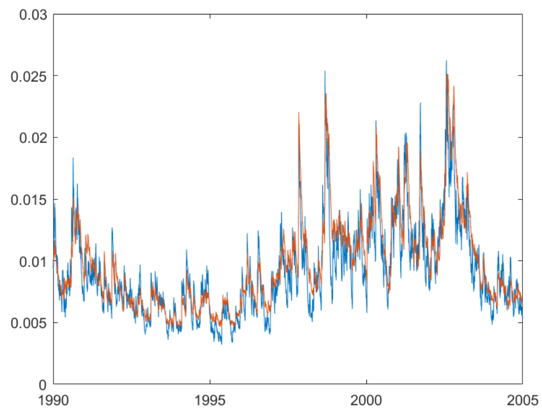
The dynamics of the conditional volatility of the presented stochastic volatility models (calculated with the parameter estimates) is shown in Figure 2, along with the volatility by the GARCH(1,1) model. Firstly, it is clear that all the presented stochastic volatility models and GARCH(1,1) model have a similar volatility dynamic. However, for the models that can capture an asymmetric relationship of return and volatility, the pattern is much more volatile. For the SV model, which can not capture the asymmetric relationship, the deviation from the GARCH(1,1) model is on average 0.00053. For all stochastic volatility models that capture the asymmetric relationship (SVA, SVAD, TSV, TSVA, TSVAD), the value of the average deviation is at least 0.00121. The TSVAD model, which has the best log-likelihood value of all models, also has the largest average deviation from the GARCH(1,1) model with 0.00175.

Figure 2: Plot of forecast conditional standard deviation for the data in Figure 1

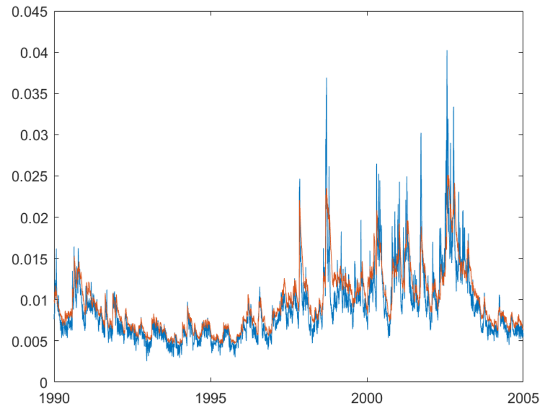
(a) SV



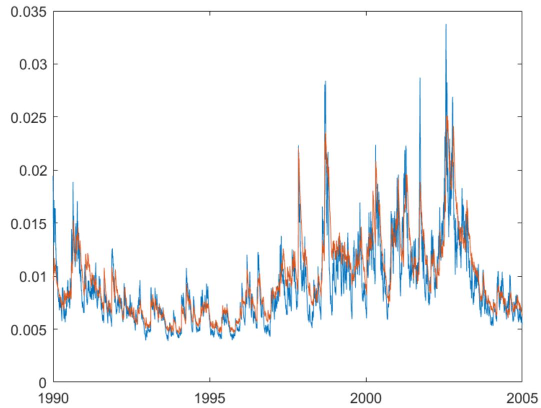
(b) SVA



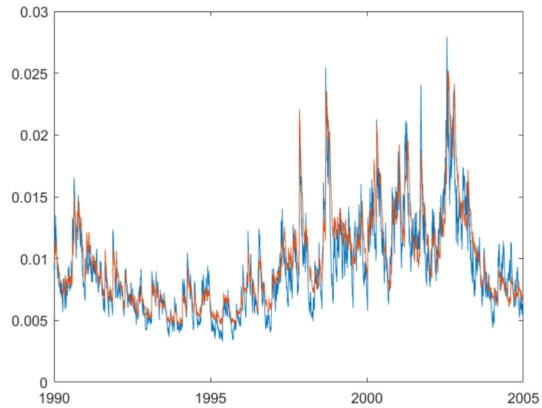
(c) TSV



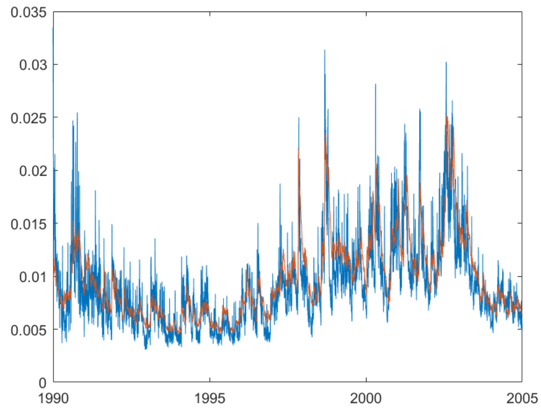
(d) TSVA



(e) SVAD



(f) TSVAD



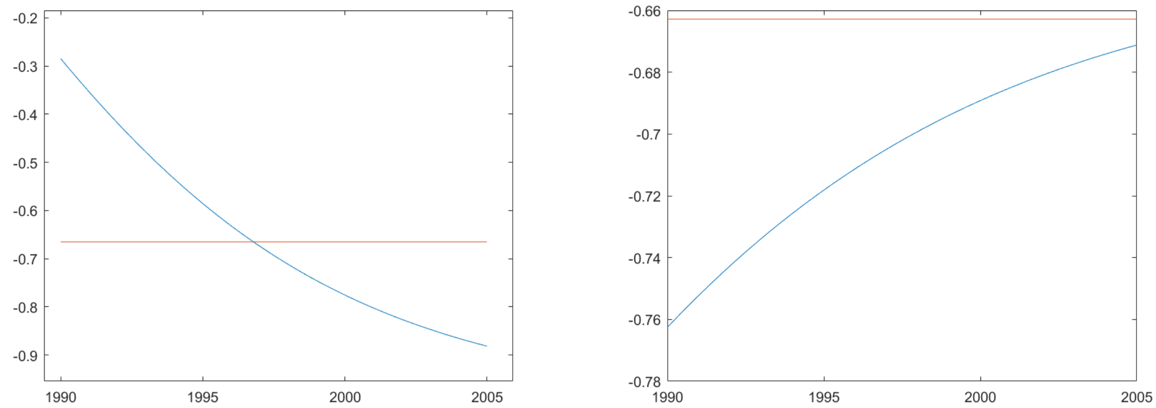
5.4 Correlation in the models

The correlations between the innovations of the return and the future volatility are -0.66539 and -0.66284 for the SVA and TSVA models, respectively, and are significant. And the log-likelihood values are thus significantly improved in both the cases. In Figure 3(a) and 3(b), the constant correlation are compared with the time-dependent correlations (calculated with the parameter estimates) for the SVA, SVAD, TSVA, and TSVAD models. Firstly, it is noticeable that the time-dependent correlations in both SVAD and TSVAD models vary strongly and reasonably. Note that significantly better log-likelihood values have been obtained in those models. We compare the constant and time-dependent correlations without threshold effect in Figure 3(a), in which the time-dependent correlation varies from -0.2846 to -0.8817, and the constant correlation value (-0.66539) lies approximately in the middle of that range. In Figure 3(b) we compare the constant and time-dependent correlation in the TSVA and TSVAD models, namely the threshold effects are included. Obviously, threshold changes values of both constant and time-dependent correlations, since threshold and correlation work complementarily. Interestingly, we see that the time-dependent correlation varies from -0.76266 to -0.67122, which seems to converge to the constant correlation (-0.66284).

In the correlated stochastic volatility models, the volatility after negative returns is higher than after positive returns. For the pure correlation models (SVA, SVAD), this can be shown mathematically. The difference depends on the magnitude of the innovation z_t of the return, since z_t is negatively correlated with v_t. Note that $v_t = \rho(t)u_{1,t} + \sqrt{1-\rho(t)^2}u_{2,t}$ and $u_{1,t} = z_t$,

Figure 3: Correlations in the SVA-, SVAD-, TSVA- and TSVAD-Models

(a) SVA-SVAD (b) TSVA-TSVAD



if only the sign of z_t in the SVA or SVAD model is used, one obtains

$$\begin{aligned}
&E[\log \sigma_{t+1}^2 | z_t < 0] - E[\log \sigma_{t+1}^2 | z_t > 0] = \sigma_v (E[v_t | z_t < 0] - E[v_t | z_t > 0]) \\
&= \sigma_v (E[\rho(t) u_{1,t} + \sqrt{1-\rho(t)^2} u_{2,t} | z_t < 0] - E[\rho(t) u_{1,t} + \sqrt{1-\rho(t)^2} u_{2,t} | z_t > 0]) \\
&= \sigma_v (E[\rho(t) z_t | z_t < 0] - E[\rho(t) z_t | z_t > 0]) \\
&= -\sigma_v \rho(t) E[|z_t|] = -\sigma_v \rho(t) \sqrt{\tfrac{2}{\pi}}.
\end{aligned}$$

For the SVA model, the volatility is 8.189% higher after negative innovations of the return than it after positive innovations. Interestingly, for the SVAD model, that value varies from 3.502% to 10.852% with time. How much higher is the volatility following negative returns than it following positive returns should be different at the different time points, this can be captured by including time-dependent correlation. Furthermore, a clear asymmetric relationship of return and volatility is captured in both models.

Vuong test

Another way to test the quality of modeling is the Vuong test [Vuong, 1989]. This is a likelihood ratio test, which tests the null hypothesis for two non-nested models whether both the models are equally close to the true distribution or whether one of the models is closer to the true distribution. The Vuong test shows for two non-nested models whether model 1 is preferred over model 2 with significance level α if:

$$Z = \frac{LR_N}{\sqrt{N}\omega_N}$$

with

$$LR_N = L_N^1 - L_N^2 - \frac{K_1 - K_2}{2} \log N$$

and

$$\omega_N^2 = \frac{1}{N} \sum_{t=1}^{N} \left[\log \frac{f_1(r_t)}{f_2(r_t)} \right]^2 - \left[\frac{1}{N} \sum_{t=1}^{N} \log \frac{f_1(r_t)}{f_2(r_t)} \right]^2$$

exceeds the (negative) (1-α) quantile of the standard normal distribution. In this case, N corresponds to the number of given returns, K_1 and K_2 correspond to the number of parameters of models 1 and 2, and L_1 and L_2 correspond to the log-likelihood values of models 1 and 2. When comparing the SVA model with the TSV model, the value of the Vuong test is $Z = 2.1149$

(p-value=0.03445). The pure correlation model is clearly superior to the pure threshold model here. When comparing the TSVA model with the SVAD model, i.e., threshold plus constant correlation together against time-dependent correlation, Z is only 0.53151 (p-value=0.59507) and thus the values are too small to evaluate which of both the models is better.

5.5 Misspecification Tests

In this paper, we use the specification tests proposed in [Smith, 2009]. The following tests are used to verify that the stochastic volatility models can completely capture the effect of past returns on conditional volatility. Here, the focus is primarily on the negative returns. For this purpose, the sign and negative size tests [Engle and Ng, 1993] are well suited. In order to apply these tests to the presented stochastic volatility models, the probability integral transformation [Rosenblatt, 1952] is used to construct a pseudo-standardized residual. If a stochastic volatility model is correctly specified, then the probability integral transform $F_t(r_{t+1})$ is an independent uniform random variable and $Z_{t+1} = \Phi^{-1}(F_t(r_{t+1}))$ is standard normally distributed, i.e., Z_{t+1} is a pseudo-standardized residual. We test whether Z_{t+1} is orthogonal to the previous negative returns. This would indicate that the stochastic volatility model fully captures the information from the value and sign of the previous returns.
For the sign test, it is of interest whether Z^2_{t+1} is uncorrelated with s_t and for the negative size test, whether Z^2_{t+1} is uncorrelated with $s_t r_t$. We take the approach using moments $m(Z_{t+1}, \theta)$ proposed in [Breunig et al., 2003]. The null hypothesis under correct specification implies that $E(m_t) = \tau_0$ holds. Thus, the following moments are considered for the tests

$$m(Z_{t+1}, \theta) = \begin{pmatrix} Z^2_{t+1} s_t \\ Z^2_{t+1} s_t r_t \end{pmatrix}.$$

For the specifications, the calculated parameter estimates are used for $\hat{\theta}$. It holds

$$\hat{\tau} = \frac{1}{T} \sum_{t=0}^{T-1} m(Z_{t+1}, \hat{\theta}),$$

which follows an asymptotic normal distribution

$$T^{\frac{1}{2}}(\hat{\tau} - \tau_0) \rightarrow N(0, V_\tau) \tag{20}$$

under suitable regulatory conditions with the covariance matrix

$$V_\tau = V_{mm} - M_\theta^{'} I_{\theta\theta}^{-1} M_\theta,$$

where $V_{mm} = \lim_{T\to\infty} E(\frac{1}{T}\sum_{t=1}^{T} m(r_t; \theta) m(r_t; \theta)^{'})$ is the asymptotic covariance matrix of the moment conditions, $I_{\theta\theta} = \lim_{T\to\infty} E(-\frac{1}{T}\frac{\delta^2}{\delta\theta\delta\theta'})$ and $M_\theta = \lim_{T\to\infty} E(\frac{\delta m(r_t;\theta)}{\delta\theta'})$ is the asymptotic Jacobian matrix of moment conditions. The results of the Wald tests are reported in Table 1.
For the analysis, the standard significance level $\alpha = 0.05$ is used. Unsurprisingly, the SV model fails the joint, sign, and negative size tests, as this model is unable to capture the asymmetric relationship of the returns and volatility. The other models that are able to capture this asymmetric relationship and pass the joint and negative size test. The sign test is passed by the TSV, TSVA and SVAD models, but the SVA model and the TSVAD model fail by a small margin (p=0.04817 and p=0.04895). When comparing the pure correlation models (SVA, SVAD), it becomes clear that the time-dependent correlation improves result of the sign test. The result suggests that Z^2_{t+1} is not correlated with s_t in the SVAD model, but correlated in the SVA model. Furthermore, one can observe that the threshold and correlation, in particular time-dependent correlation, adversely affect with each other, these indicate again that threshold and correlation effect are complementary.

6 Conclusion

In this work we have proposed the SVAD and TSVAD models to capture the asymmetric relationship between returns and volatility. These new models are constructed by including time-dependent correlation into the SVA and TSVAD models instead of constant correlation, respectively. This is to say that they nest the basis SV, SVA, TSV and TSVA models. We show how the model parameters can be estimated by using maximum likelihood with a nonlinear numerical integration-based filter. Our simulation study results using the S&P 500 returns show that using time-dependent correlation rather than constant correlation leads to a significant improvement in modelling. Based on the values of the log-likelihood, AIC and BIC, our SVAD model is preferred over the SVA model. Furthermore, our most comprehensive TSVAD model can be better fitted to the data than all other models.

References

[Black, 1976] Black, F. (1976). Studies of stock price volatility changes. In *Proceedings of the Business and Economic Statistical Section*. American Statistical Association, Washington DC.

[Breunig et al., 2003] Breunig, R., Najarian, S., and Pagan, A. (2003). Specification testing of markov switching models. *Oxf. Bull. Econ. Stat.*, 65:703–725.

[Buraschi et al., 2010] Buraschi, A., Porchia, P., and Trojani, F. (2010). Correlation risk and optimal portfolio choice. *J. Finance*, 65(1):393–420.

[Campbell and Hentschel, 1992] Campbell, J. and Hentschel, L. (1992). No news is good news: An asymmetric model of changing volatility in stock returns. *J. Financ. Econ.*, 31(3):281–318.

[Chesney and Scott, 1989] Chesney, M. and Scott, L. (1989). Pricing european currency options: A comparison of the modified black-scholes model and a random variance model. *J. Finan. Quant. Anal.*, 24(3):267–284.

[Christie, 1982] Christie, A. A. (1982). The stochastic behavior of common stock variances:value, leverage and interest rate effects. *J. Financ. Econ.*, 10(4):407–432.

[Duffie et al., 2003] Duffie, D., Filipovic, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *Ann. Appl. Probab.*, 13(3):984–1053.

[Engle and Ng, 1993] Engle, R. F. and Ng, V. K. (1993). Measuring and testing the impact of news on volatility. *J. Financ. Econ.*, 48(5):1749–1778.

[French et al., 1987] French, K. R., Schwert, G. W., and Stambaugh, R. F. (1987). Expected stock returns and volatility. *J. Financ. Econ.*, 19(1):3–29.

[Fridman and Harris, 1998] Fridman, M. and Harris, L. (1998). A maximum likelihood approach for non-gaussian stochastic volatility models. *J. Bus. Econ. Stat.*, 16(3):284–291.

[Glosten et al., 1993] Glosten, L. R., Jagannathan, R., and Runkle, D. E. (1993). On the relation bewteen the expecated value and the volatility of the nominal excess return on stocks. *J. Finance*, 48(5):1779–1801.

[Goetzmann et al., 2005] Goetzmann, W. N., Li, L., and Rouwenhorst, K. G. (2005). Long-term global market correlations. *J. Bus.*, 78(1):1–38.

[Harvey et al., 1994] Harvey, A. C., Ruiz, E., and Shephard, N. (1994). Multivariate stochastic variance models. *J. Bus. Econ. Stat.*, 61(2):247–264.

[Harvey and Shephard, 1996] Harvey, A. C. and Shephard, N. (1996). Estimation of an asymmetric stochastic volatility model for asset returns. *J. Bus. Econ. Stat.*, 14(4):429–434.

[Jacquier et al., 2004] Jacquier, E., Polson, N. G., and Rossi, P. E. (2004). Bayesian analysis of stochastic volatility models with fat-tails and correlated errors. *J. Econom.*, 122(1):185–212.

[Kahneman and Tversky, 1979] Kahneman, D. and Tversky, A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica*, 47(2):263–291.

[Kitagawa, 1987] Kitagawa, G. (1987). Non-gaussian state-space modeling of nonstationary time series. *J. Am. Stat. Assoc.*, 82(400):1032–1041.

[Longin and Solnik, 1995] Longin, F. and Solnik, B. (1995). Is the correlation in international equity returns constant: 1960-1990? *J. Int. Money Finance*, 14(1):3–26.

[Meyer and Yu, 2000] Meyer, A. and Yu, J. (2000). Bugs for a bayesian analysis of stochastic volatility models. *J. Econom.*, 3(1):198–215.

[Nelson, 1991] Nelson, D. (1991). Conditional heteroscedasticity in stock returns: A new approach. *Econometrica*, 59:347–370.

[Omori et al., 2007] Omori, Y., Chib, S., Shephard, N., and Nakajima, J. (2007). Stochastic volatility with leverage: fast and efficient likelihood inference. *J. Econom.*, 140(2):425–449.

[Rosenblatt, 1952] Rosenblatt, M. (1952). Remarks on a multivariate transformation. *Ann. Math. Stat.*, 23(3):470–472.

[Schwert, 1989] Schwert, G. W. (1989). Why does stock market volatility change over time? *J. Finance*, 44(5):1115–1153.

[Smith, 2009] Smith, D. R. (2009). Asymmetry in stochastic volatility models: threshold or correlation? *Stud. Nonlinear Dyn. Econ.*, 13(3).

[So et al., 2003] So, M. K., Li, W., and Lam, K. (2003). On a threshold stochastic volatility model. *Int. J. Forecast.*, 22:473–500.

[Taylor, 1986] Taylor, S. (1986). *Modelling Financial Time Series*. John Wiley and Sons, UK.

[Teng and Clevenhaus, 2019] Teng, L. and Clevenhaus, A. (2019). Accelerated implementation of the adi for the heston model with stochastic correlation. *J. Comput. Sci-neth.*, 36.

[Teng et al., 2015a] Teng, L., Ehrhardt, M., and Günther, M. (2015a). Option pricing with dynamically correlated stochastic interest rate. *Acta. Math. Uni. Comenianae*, LXXXIV(2):179–190.

[Teng et al., 2015b] Teng, L., Ehrhardt, M., and Günther, M. (2015b). The pricing of quanto options under dynamic correlation. *J. Comput. Appl. Math.*, 275:304–310.

[Teng et al., 2016a] Teng, L., Ehrhardt, M., and Günther, M. (2016a). The dynamic correlation model and its application to the heston model. In Glau, K., Grbac, Z., Scherer, M., and Zagst, R., editors, *Innovations in Derivatives Markets*. Springer, Cham.

[Teng et al., 2016b] Teng, L., Ehrhardt, M., and Günther, M. (2016b). Modelling stochastic correlation. *J. Math. Ind.*, 6(1):1–18.

[Teng et al., 2016c] Teng, L., Ehrhardt, M., and Günther, M. (2016c). On the heston model with stochastic correlation. *Int. J. Theor. Appl. Finan.*, 19(06):16500333.

[Teng et al., 2018a] Teng, L., Ehrhardt, M., and Günther, M. (2018a). Numerical simulation of the heston model with stochastic correlation. *Int. J. Financial Stud.*, 6(1)(3).

[Teng et al., 2018b] Teng, L., Ehrhardt, M., and Günther, M. (2018b). Quanto pricing in stochastic correlation models. *Int. J. Theor. Appl. Finan.*, 21(05):1850038.

[Teng et al., 2016d] Teng, L., van Emmerich, C., Ehrhardt, M., and Günther, M. (2016d). A versatile approach for stochastic correlation using hyperbolic functions. *Int. J. Comput. Math.*, 93(3):524–539.

[Teng et al., 2020] Teng, L., Wu, X., Günther, M., and Ehrhardt, M. (2020). A new methodology to create valid time-dependent correlation matrices via isospectral flows. *ESAIM: (M2AN)*, 54(1):361–371.

[Tse, 2000] Tse, Y. K. (2000). A test for constant correlation in a multivariate garch model. *J. Econom.*, 98(1):107–127.

[Uhlenbeck and Ornstein, 1930] Uhlenbeck, G. E. and Ornstein, L. S. (1930). On the theory of brownian motion. *Phys. Rev.*, 36:823–841.

[Vuong, 1989] Vuong, . (1989). Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica*, 57:307–333.

[Wiggins, 1989] Wiggins, J. B. (1989). Option values under stochastic volatility: theory and empirical estimates. *J. Financ. Econ.*, 19:351–372.

[Wu and Zhou, 2014] Wu, X. and Zhou, H. (2014). A triple-threshold leverage stochastic volatility model. *Stud. Nonlinear Dyn. Econ.*, 19:483–500.

[Yu, 2005] Yu, J. (2005). On leverage in a stochastic volatility model. *J. Econom.*, 127(2):165–178.