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Riesz bases of port-Hamiltonian systems^{*}

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Abstract

The location of the spectrum and the Riesz basis property of well-posed homogeneous infinite-dimensional linear port-Hamiltonian systems on a 1D spatial domain are studied. It is shown that the Riesz basis property is equivalent to the fact that system operator generates a strongly continuous group. Moreover, in this situation the spectrum consists of eigenvalues only, located in a strip parallel to the imaginary axis and they can be decomposed into finitely many sets having each a uniform gap.

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1 Introduction

It is well-known that the eigenvectors of a compact self-adjoint operator form an orthonormal basis of the underlying Hilbert space. In the 1960s Dunford and Schwarz [DS71] introduced the more general notion of spectral operators. Further, Curtain [Cur84] analysed discrete spectral operators, i.e., spectral operators with compact resolvent, and the class of Riesz spectral operators was formulated in [CZ95] and extended in [GZ01] to characterize also operators with multiple eigenvalues. For Riesz spectral operator its eigenvectors still form a basis, but this basis is assumed to be Riesz basis. Since a Riesz basis is isomorphic to an orthonormal basis, many of the nice properties of compact self-adjoint operators carry over to Riesz spectral operators. For instance, solutions of the abstract differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, with A a Riesz spectral operator, can still be described by an eigenfunction expansion of non-harmonic Fourier series. This enables that many properties of these infinite-dimensional

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systems such as stability, stabilizability and controllability can be characterized in an elegant manner, see e.g. [CZ95, CZ20].

In this article, we investigate the Riesz basis property of a special class of infinite-dimensional systems, namely port-Hamiltonian systems on a 1D spatial domain. Here by the Riesz basis property we mean that the associated system operator is a discrete Riesz spectral operator, see Definition 2.2. Port-based network modelling of complex physical systems leads to port-Hamiltonian systems. For finite-dimensional systems there is by now a well-established theory [vdS06, EMvdS07, DMSB09]. The port-Hamiltonian approach has been extended to the infinite-dimensional situation by a functional analytic approach in [Vil07, ZLMV10, JZ12, JMZ15, Aug16, JZ18]. This approach has been successfully used to derive simple verifiable conditions for well-posedness [LGZM05, Vil07, ZLMV10, JZ12, JMZ15, JK19b], stability [JZ12, AJ14] and stabilization [RZLG17, RLGMZ14, AJ14, SZ18] and robust regulation [HP18]. The study of the Riesz basis property for infinite-dimensional port-Hamiltonian systems has started with the thesis by Villegas [Vil07, Chapter 4]. Using results on first order eigenvalue problems by Tretter [Tre00a, Tre00b], he obtained a sufficient condition. However, it is not easy to see when this technical sufficient condition is satisfied.

Many systems have a Riesz basis of eigenfunctions, see e.g. [GX04, XG03]. In the monograph [GW19, Section 4.3] B.-Z. Guo and J.-M. Wang study the Riesz basis property for a closely related class of systems, that is, hyperbolic systems of the form $\frac{\partial x}{\partial t} = K(\zeta)\frac{\partial x}{\partial \zeta} + C(\zeta)x$ with K and C diagonal. Note, that (almost) every port-Hamiltonian system on a one dimensional spatial domain can be transformed into a hyperbolic system of this form. However, in general not with a diagonal C . Furthermore, the boundary conditions will be more general, see [ZLMV10, JZ12] or the proof of Lemma 5.1. Our main result thus generalizes their theorem [GW19, Theorem 4.11]. We remark, that in this situation the notions of Riesz basis of subspaces and Riesz basis with parentheses are equivalent. Moreover, in [XW11] the Riesz basis property is investigated for operators perturbed by output feedback.

Our main result shows that a linear infinite-dimensional port-Hamiltonian system on a 1D spatial domain has the Riesz basis property if and only if the system operator generates a strongly continuous group. Important to note here is that we don't need constant coefficients, nor extra assumption. Since the group property is equivalent to a simple matrix condition, our results enable us to very quickly check whether the Riesz basis property holds, see also our example section. Our proof combines methods from complex analysis, differential equations and mathematical systems theory. In particular, we use the fact that every well-posed port-Hamiltonian control system (12) is exactly controllable in finite time. We refer the reader to Section 4 for the definition of exact controllability.

Notation: X and Y are complex and separable Hilbert space. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. To shorten notation we write $\mathcal{L}(X) := \mathcal{L}(X, X)$. Throughout this article we assume that

$A : \mathcal{D}(A) \subset X \rightarrow X$ is a closed, densely defined, linear operator. The set $\sigma(A)$ refers to the spectrum of A and $\sigma_p(A)$ to its point spectrum. We denote by $\rho(A) := \mathbb{C} \setminus \sigma(A)$ the resolvent set of A , and for each $s \in \rho(A)$ we denote the resolvent of A by $(s - A)^{-1} := (sI - A)^{-1}$. The growth bound of a C_0 -semigroup $(T(t))_{t \geq 0}$ with generator A is denoted by $\omega_0(A)$ and $s(A) := \sup_{s \in \sigma(A)} \operatorname{Re} s$ is the *spectral bound* of A . $H^1((0, 1); \mathbb{C}^d)$ denotes the first order Sobolev space on the interval $(0, 1)$.

2 Main Result

In this section we formulate our main result. We start with the definition of discrete Riesz spectral operators.

Definition 2.1. For an operator A on X we call $\gamma \subset \sigma(A)$ a *compact spectral set* if γ is a compact subset of \mathbb{C} which is open and closed in $\sigma(A)$. The *spectral projection* on the spectral subset γ is defined as

$$E(\gamma) = \frac{1}{2\pi i} \int_{\Gamma} (s - A)^{-1} ds,$$

where Γ is a closed Jordan curve containing every point of γ and no point of $\sigma(A) \setminus \gamma$.

In this article operators with compact resolvent are of particular interest. The spectrum of these operators is a denumerable set of points with no finite accumulation point, c.f. [DS71, Lemma XIX.2]. Furthermore, every point in the spectrum is an eigenvalue which has finite algebraic as well as finite geometric multiplicity, c.f. [GGK90, Theorem XV.2.3]. If $(s_n)_{n \in \mathbb{N}}$ is the spectrum of an operator with compact resolvent we write $E_n := E((s_n))$, $n \in \mathbb{N}$.

Definition 2.2. An operator A with compact resolvent and $\sigma(A) = (s_n)_{n \in \mathbb{N}}$ is a *discrete Riesz spectral operator*, if

1. for every $n \in \mathbb{N}$ there exists $N_n \in \mathcal{L}(X)$ such that $AE_n = (s_n + N_n)E_n$,
2. the sequence of closed subspaces $(E_n(X))_{n \in \mathbb{N}}$ is a *Riesz basis of subspaces* of X , that is, $\operatorname{span}(E_n(X))_{n \in \mathbb{N}}$ is dense and there exists an isomorphism $T \in \mathcal{L}(X)$, such that $(TE_n(X))_{n \in \mathbb{N}}$ is system of pairwise orthogonal subspaces of X .
3. $N := \sum_{n \in \mathbb{N}} N_n$ is bounded and nilpotent.

Remark 2.3. Discrete Riesz spectral operator are spectral operators in the sense of Dunford and Schwartz [DS71]. However, we additionally assume that the operator has a compact resolvent, which are discrete operators in the sense of Dunford and Schwartz [DS71, Definition XIX.1]. Every discrete Riesz spectral operator is a Riesz spectral operator in the sense of Guo and Zwart [GZ01] and these are again spectral operator in the sense of Dunford and Schwartz.

In Curtain and Zwart [CZ95] a slightly stronger notion is considered, where all eigenvalues have to be simple. However, they do not require that the operator has a compact resolvent.

Remark 2.4. If a sequence of vectors $(x_n)_{n \in \mathbb{N}}$ in X is a *Riesz basis* of X , that is, there is an isomorphism $T \in \mathcal{L}(X)$, such that $(Tx_n)_{n \in \mathbb{N}}$ is an orthonormal basis of X , then clearly $\{\text{span } x_n\}_{n \in \mathbb{N}}$ is a Riesz basis of subspaces of X .

Remark 2.5. If A is a discrete Riesz operator, then clearly E_n commute with A and A is equivalent to the infinite matrix

$$A = \text{diag}(A_1, A_2, \dots, A_n, \dots),$$

where A_n is a square matrix which corresponds to the restriction $A|_{E_n X}$ of A . Then $A = \sum_{n \in \mathbb{N}} i_n A_n E_n$, where i_n is the (natural) inclusion operator and A_n is identified with $(s_n + N_n)|_{E_n}$.

We consider first order linear port-Hamiltonian systems on a one-dimensional spatial domain of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) (\mathcal{H}(\zeta)x(\zeta, t)), \\ x(\zeta, 0) &= x_0(\zeta), \\ 0 &= \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \end{aligned} \quad (1)$$

where $\zeta \in [0, 1]$ and $t \geq 0$ and the Assumption 2.6 is fulfilled. Equation (1) describes a special class of port-Hamiltonian systems, which cover in particular the wave equation, the transport equation and the Timoshenko beam as well as coupled systems. For more information we refer to [JZ12, JZ18].

Assumption 2.6. The $d \times d$ Hermitian matrix P_1 is invertible, P_0 is a $d \times d$ skew-symmetric matrix, $\begin{bmatrix} W_1 & W_0 \end{bmatrix}$ is a full row rank $d \times 2d$ -matrix, and $\mathcal{H}(\zeta)$ is a positive $d \times d$ Hermitian matrix for a.e. $\zeta \in (0, 1)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C}^{d \times d})$. Thus, the matrix $P_1 \mathcal{H}(\zeta)$ can be diagonalized as $P_1 \mathcal{H}(\zeta) = S^{-1}(\zeta) \Delta(\zeta) S(\zeta)$, where $\Delta(\zeta)$ is a diagonal matrix and $S(\zeta)$ is an invertible matrix for a.e. $\zeta \in (0, 1)$. We suppose the technical assumption that $S^{-1}, S, \Delta : [0, 1] \rightarrow \mathbb{C}^{d \times d}$ are continuously differentiable.

Definition 2.7. Let P_0, P_1, \mathcal{H} satisfy Assumption 2.6 and $X := L^2((0, 1); \mathbb{C}^d)$. Then the operator $A : \mathcal{D}(A) \subset X \rightarrow X$ defined by

$$Ax := \left(P_1 \frac{d}{d\zeta} + P_0 \right) (\mathcal{H}x), \quad x \in \mathcal{D}(A), \quad (2)$$

$$\mathcal{D}(A) := \left\{ x \in X \mid \mathcal{H}x \in H^1((0, 1); \mathbb{C}^d) \text{ and } \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0 \right\}, \quad (3)$$

is called *port-Hamiltonian operator*.

We equip the space $X := L^2((0, 1); \mathbb{C}^d)$ with the energy norm $\sqrt{\langle \cdot, \mathcal{H} \cdot \rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2((0, 1); \mathbb{C}^d)$. Due to our Assumption 2.6 the energy norm is equivalent to the standard norm on $L^2((0, 1); \mathbb{C}^d)$. Furthermore, we emphasize that a port-Hamiltonian operator which generate a C_0 -semigroup is closed and that its resolvent is compact, see [Aug16]. Let A generate a C_0 -semigroup $(T(t))_{t \geq 0}$ on X . Then a closed subspace $V \subset X$ is called $(T(t))_{t \geq 0}$ -invariant, if $T(t)V \subseteq V$ for all $t \geq 0$. In this case the restriction $(T(t)|_V)_{t \geq 0}$ is again a C_0 -semigroup with generator $A|_V$ on V , c.f. [CZ95, Lemma 2.5.3]. We say that a set $(s_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ has *uniform gap*, if $\inf_{n \neq m} |s_n - s_m| > 0$ for $n, m \in \mathbb{N}$. By $Z^+(\zeta)$, we denote the span of the eigenvectors of $P_1 \mathcal{H}(\zeta)$ corresponding to the positive eigenvalues of $P_1 \mathcal{H}(\zeta)$ and by $Z^-(\zeta)$ the span of the eigenvectors of $P_1 \mathcal{H}(\zeta)$ corresponding to the negative eigenvalues of $P_1 \mathcal{H}(\zeta)$.

We are now in the position to formulate our main result.

Theorem 2.8. *Let A be a port-Hamiltonian operator and the generator of a C_0 -semigroup. Then the following is equivalent:*

1. *A is a discrete Riesz spectral operator.*
2. *A is the generator of a C_0 -group.*
3. *$W_1 \mathcal{H}(1)Z^+(1) \oplus W_0 \mathcal{H}(0)Z^-(0) = W_1 \mathcal{H}(1)Z^-(1) \oplus W_0 \mathcal{H}(0)Z^+(0) = \mathbb{C}^d$.*

If one of the equivalent conditions are satisfied, then $\sigma(A) = \sigma_p(A)$ lie in a strip parallel to the imaginary axis, the eigenvalues (counted according to the algebraic multiplicity) can be decomposed into finitely many sets each having a uniform gap and A satisfies the spectrum determined growth assumption, that is, $\omega_0(A) = s(A)$.

The proof of Theorem 2.8 will be given in Section 5.

Remark 2.9. It is particularly easy to check if a port-Hamiltonian operator A is the generator of a unitary C_0 -group, c.f. [JMZ15, Theorem 1.1]. This is actually the case if and only if $\widetilde{W}_B \Sigma \widetilde{W}_B^* = 0$, where $\widetilde{W}_B := [w_1 \ w_0] \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1}$ and $\Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. In this case A is even a skew-adjoint operator by Stone's Theorem, c.f. [EN00, Theorem VII.3.24], which implies that the normalized eigenvectors form an orthonormal basis of X .

3 Preliminaries on discrete Riesz spectral operators

Riesz bases of subspace have the following useful characterizations.

Proposition 3.1. [GZ01, Definition 1.4] *Let A be an operator with compact resolvent and $\sigma(A) = (s_n)_{n \in \mathbb{N}}$. Then the sequence of subspaces $(E_n(X))_{n \in \mathbb{N}}$ is a Riesz basis of subspaces of X if and only if there exists positive constants m_1*

and m_2 such that it holds

$$m_1 \|x\|^2 \leq \sum_{n \in \mathbb{N}} \|E_n x\|^2 \leq m_2 \|x\|^2, \quad x \in X.$$

Lemma 3.2, Lemma 3.3, Proposition 3.4 and Proposition 3.5 will be useful for the proof of the main result.

Lemma 3.2. *Let A be a discrete Riesz spectral operator and $M := \|N\|$, where N is given by Definition 2.2. Then there exists $C > 0$ such that for $s \in \rho(A)$ with $d(s, \sigma(A)) > M$ we have*

$$\|(s - A)^{-1}\| \leq \frac{C}{d(s, \sigma(A))}, \quad (4)$$

where $d(s, \sigma(A))$ denotes the distance from s to the spectrum of A .

Proof. Let $\sigma(A) = (s_n)_{n \in \mathbb{N}}$, E_n, N_n, N as in Definition 2.2 and $s \in \rho(A)$ with $d(s, \sigma(A)) > M$ be arbitrary. Since E_n is a spectral projection, it commutes with A and the resolvent of A . By the definition of a discrete spectral operator we have

$$s - A = \sum_{n=1}^{\infty} ((s - s_n) - N_n) E_n$$

and identifying $(s - s_n) - N_n$ with the matrix corresponding to $((s - s_n) - N_n)|_{E_n X}$ we obtain

$$(s - A)^{-1} = \sum_{n=1}^{\infty} ((s - s_n) - N_n)^{-1} E_n.$$

Then it holds for $x \in X$

$$\begin{aligned} \|(s - A)^{-1} x\|^2 &= \left\| \sum_{n \in \mathbb{N}} ((s - s_n) - N_n)^{-1} E_n x \right\|^2 \\ &\leq \sum_{n \in \mathbb{N}} \|((s - s_n) - N_n)^{-1} E_n x\|^2 \\ &\leq \sup_{n \in \mathbb{N}} \|((s - s_n) - N_n)^{-1}\|^2 \sum_{n \in \mathbb{N}} \|E_n x\|^2 \\ &\leq m_2 \sup_{n \in \mathbb{N}} \|((s - s_n) - N_n)^{-1}\|^2 \|x\|^2, \end{aligned}$$

where m_2 is the positive constant of the Riesz basis of subspaces $(E_n)_{n \in \mathbb{N}}$, c.f. Lemma 3.1. Using

$$((s - s_n) - N_n) = (s - s_n) \left(I - \frac{1}{(s - s_n)} N_n \right)$$

we get

$$((s - s_n) - N_n)^{-1} = \frac{1}{(s - s_n)} \sum_{j=0}^{k_n} \frac{1}{(s - s_n)^j} N_n^j,$$

where k_n denotes the degree of nilpotency of N_n . Thus, for $s \in \rho(A)$ such that $d(s, \sigma(A)) > M$, it holds

$$\begin{aligned} \|((s - s_n) - N_n)^{-1}\| &\leq \sum_{j=1}^{k_n+1} \frac{1}{|(s - s_n)|^j} M^{j-1} \leq \sum_{j=1}^{k_n+1} \frac{1}{d(s, \sigma(A))^j} M^{j-1} \\ &\leq \frac{1}{d(s, \sigma(A))} \sum_{j=0}^{\infty} \left(\frac{M}{d(s, \sigma(A))} \right)^j, \end{aligned}$$

which concludes the proof. \square

Lemma 3.3. *Let A be a discrete Riesz spectral operator and generator of a C_0 -semigroup, and $P \in \mathcal{L}(X)$. Then there exist constants $K, M > 0$ such that for $s \in \rho(A)$ with $d(s, \sigma(A)) > M$ we have $s \in \rho(A + P)$ and*

$$\|(s - (A + P))^{-1}\| \leq \frac{K}{d(s, \sigma(A))}.$$

Proof. By Lemma 3.2 there exists $M_1, C > 0$ such that

$$\|(s - A)^{-1}\| \leq \frac{C}{d(s, \sigma(A))},$$

for $s \in \rho(A)$ with $d(s, \sigma(A)) > M_1$. Set $M := \max\{M_1, 2\|P\|C\}$. Let $s \in \rho(A)$ with $d(s, \sigma(A)) > M$. Then $I - P(s - A)^{-1}$ is invertible and we obtain

$$\begin{aligned} \|(s - (A + P))^{-1}\| &= \|(s - A)^{-1}[I - P(s - A)^{-1}]^{-1}\| \\ &\leq \frac{C}{d(s, \sigma(A))} \frac{1}{1 - \|P\| \|(s - A)^{-1}\|} \\ &\leq \frac{2C}{d(s, \sigma(A))}. \end{aligned}$$

\square

Proposition 3.4. *[DS71] Let A be an operator with $\sigma(A) = (s_n)_{n \in \mathbb{N}}$ such that the family of spectral subspaces is a Riesz basis of subspaces of X . Then A has the representation $A = S + N$, where the scalar part S is defined as*

$$\begin{aligned} Sx &:= \sum_{n \in \mathbb{N}} s_n E_n x, \\ \mathcal{D}(S) &= \{x \in X \mid \sum_{n=1}^{\infty} \|s_n E_n x\|^2 < \infty\}, \end{aligned}$$

and $N_n := NE_n = (A - s_n)E_n$. Furthermore, N_n is quasi-nilpotent, i.e., $\sigma(N_n) = \{0\}$ for all $n \in \mathbb{N}$.

Proposition 3.5. *Let A be a generator of C_0 -group on X with compact resolvent. The eigenvalues are counted with algebraic multiplicity. If the following conditions are both fulfilled,*

- I. The span of the (generalized) eigenvectors form a dense set in X ;
- II. The eigenvalues can be decomposed into finitely many sets each having a uniform gap;

then A is a discrete Riesz spectral operator.

Proof. By [Zwa10, Theorem 1.1 and Theorem 1.6] it follows that the family of spectral subspaces is a Riesz basis of subspaces of X .

Thus, A has the representation $A = S + N$, c.f. Proposition 3.4 where S denotes the scalar part of the spectral operator A and $N := \sum_{n \in \mathbb{N}} N_n$, where $N_n = (A - s_n)E_n$ and N_n is quasi-nilpotent. To prove that A is a discrete Riesz spectral operator it remains to prove that N is bounded and nilpotent.

We can identify N_n with a square matrix corresponding to $N_n|_{E_n X}$ and thus N_n is bounded and nilpotent.

Since the eigenvalues A can be decomposed into finitely many sets each having a uniform gap, N is nilpotent.

Finally, we verify that N is bounded. Without loss of generality we assume that A generates an exponentially stable C_0 -group. Then by [LW83], there exists an invertible and positive operator $L \in \mathcal{L}(X)$ such that

$$\langle Ax, Lx \rangle + \langle x, LAx \rangle = -\langle x, x \rangle \quad \forall x \in \mathcal{D}(A). \quad (5)$$

We define $A_n := AE_n = (s_n + N_n)E_n$, where E_n denotes the n th spectral projection, and we identify A_n and N_n with the corresponding matrices on $E_n X$. From now on we fix n . Then we get for $x \in X$

$$\langle AE_n x, LE_n x \rangle + \langle E_n x, LAE_n x \rangle = -\langle E_n x, E_n x \rangle$$

or equivalently

$$\langle A_n E_n x, L_n E_n x \rangle + \langle E_n x, L_n A_n E_n x \rangle = -\langle E_n x, E_n x \rangle, \quad (6)$$

where $L_n := E_n^* L E_n = L_n^*$. As L is self-adjoint, L_n is self-adjoint as well. Again we identify L_n with the corresponding matrix on $E_n X$ and obtain on $E_n X$

$$(s_n + N_n)^* L_n + L_n (s_n + N_n) = -I.$$

Thus we have

$$N_n^* L_n + L_n N_n = -I + r_n L_n \quad (7)$$

with $r_n := -2\operatorname{Re} s_n$. Multiplying (7) from the right by $N_n^{k_n-j+1}$ and from the left by $(N_n^*)^{k_n-j}$ with $j = 2, 3, \dots, k_n$ gives

$$\begin{aligned} (N_n^*)^{k_n-j+1} L_n N_n^{k_n-j+1} + (N_n^*)^{k_n-j} L_n N_n^{k_n-j+2} \\ = -(N_n^*)^{k_n-j} N_n^{k_n-j+1} + r_n (N_n^*)^{k_n-j} L_n N_n^{k_n-j+1} \end{aligned}$$

and thus it holds

$$\begin{aligned} \|L_n^{1/2} N_n^{k_n-j+1}\|^2 \leq \|N_n^{k_n-j}\| \|N_n^{k_n-j+1}\| + |r_n| \|L_n\| \|N_n^{k_n-j}\| \|N_n^{k_n-j+1}\| \\ + \|N_n^{k_n-j}\| \|L_n\| \|N_n^{k_n-j+2}\|. \end{aligned} \quad (8)$$

Since L_n is boundedly invertible on $E_n X$, we get

$$m \|N_n^{k_n-j+1}\|^2 \leq \|L_n^{1/2} N_n^{k_n-j+1}\|^2 \text{ for some } m \text{ independent of } n. \quad (9)$$

For $j = 2$ we use $N_n^{k_n} = 0$ and obtain

$$m \|N_n^{k_n-1}\| \leq \|N_n^{k_n-2}\| + |r_n| \|L_n\| \|N_n^{k_n-2}\|.$$

Since A is the generator of a C_0 -group, we have $R := \sup_{n \in \mathbb{N}} |r_n| < \infty$. Moreover, with $M := \sup_{n \in \mathbb{N}} \|L_n\| < \infty$ and $C := 1 + RM$, this implies

$$\|N_n^{k_n-1}\| \leq \frac{C}{m} \|N_n^{k_n-2}\|. \quad (10)$$

For $j = 3, \dots, k_n$ we get using (9), (8), and (10) and by induction over j

$$\|N_n^{k_n-j+1}\| \leq \|N_n^{k_n-j}\| \sum_{l=1}^{j-1} \frac{C}{m^l} M^{l-1}. \quad (11)$$

In particular, for $j = k_n$ and using $k_n \leq K$, this implies

$$\|N_n\| \leq \sum_{l=1}^{k_n-1} \frac{C}{m^l} M^{l-1} \leq \sum_{l=1}^{K-1} \frac{C}{m^l} M^{l-1}.$$

Together with Proposition 3.1 this implies that N is bounded. \square

4 Review on port-Hamiltonian systems

The proof of our main result Theorem 2.8 requires some methods and results from port-Hamiltonian control systems. We define the (*maximal*) *port-Hamiltonian operator* by

$$\mathfrak{A}x := \left(P_1 \frac{d}{d\zeta} + P_0 \right) (\mathcal{H}x), \quad x \in \mathcal{D}(\mathfrak{A}),$$

on $X := L^2((0, 1); \mathbb{C}^d)$ with the domain $\mathcal{D}(\mathfrak{A}) := \{x \in X \mid \mathcal{H}x \in H^1((0, 1); \mathbb{C}^d)\}$. We equip the linear port-Hamiltonian system 1 with a boundary control u , that is, we consider

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) (\mathcal{H}(\zeta)x(\zeta, t)), \\ x(\zeta, 0) &= x_0(\zeta), \\ u(t) &= \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \end{aligned} \quad (12)$$

where $\zeta \in [0, 1]$ and $t \geq 0$ and the Assumption 2.6 is fulfilled.

Definition 4.1. We call the port-Hamiltonian system (12) a *well-posed control system* if A generates a C_0 -semigroup on X and there exist $\tau > 0$ and $m_\tau \geq 0$ such that for all $x_0 \in \mathcal{D}(\mathfrak{A})$ and $u \in C^2([0, \tau]; \mathbb{C}^d)$ with $u(0) = \begin{bmatrix} (\mathcal{H}x_0)(1,0) \\ (\mathcal{H}x_0)(0,0) \end{bmatrix}$ the classical solution x of (12) satisfies

$$\|x(\tau)\|_X^2 \leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(t)\|^2 dt \right).$$

There exists a rich literature on well-posed control systems, see e.g. Staffans [Sta05] and Tucsnak and Weiss [TW09]. In general, it is not easy to show that a control system is well-posed. However, for the port-Hamiltonian control system (12) well-posedness is already satisfied if A generates a C_0 -semigroup, c.f. [ZLMV10, Theorem 3.3] and [JZ12, Theorem 13.2.2].

In the following, we assume that the port-Hamiltonian system (12) is a well-posed control system.

Well-posedness implies that for every initial condition $x_0 \in X$ and every L^2 control function u the port-Hamiltonian control system has a unique mild solution. In order to define the mild solution we need to introduce some notation. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A . This implies, $X \subset X_{-1}$ and X is continuously embedded and dense in X_{-1} . Moreover, let $(T(t))_{t \geq 0}$ be the C_0 -semigroup generated by A with growth bound $\omega_0(T)$. The C_0 -semigroup $(T(t))_{t \geq 0}$ extends uniquely to a C_0 -semigroup $(T_{-1}(t))_{t \geq 0}$ on X_{-1} whose generator A_{-1} , with domain equal to X , is an extension of A , see e.g. [EN00].

Well-posedness implies the existence of $\tilde{B} \in \mathcal{L}(\mathbb{C}^d, X)$ with $\text{ran } \tilde{B} \subset \mathcal{D}(\mathfrak{A})$ and $\mathfrak{A}\tilde{B} \in \mathcal{L}(\mathbb{C}^d, X)$ such that the unique mild solution of (12) can be defined by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)(\mathfrak{A}\tilde{B} - A_{-1}\tilde{B})u(s) ds,$$

see [JZ18].

The port-Hamiltonian control system can be written equivalently in the standard control operator formulation

$$\dot{x}(t) = A_{-1}x(t) + Bu(t) \quad x(0) = x_0, \quad t \geq 0, \quad (13)$$

where $B \in \mathcal{L}(\mathbb{C}^d, X_{-1})$ given by

$$B := \mathfrak{A}\tilde{B} - A_{-1}\tilde{B}. \quad (14)$$

The operator B is an *admissible control operator* for $(T(t))_{t \geq 0}$, that is,

$$\Phi_{t_f} u := \int_0^{t_f} T_{-1}(t_f - s)Bu(s) ds \in X$$

for every $u \in L^2((0, t_f); \mathbb{C}^d)$. Admissibility implies that the mild solution of (12) satisfies $x \in C([0, t_f]; X)$ for every initial condition $x_0 \in X$ and every $u \in L^2((0, t_f); \mathbb{C}^d)$.

Well-posed port-Hamiltonian control systems are always exactly controllable in finite time as the following proposition shows.

Proposition 4.2. [JK19a] *Every well-posed port-Hamiltonian control system (12) is exactly controllable in finite time, that is, there exists a time $\tau > 0$ such that for all $x_1 \in X$ there exists a control function $u \in L^2((0, \tau); \mathbb{C}^d)$ such that the corresponding mild solution satisfies $x(0) = 0$ and $x(\tau) = x_1$.*

As a consequence of exact controllability we obtain that the eigenspaces span the state space.

Proposition 4.3. *Consider a well-posed port-Hamiltonian control system (12) and assume that A generates a C_0 -group on X . Let $\sigma(A) = (s_n)_{n \in \mathbb{N}}$. Then*

$$X = \overline{\text{span}_{n \in \mathbb{N}} E((s_n)) X},$$

where $E((s_n))$ is introduced in Definition 2.1.

Proof. Follows from [JZ99, Lemma 7.3] together with Proposition 4.2. \square

5 Proof of the Main Result

This section is devoted to the proof of Theorem 2.8. Let us first assume that one of the conditions of the theorem is satisfied. As the resolvent of A is compact the spectrum consists of isolated eigenvalues only. That the eigenvalues lie in a strip parallel to the imaginary axis and that they can be decomposed into finitely many sets each having a uniform gap will be shown in the proof of the implication $2) \Rightarrow 1)$. Finally, $\omega_0(A) = s(A)$ is implied by [GZ01, Theorem 2.12].

5.1 Proof of the equivalence $2) \Leftrightarrow 3)$ of Theorem 2.8

The operator A generates a C_0 -group, if and only if A and $-A$ generates a C_0 -semigroup, [EN00, Section II.3.11]. In [JMZ15, Theorem 1.5] shows that A is the generator of a C_0 -semigroup if and only if $W_1 \mathcal{H}(1)Z^+(1) \oplus W_0 \mathcal{H}(0)Z^-(0) = \mathbb{C}^d$. Since $\mathcal{D}(-A) = \mathcal{D}(A)$ and $\sigma_p(P_1 \mathcal{H}) = -\sigma_p(-P_1 \mathcal{H})$ it follows that $-A$ generates a C_0 -semigroup if and only if $W_1 \mathcal{H}(1)Z^-(1) \oplus W_0 \mathcal{H}(0)Z^+(0) = \mathbb{C}^d$.

5.2 Proof of the implication $2) \Rightarrow 1)$ of Theorem 2.8

The following lemma will be useful.

Lemma 5.1. *Let $s \in \mathbb{C}$ and P_1, P_0 and \mathcal{H} fulfil the condition for a port-Hamiltonian operator in Assumption 2.6. Then the solutions of the system of ordinary differential equations*

$$sx(\zeta) = \left(P_1 \frac{d}{d\zeta} + P_0 \right) (\mathcal{H}x)(\zeta), \quad \zeta \in [0, 1], \quad (15)$$

denoted by $x(\zeta) = \Psi^s(\zeta)x(0)$, satisfy

$$\tilde{M}e^{-|\text{Re } s|\tilde{c}_0\zeta} \|v\| \leq \|\Psi^s(\zeta)v\| \leq Me^{|\text{Re } s|c_0\zeta} \|v\|, \quad v \in \mathbb{C}^d, \quad \zeta \in [0, 1],$$

with constants $M, \tilde{M} > 0$, and $\tilde{c}_0, c_0 \geq 0$ independent of s and ζ .

Proof. Writing $\tilde{x} = \mathcal{H}x$, (15) can be equivalently written as

$$\mathcal{H}(\zeta)P_1\tilde{x}'(\zeta) = s\tilde{x}(\zeta) - \mathcal{H}(\zeta)P_0\tilde{x}(\zeta).$$

We write $s = i\omega + r$ with $\omega, r \in \mathbb{R}$ and diagonalize $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, see Assumption 2.6. Thus, $\mathcal{H}(\zeta)P_1 = S^*(\zeta)\Delta(\zeta)S^{-*}(\zeta)$ and we get

$$\Delta(\zeta)S^{-*}(\zeta)\tilde{x}'(\zeta) = i\omega S^{-*}(\zeta)\tilde{x}(\zeta) + (rI - S^{-*}(\zeta)\mathcal{H}(\zeta)P_0S^*(\zeta))S^{-*}(\zeta)\tilde{x}(\zeta).$$

Using the substitution $z = S^{-*}\tilde{x}$ gives the equivalent differential equation

$$z'(\zeta) = i\omega\Delta^{-1}(\zeta)z(\zeta) + (r\Delta^{-1}(\zeta) + Q(\zeta))z(\zeta), \quad (16)$$

where

$$Q(\zeta) := -\Delta^{-1}(\zeta)S^{-*}(\zeta)\mathcal{H}(\zeta)P_0S^*(\zeta) - (S^{-*})'(\zeta)S^*(\zeta).$$

Thus, equation (15) is equivalent to equation (16). Due to the fact that $P_1\mathcal{H}(\zeta)$ has real eigenvalues, $\Delta(\zeta)$ is a diagonal, real matrix and $i\omega\Delta^{-1}(\zeta)$ is a diagonal, purely imaginary matrix. We write $\Delta^{-1}(\zeta) = \text{diag}_{k=1,\dots,n}(\alpha_k(\zeta))$ with $\alpha_k(\zeta) : [0, 1] \rightarrow \mathbb{R}$ and define $\Phi_\omega(\zeta) = \text{diag}(\exp(-i\omega \int_0^\zeta \alpha_k(\tau)d\tau))$ which satisfies

$$\|\Phi_\omega(\zeta)\|_{\mathcal{L}(\mathbb{C}^d)} = 1, \quad \zeta \in [0, 1].$$

Multiplying (16) with $\Phi_\omega(\zeta)$, we get

$$\Phi_\omega(\zeta)z'(\zeta) - i\omega\Delta^{-1}(\zeta)\Phi_\omega(\zeta)z(\zeta) = \Phi_\omega(\zeta)(r\Delta^{-1}(\zeta) + Q(\zeta))z(\zeta)$$

or equivalently

$$(\Phi_\omega(\zeta)z(\zeta))' = (r\Delta^{-1}(\zeta) + \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta))\Phi_\omega(\zeta)z(\zeta).$$

Using the substitution $y = \Phi_\omega z$, this ordinary differential equation becomes

$$y' = (r\Delta^{-1}(\zeta) + Q_\omega(\zeta))y(\zeta), \quad (17)$$

where $Q_\omega(\zeta) := \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta)$. There exist constants $c_0, c_1 \geq 0$, independent of ω , such that

$$2 \max_{\zeta \in [0, 1]} \|r\Delta^{-1}(\zeta) + Q_\omega(\zeta)\| \leq |r|c_0 + c_1. \quad (18)$$

The solution y of (17) satisfies

$$\frac{d}{d\zeta} \|y(\zeta)\|^2 = y(\zeta)^* [(r\Delta^{-1}(\zeta) + Q_\omega(\zeta)) + (r\Delta^{-1}(\zeta) + Q_\omega(\zeta))^*] y(\zeta). \quad (19)$$

This together with (18) implies

$$-(|r|c_0 + c_1)\|y(\zeta)\|^2 \leq \frac{d}{d\zeta} \|y(\zeta)\|^2 \leq (|r|c_0 + c_1)\|y(\zeta)\|^2,$$

and thus

$$e^{-(|r|c_0 + c_1)\zeta} \|y(0)\|^2 \leq \|y(\zeta)\|^2 \leq e^{(|r|c_0 + c_1)\zeta} \|y(0)\|^2.$$

As the mapping $x \mapsto y$ is boundedly invertible on $L^2((0, 1), \mathbb{C}^d)$, with norm independent on ω , the statement follows. \square

Next we state some results from complex analysis.

Definition 5.2. [AI95, II.1.27] An entire function f is called an *entire function of exponential type*, if there exist constants C and T such that $|f(s)| \leq Ce^{T|s|}$ for all $s \in \mathbb{C}$. Further, an entire function f of exponential type is said to be of *sine type*, if

1. the zeros of f lie in a strip $\{s \in \mathbb{C} \mid |\operatorname{Im} s| \leq h\}$ for some $h \geq 0$.
2. there exist $\hat{\omega} \in \mathbb{R}$ and positive constants c and C such that $c \leq |f(r + i\hat{\omega})| \leq C$ for every $r \in \mathbb{R}$ holds.

Proposition 5.3. [AI95, Proposition II.1.28] (Levin 1961) *If f is of sine type, then its set of zeros counted with algebraic multiplicity is a finite unification of sets each having a uniform gap.*

Lemma 5.4. *A complex number $s \in \mathbb{C}$ is an eigenvalue of a port-Hamiltonian operator A if and only if*

$$\det [W_1 \mathcal{H}(1) \Psi^s(1) + W_0 \mathcal{H}(0)] = 0,$$

where Ψ^s is described in Lemma 5.1.

Proof. For every $x(0) \in \mathbb{C}^d$ there exists a solution of the differential equation $sx(\zeta) = (P_1 \frac{d}{d\zeta} + P_0)(\mathcal{H}x)(\zeta)$, $\zeta \in [0, 1]$. The complex number s is an eigenvalue of A if and only if $x \in \mathcal{D}(A)$ and $Ax = sx$. Using Lemma 5.1 this is equivalent to

$$x(\zeta) = \Psi^s(\zeta)x(0) \text{ and } \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0.$$

Inserting the first equation in the second, we get that s is an eigenvalue of A if and only if $\det [W_1 \mathcal{H}(1) \Psi^s(1) + W_0 \mathcal{H}(0)] = 0$. \square

Now we are in the situation to give the proof of the implication $2) \Rightarrow 1)$.

Proof of the implication $2) \Rightarrow 1)$ of Theorem 2.8. Assertion 2) implies that the eigenvalues lie in a strip parallel to the imaginary axis. Thus, thanks to Proposition 4.3 and Proposition 3.5 it suffices to show that the eigenvalues $(s_n)_{n \in \mathbb{N}}$ of A (counted according to the algebraic multiplicity) can be decomposed into finitely many sets having each a uniform gap.

Using Proposition 5.3, this is implied by the existence of an entire function g of exponential type with

- i) g has exactly the zeros $(s_n)_{n \in \mathbb{N}}$ and
- ii) there exist $r, c, C > 0$ such that for every $\omega \in \mathbb{R}$: $c \leq |g(r + i\omega)| \leq C$.

Note that $f : \mathbb{C} \rightarrow \mathbb{C}$, defined by $f(s) := g(is)$ is a sine-type function if g is an entire function of exponential type and satisfying the above conditions i) and ii). We define $g : \mathbb{C} \rightarrow \mathbb{C}$ by

$$g(s) := \det [W_1 \mathcal{H}(1) \Psi^s(1) + W_0 \mathcal{H}(0)]. \quad (20)$$

By Lemma 5.4, a complex number $s \in \mathbb{C}$ is an eigenvalue of the operator A if and only if $g(s) = 0$.

Ψ^s described in Lemma 5.1 is an entire function, c.f. [Was87, Theorem 24.1] and thus g is an entire function as well. Clearly, g has the zeros $(s_n)_{n \in \mathbb{N}}$. Since the determinant of a matrix equals the product of its eigenvalues and every eigenvalue is smaller or equal the norm of the matrix, it yields $|g(s)| \leq \| [W_1 \mathcal{H}(1) \Psi^s(1) + W_0 \mathcal{H}(0)] \|$. Using Lemma 5.1 it holds $|g(s)| \leq c_2 e^{|\operatorname{Re} s| c_3}$ for some constants $c_2, c_3 \geq 0$, and thus, g is bounded on lines parallel to the imaginary axis and grows at most exponentially.

Next, we show that g is bounded away from zero on some line parallel to the imaginary axis. Since the control operator B of the port-Hamiltonian system (12) is admissible, see Section 4, it yields that for $\omega > \omega_0(A)$ exists a constant $M_\omega > 0$ such that

$$\|(s - A_{-1})^{-1} B\|_{\mathcal{L}(\mathbb{C}^d, X)} \leq \frac{M_\omega}{\sqrt{\operatorname{Re} s - \omega}} \quad \text{for } \operatorname{Re} s \geq \omega, \quad (21)$$

see [TW14, Proposition 4.4.6]. Let $r > \omega_0(A)$ and we assume that g is not bounded away from zero on $r + i\mathbb{R}$, i.e., there exists a sequence $\omega_k \in \mathbb{R}$ such that $g(r + i\omega_k) \rightarrow 0$. Since all zeros of g have real part less or equal to the growth bound $\omega_0(T)$ of the C_0 -semigroup generated by A , it holds true that $g(r + i\omega_k) \neq 0$. Let u_0 be an arbitrary vector in \mathbb{C}^d . By [JZ12, Theorem 12.1.3] the solution $x_{u_0}^{r+i\omega_k}$ of

$$\begin{aligned} (r + i\omega_k)x(\zeta) &= \left(P_1 \frac{d}{d\zeta} + P_0 \right) (\mathcal{H}x)(\zeta) \\ [W_1 \quad W_0] \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} &= u_0, \end{aligned}$$

is given by

$$\begin{aligned} x_{u_0}^{r+i\omega_k} &= ((r + i\omega_k) - A_{-1})^{-1} (\mathfrak{A}\tilde{B} - (r + i\omega_k)\tilde{B}) + \tilde{B} u_0 \\ &= ((r + i\omega_k) - A_{-1})^{-1} B u_0 \end{aligned}$$

and $x_{u_0}^{r+i\omega_k}(0)$ fulfils

$$[W_1 \mathcal{H}(1) \Psi^{r+i\omega_k}(1) + W_0 \mathcal{H}(0)] x_{u_0}^{r+i\omega_k}(0) = u_0.$$

Let $M_k := [W_1 \mathcal{H}(1) \Psi^{r+i\omega_k}(1) + W_0 \mathcal{H}(0)]$. Since $g(r + i\omega_k) = \det(M_k) \rightarrow 0$ and $1 = \det(I) = \det(M_k) \det(M_k^{-1})$, we have $\det(M_k^{-1}) \rightarrow \infty$. Hence, M_k^{-1} has an eigenvalue ν_k with $|\nu_k| \rightarrow \infty$. Choose $u_{k,0}$ as a normalized eigenvector to ν_k . Then it yields

$$x_{u_{k,0}}^{r+i\omega_k}(0) = M_k^{-1} u_{k,0} = \nu_k u_{k,0},$$

which implies $\|x_{u_{k,0}}^{r+i\omega_k}(0)\|_{\mathbb{C}^d} \rightarrow \infty$. We note that the function $x_{u_{k,0}}^{r+i\omega_k}$ is given by

$$x_{u_{k,0}}^{r+i\omega_k}(\zeta) = \Psi^{r+i\omega_k}(\zeta) x_{u_{k,0}}^{r+i\omega_k}(0).$$

Using that the inverse of $\Psi^{r+i\omega_k}$ is a bounded function, see Lemma 5.1, we get

$$\|x_{u_{k,0}}^{r+i\omega_k}\|_{L^2((0,1);\mathbb{C}^d)} \rightarrow \infty. \quad (22)$$

However, since we also have $x_{u_{k,0}}^{r+i\omega_k} = ((r+i\omega_k) - A_{-1})^{-1}Bu_{k,0}$, equation (22) is in contradiction with the uniform boundedness of $((r+i\omega_k) - A_{-1})^{-1}Bu_{0,k}$, see equation (21). Thus the entire function g is of exponential type and satisfies condition *i*) and *ii*). This concludes the proof. \square

5.3 Proof of the implication 1) \Rightarrow 2) of Theorem 2.8

We start with some characterizations of the resolvent and the spectrum of port-Hamiltonian operators.

Lemma 5.5. *Let $\Lambda \in C([0, 1]; \mathbb{C}^d)$ with $\Lambda(\zeta)$ diagonal, invertible und positive definite for every $\zeta \in [0, 1]$, $Q \in \mathbb{C}^{n \times n}$ singular and $A : \mathcal{D}(A) \subset L^2((0, 1); \mathbb{C}^d) \rightarrow L^2((0, 1); \mathbb{C}^d)$ defined by $Ax = \Lambda x'$ and $\mathcal{D}(A) = \{x \in H^1((0, 1); \mathbb{C}^d) \mid x(1) + Qx(0) = 0\}$. Then there exist real constants $\gamma < 0$ and $a, b > 0$ such that $\|(s - A)^{-1}\| \geq ae^{b|s|}$ for real $s \in \rho(A)$ with $s < \gamma$.*

Proof. Let $0 \neq x(0) \in \ker Q$. We define $F_s(\zeta) := s \int_0^\zeta \Lambda^{-1}(\tau) d\tau$ for $s \in \mathbb{R}$ and $\zeta \in [0, 1]$. Note that F_s and Λ commute as both are diagonal. There exists $\gamma_0 < 0$ such that $I - e^{2F_s(1)}$ is invertible if $s < \gamma_0$. Further, let $s \in \rho(A)$ with $s < \gamma_0$ and define

$$g(\zeta) = 2se^{F_s(1)-F_s(\zeta)}g_0 = 2se^{s \int_\zeta^1 \Lambda^{-1}(\tau) d\tau}g_0 \quad (23)$$

with $g_0 := e^{F_s(1)}(I - e^{2F_s(1)})^{-1}x(0)$. Then the solution of

$$\Lambda x' = sx + g \quad (24)$$

is given by

$$\begin{aligned} x(\zeta) &= e^{F_s(\zeta)}x(0) + \int_0^\zeta e^{F_s(\zeta)-F_s(\tau)}\Lambda^{-1}(\tau)g(\tau)d\tau \\ &= e^{F_s(\zeta)}x(0) - e^{F_s(\zeta)+F_s(1)} \int_0^\zeta (-2s\Lambda^{-1}(\tau))e^{-2F_s(\tau)}d\tau g_0 \\ &= e^{F_s(\zeta)}x(0) - e^{F_s(\zeta)+F_s(1)}(e^{-2F_s(\zeta)} - I)g_0 \\ &= e^{F_s(\zeta)}(I - e^{2F_s(1)})e^{-F_s(\zeta)}g_0 - e^{-F_s(\zeta)+F_s(1)}g_0 + e^{-F_s(\zeta)+F_s(1)}g_0 \\ &= e^{F_s(\zeta)-F_s(1)}g_0 - \frac{1}{2s}g(\zeta). \end{aligned}$$

In particular $x(1) = 0$ and thus $(s-A)^{-1}g = x$. As $\Lambda \in C([0, 1]; \mathbb{C}^d)$ is a diagonal and invertible matrix-valued function with positive entries on its diagonal, there exists $\lambda_0 > 0$ such that the diagonal elements of Λ are bounded by λ_0 . First using (23) we can estimate

$$\|g\|_{L^2((0,1);\mathbb{C}^d)} \leq c\sqrt{|s|}\|g_0\|, \quad c > 0,$$

and

$$\begin{aligned}
\|x\|_{L^2(0,1;\mathbb{C}^d)} &\geq \left\| e^{F_s(\cdot) - F_s(1)} g_0 \right\|_{L^2(0,1;\mathbb{C}^d)} - \frac{1}{2|s|} \|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&= \left(\int_0^1 \left\| e^{-s \int_\tau^1 \Lambda^{-1}(\sigma) d\sigma} g_0 \right\|^2 d\tau \right)^{\frac{1}{2}} - \frac{1}{2|s|} \|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&\geq \left(\int_0^1 e^{2|s|(1-\tau)\lambda_0} \|g_0\|^2 d\tau \right)^{\frac{1}{2}} - \frac{1}{2|s|} \|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&\geq \frac{1}{4|s|^2 \lambda_0} \left(e^{2|s|\lambda_0} - 1 \right) \|g\|_{L^2((0,1);\mathbb{C}^d)} - \frac{1}{2|s|} \|g\|_{L^2((0,1);\mathbb{C}^d)}.
\end{aligned}$$

This completes the lemma. \square

Lemma 5.6. *Let A be a port-Hamiltonian operator, which generates a C_0 -semigroup. Furthermore, let A be a discrete Riesz spectral operator and let $(s_n)_{n \in \mathbb{N}}$ denote its eigenvalues. Then there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$ within the ball $\{s \in \mathbb{C} \mid |s - s_n| \leq K |\operatorname{Re} s_n|^2\}$ there lie at most d eigenvalues.*

Proof. Without loss of generality we assume that A generates an exponentially stable C_0 -semigroup. By Proposition 4.2 the corresponding port-Hamiltonian control system (12) with control operator B is exactly controllable in finite time. Then the dual system, described by A^* and B^* , is exactly observable and it yields due to the Hautus Test, c.f. [RW94], that there exists a positive constant m such that

$$\|(s - A^*)x\|^2 + |\operatorname{Re} s| \|B^*x\|^2 \geq m |\operatorname{Re} s|^2 \|x\|^2, \quad \operatorname{Re} s < 0, x \in \mathcal{D}(A^*). \quad (25)$$

No matter that there may exist generalized eigenvectors, we consider only eigenvectors corresponding to different eigenvalues of A^* . As $\sigma(A) = \overline{\sigma(A^*)}$, it suffices to prove the statement for A^* . Choose arbitrary e_1, \dots, e_{d+1} normed eigenvectors of the operator A^* to the eigenvalues $\lambda_1, \dots, \lambda_{d+1}$ with $\lambda_n \neq \lambda_m$ for $n \neq m \in 1, \dots, d+1$.

Since $B^* \in \mathcal{L}(\mathcal{D}(A^*), \mathbb{C}^d)$ the $d+1$ vectors B^*e_n are linearly dependent in \mathbb{C}^d , i.e., there exists scalars $a_1, \dots, a_{d+1} \in \mathbb{C}$ with $\sum_{n=1}^{d+1} |a_n|^2 = 1$ such that

$$a_1 B^*e_1 + \dots + a_{d+1} B^*e_{d+1} = 0. \quad (26)$$

Consider $x = \sum_{n=1}^{d+1} a_n e_n$. Then $x \in \mathcal{D}(A^*)$ with $B^*x = 0$, and

$$\|x\|^2 = \left\| \sum_{n=1}^{d+1} a_n e_n \right\|^2 \geq m_1 > 0,$$

by Proposition 3.1. It follows with the Hautus Test (25) at the point $s = \lambda_{d+1}$

$$\begin{aligned} m m_1 |\operatorname{Re} \lambda_{d+1}|^2 &\leq \|(\lambda_{d+1} - A^*)x\|^2 = \left\| \sum_{n=1}^d a_n (\lambda_{d+1} - \lambda_n) e_n \right\|^2 \\ &\leq m_2 \sum_{n=1}^d |a_n|^2 |\lambda_{d+1} - \lambda_n|^2. \end{aligned}$$

Thus,

$$\frac{d+1}{d+1} \frac{m m_1}{m_2} |\operatorname{Re} \lambda_{d+1}|^2 \leq \sum_{n=1}^d |\lambda_{d+1} - \lambda_n|^2.$$

Since the eigenvalues are arbitrary chosen, this implies that in the ball with radius $\frac{m m_1}{(d+1) m_2} |\operatorname{Re} \lambda_{d+1}|^2$ around λ_{d+1} lies at most d eigenvalues. Hence, the statement follows. \square

Now we are in the position to prove the implication 1) \Rightarrow 2).

Proof of the implication 1) \Rightarrow 2) of Theorem 2.8: We assume that A does not generate a C_0 -group.

Since A is a port-Hamiltonian operator, we denote by S the matrix-valued function such that $P_1 \mathcal{H}(\zeta) = S^{-1}(\zeta) \Delta(\zeta) S(\zeta)$, see Assumption 2.6. Since the eigenvalues of $P_1 \mathcal{H}(\zeta)$ and $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ are the same, it follows by Sylvester's law of inertia that the number of positive and negative eigenvalues of $P_1 \mathcal{H}(\zeta)$ equal those of P_1 . Let d_1 denote the number of positive and $d_2 = d - d_1$ the number of negative eigenvalues of P_1 . Thus without loss of generality Δ can be written as $\Delta(\zeta) = \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix}$, where $\Lambda(\zeta) \in \mathbb{C}^{d_1 \times d_1}$ correspond to the positive eigenvalues and $\Theta(\zeta) \in \mathbb{C}^{d_2 \times d_2}$ to the negative ones.

Let $\mathcal{S} \in \mathcal{L}(X)$ be the multiplication operator $(\mathcal{S}x)(\zeta) := S(\zeta)x(\zeta)$. By assumption \mathcal{S} is invertible and we obtain

$$\begin{aligned} \mathcal{S} \mathcal{A} \mathcal{S}^{-1} z(\zeta) &= \Delta(\zeta) (z(\zeta))' \\ &\quad + \Delta'(\zeta) z(\zeta) + S(\zeta) (S^{-1})' \Delta(\zeta) z(\zeta) + S(\zeta) P_0 \mathcal{H}(\zeta) S^{-1}(\zeta) z(\zeta) \\ \mathcal{D}(\mathcal{S} \mathcal{A} \mathcal{S}^{-1}) &= \{z \in H^1((0, 1); \mathbb{C}^d) \mid [W_1 \quad W_0] \begin{bmatrix} (\mathcal{H} S^{-1} z)(1) \\ (\mathcal{H} S^{-1} z)(0) \end{bmatrix} = 0\}. \end{aligned}$$

The operator $\mathcal{S} \mathcal{A} \mathcal{S}^{-1}$ generates a C_0 -semigroup, too. We split the variable $z(\zeta) = \begin{bmatrix} z^+(\zeta) \\ z^-(\zeta) \end{bmatrix} \in \mathbb{C}^d$ with $z^+(\zeta) \in \mathbb{C}^{d_1}$ and $z^-(\zeta) \in \mathbb{C}^{d_2}$, and we define $W_1 \mathcal{H}(1) S^{-1}(1) =: [V_1 \quad V_2]$ and $W_0 \mathcal{H}(0) S^{-1}(0) =: [U_1 \quad U_2]$, where $U_1, V_1 \in \mathbb{C}^{d \times d_1}$ and $U_2, V_2 \in \mathbb{C}^{d \times d_2}$. Then, $z \in \mathcal{D}(\mathcal{S} \mathcal{A} \mathcal{S}^{-1})$ if and only if $z \in H^1((0, 1); \mathbb{C}^d)$

and

$$\begin{aligned}
0 &= \begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}S^{-1}z)(1) \\ (\mathcal{H}S^{-1}z)(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(1) \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(0) \end{bmatrix} + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(1) \end{bmatrix} \\
&= K \begin{bmatrix} \Lambda(1)z^+(1) \\ \Theta(0)z^-(0) \end{bmatrix} + Q \begin{bmatrix} \Lambda(0)z^+(0) \\ \Theta(1)z^-(1) \end{bmatrix},
\end{aligned}$$

where $K := \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix}$ and $Q := \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda(0)^{-1} & 0 \\ 0 & \Theta(1)^{-1} \end{bmatrix}$. Since $\mathcal{S}AS^{-1}$ is the generator of a C_0 -semigroup and this property is invariant under bounded perturbations, K is invertible, see [JZ12, Theorem 13.3.1].

Let $\mathcal{T} \in \mathcal{L}(X)$ be defined by $\mathcal{T} \begin{bmatrix} z^+(\zeta) \\ z^-(\zeta) \end{bmatrix} := \begin{bmatrix} z^+(\zeta) \\ z^-(1-\zeta) \end{bmatrix}$. Clearly, \mathcal{T} is invertible. Then the operator $\mathcal{A} := \mathcal{T}\mathcal{S}AS^{-1}\mathcal{T}^{-1}$ on X is given by

$$\begin{aligned}
\mathcal{A}z(\zeta) &= \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & -\Theta(1-\zeta) \end{bmatrix} z'(\zeta) + R(\zeta)z(\zeta) \\
\mathcal{D}(\mathcal{A}) &= \left\{ z \in H^1((0,1); \mathbb{C}^d) \mid K \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix} z(1) + Q \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix} z(0) = 0 \right\},
\end{aligned}$$

where $z \mapsto R(\cdot)z(\cdot)$ is a bounded multiplication operator on X . Let $\tilde{K} := K \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix}$ and $\tilde{Q} := Q \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix}$. By assumption, the matrix \tilde{K} is invertible. As a bounded perturbation of a generator of C_0 -group generates again a C_0 -group, we obtain that the operator

$$\begin{aligned}
\tilde{A}z(\zeta) &= \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & -\Theta(1-\zeta) \end{bmatrix} z'(\zeta) \\
\mathcal{D}(\tilde{A}) &= \left\{ z \in H^1((0,1); \mathbb{C}^d) \mid z(1) = \tilde{K}^{-1}\tilde{Q}z(0) \right\}
\end{aligned}$$

generates a C_0 -semigroup, but not a C_0 -group. In particular, $Q_1 := \tilde{K}^{-1}\tilde{Q}$ is singular.

Since A is a discrete Riesz spectral operator, due to Lemma 5.6, there is a $K > 0$ such that in the ball with radius $K|\operatorname{Re} s|^2$ around every eigenvalue s of A lie at most d eigenvalues. Thus there exist sequences $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \rightarrow -\infty$ and $r_n \rightarrow \infty$ such that the ball with center t_n and radius r_n lie in $\rho(A)$. By Lemma 3.3 we get

$$\|(t_n - \tilde{A})^{-1}\| \rightarrow 0.$$

However, for the operator \tilde{A} the Lemma 5.5, is applicable which implies that $\|(t_n - \tilde{A})^{-1}\| \rightarrow \infty$. This leads to a contradiction. \square

6 Examples

We consider the one-dimensional wave equation with boundary feedback as in [XW11], but we allow for spatial dependent mass density and Young's modulus, given by

$$\begin{aligned} w_{tt}(\zeta, t) &= \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} (T(\zeta) w_\zeta(\zeta, t)), \quad \zeta \in [0, 1], t \geq 0, \\ w(0, t) &= 0 \\ u(t) &= T(1) w_\zeta(1, t) \\ y(t) &= w_t(1, t), \\ u(t) &= -\kappa y(t), \kappa > 0, \end{aligned} \tag{27}$$

where $\zeta \in [0, 1]$ is the spatial variable, $w(\zeta, t)$ describes the displacement of the point ζ of the string at time t , $T(\zeta) > 0$ is the Young's modulus of the string, $\rho(\zeta) > 0$ is the mass density, and $\kappa > 0$. We model this system as a port-Hamiltonian system. Therefore we introduce the state variable $x = \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix}$ with $x_1 = \rho(\zeta) \frac{\partial w}{\partial t}$ (momentum), $x_2 = \frac{\partial w}{\partial \zeta}$ (strain) and the state space $X = L^2((0, 1); \mathbb{C}^d)$. Since the meaning of $w(0, t) = 0$ is that in the point $\zeta = 0$ the string is fixed for all times, we model this boundary condition as $\frac{\partial w}{\partial t} w(0, t) = 0$. Then the closed-loop system (27) can equivalently be written as

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} &= P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} \right), \\ 0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ \kappa & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H}(1)x(1, t) \\ \mathcal{H}(0)x(0, t) \end{bmatrix}, \end{aligned} \tag{28}$$

where $P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}$, $W_1 = \begin{bmatrix} 0 & 0 \\ \kappa & 1 \end{bmatrix}$, $W_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\kappa > 0$. We define the corresponding port-Hamiltonian operator A

$$\begin{aligned} Ax &:= P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x), \\ \mathcal{D}(A) &= \{x \in X \mid \mathcal{H}x \in H^1((0, 1); \mathbb{C}^2) \text{ and } [W_1 \quad W_0] \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0\} \end{aligned}$$

and remind that due to [JZ12, Theorem 13.2.2] the system has a unique mild solution if A generates a C_0 -semigroup. To show that the port-Hamiltonian operator A is a discrete Riesz spectral operator, it is sufficient due to Theorem 2.8 to prove that A generates a C_0 -group. So we have only to check that $W_1 \mathcal{H}(1) Z^+(1) \oplus W_0 \mathcal{H}(0) Z^-(0) = \mathbb{C}^d$ and $W_1 \mathcal{H}(1) Z^-(1) \oplus W_0 \mathcal{H}(0) Z^+(0) = \mathbb{C}^d$, where $Z^+(\zeta)$ denotes the span of the eigenvectors of $P_1 \mathcal{H}(\zeta)$ corresponding to the positive eigenvalues of $P_1 \mathcal{H}(\zeta)$ and $Z^-(\zeta)$ is the span of the eigenvectors of $P_1 \mathcal{H}(\zeta)$ corresponding to the negative eigenvalues of $P_1 \mathcal{H}(\zeta)$. Defining $\gamma =$

$$P_1 \mathcal{H} = \underbrace{\begin{bmatrix} \gamma & -\gamma \\ \rho^{-1} & \rho^{-1} \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} (2\gamma)^{-1} & \rho/2 \\ (2\gamma)^{-1} & \rho/2 \end{bmatrix}}_S.$$

It is easy to see that $Z^+(\zeta) = \text{span} \begin{bmatrix} T(\zeta) \\ \gamma(\zeta) \end{bmatrix}$ and $Z^-(\zeta) = \text{span} \begin{bmatrix} -T(\zeta) \\ \gamma(\zeta) \end{bmatrix}$. Then it holds for $\kappa \neq -\frac{T(1)}{\gamma(1)}$ and $\kappa \neq \frac{T(1)}{\gamma(1)}$

$$W_1\mathcal{H}(1)Z^+(1) \oplus W_0\mathcal{H}(0)Z^-(0) = \begin{bmatrix} 0 \\ \kappa\gamma(1) + T(1) \end{bmatrix} \oplus \begin{bmatrix} -\gamma(0) \\ 0 \end{bmatrix} = \mathbb{C}^2,$$

$$W_1\mathcal{H}(1)Z^-(1) \oplus W_0\mathcal{H}(0)Z^+(0) = \begin{bmatrix} 0 \\ -\kappa\gamma(1) + T(1) \end{bmatrix} \oplus \begin{bmatrix} \gamma(0) \\ 0 \end{bmatrix} = \mathbb{C}^2,$$

Thus, A is a discrete Riesz spectral operator for $\kappa \neq -\frac{T(1)}{\gamma(1)}$ and $\kappa \neq \frac{T(1)}{\gamma(1)}$.

In contrast to [XW11] we do not need determine the eigenvalues exactly, which is only possible if ρ and T are constant. For spacial varying coefficients like $\rho(\zeta) = e^\zeta$ and $T(\zeta) = \zeta + 1$ the problem is not analytically solvable, see [JZ12, Exercise 12.1].

6.1 Timoshenko-beam

In [JZ12] it is shown that the Timoshenko beam with boundary damping can be formulated as port-Hamiltonian system: It can be written in the form of (12),

$$\text{with } P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 1 \\ 0 & 0 & 0 & \frac{1}{I_\sigma(\zeta)} \end{bmatrix} \text{ and } P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ where}$$

the $K(\zeta)$ denotes the shear modulus, $EI(\zeta)$ is the product of Young's modulus of elasticity and the moment of inertia of a cross section, $\rho(\zeta)$ is the mass per unit length and $I_\rho(\zeta)$ denotes the rotary moment of inertia of a cross section. All these physical parameters are positive and continuously differentiable functions of ζ . To model the fact that the beam is clamped in at $\zeta = 0$ and controlled at $\zeta = 1$ by the force and moment feedback, we add the boundary condition $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [w_1 \ w_0] \begin{bmatrix} (\mathcal{H}\mathbf{x})(1,t) \\ (\mathcal{H}\mathbf{x})(0,t) \end{bmatrix}$ with $[w_1 \ w_0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha_2 & 0 & 0 & 0 & 0 \end{bmatrix}$ and α_1, α_2 are given positive gain feedback constants.

Xu and Feng dedicate the paper [XF02] to this example and they proved under the extra assumption that all physical constants are independent of ζ that the eigenvectors and generalized eigenvectors of the operator form a Riesz basis. This example is also revisited in [Vil07] using another approach. Using our main theorem, we can easily verify that the associated system operator is a discrete Riesz spectral operator.

7 Closing remarks and open problems

We have shown that a port-Hamiltonian system of the form (1) is a Riesz spectral operator if and only if it generates a C_0 -group. Many (hyperbolic) systems can be written into this form, with as main exception the Euler-Bernoulli beam equation. Of course the basis property of this equation is well-studied, and many results are known, see e.g. [GW19]. However, we assert that the main result of this paper does not hold for the Euler-Bernoulli beam equation.

In Theorem 2.8 we have shown that if the port-Hamiltonian systems (1) is a Riesz spectral operator, then the eigenvalues (counted according to the algebraic multiplicity) can be decomposed into finitely many sets each having a uniform gap. If we count the eigenvalues without multiplicity, then [JZ01a, Theorem 2] shows that the eigenvalues can be decomposed into at most d sets each having a uniform gap. We claim that this results holds true if we count the eigenvalues according to the algebraic multiplicity.

One may ask whether the main theorem of this paper (Theorem 2.8) holds if we drop the assumption that P_0 is skew-symmetric. Our proof uses the fact that every well-posed port-Hamiltonian control system (12) is exactly controllable in finite time and this property is only known in the case that P_0 is skew-symmetric. Thus, if P_0 is an arbitrary $d \times d$ -matrix, then our proof carries over to this more general case, provided we add the assumption that the corresponding port-Hamiltonian system is exactly controllable in finite-time. However, we assert that even when P_0 is not skew-symmetric, the system (12) is exactly controllable in finite time, and thus this extra assumption would not be needed.

B.-Z. Guo and J.-M. Wang [GW19] studied the Riesz basis property for a closely related class of systems, that is, hyperbolic systems of the form $\frac{\partial x}{\partial t} = K(\zeta) \frac{\partial x}{\partial \zeta} + C(\zeta)x$ with K and C diagonal. They showed that for their class of systems the state space can be split into two parts, one part generated by a C_0 -group and the other generated by an operator without spectrum, see [GW19, Theorem 4.10]. In particular, the spectrum of their operators always lie in a strip parallel to the imaginary axis. This result does not generalize to our system class as the following example shows.

Example 7.1. We consider the port-Hamiltonian system

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix}, \\ 0 &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(1, t) \\ x(0, t) \end{bmatrix}. \end{aligned}$$

The system operator generates a C_0 -semigroup, but there exists a sequence of eigenvalues which real parts converge to $-\infty$.

The following example shows that the equivalence 1) \Leftrightarrow 2) in Theorem 2.8 does not hold for generators A of C_0 -semigroups $(T(t))_{t \geq 0}$ on Hilbert spaces even if we additionally assume that there exists admissible control operator $B \in \mathcal{L}(\mathbb{C}^d, X_{-1})$ for $(T(t))_{t \geq 0}$ such that the control system $\dot{x}(t) = Ax(t) + Bu(t)$ is exactly controllable in finite time.

Example 7.2. Let $A : \mathcal{D}(A) \subset \ell^2 \rightarrow \ell^2$ be defined by $(Ax)_n = (s_n x_n)_n$, $(s_n)_{n \in \mathbb{N}} = (-2^n)_{n \in \mathbb{N}}$, and $\mathcal{D}(A) = \{x \in \ell^2(\mathbb{N}) \mid \sum_{n \in \mathbb{N}} (1 + |s_n|^2) |x_n|^2 < \infty\}$. Clearly, A is a discrete Riesz spectral operator, generates a C_0 -semigroup, but not a C_0 -group. Here $X_{-1} = \ell^2_{-1} = \{(x_n)_{n \in \mathbb{N}} \mid \sum_{n \in \mathbb{N}} \frac{|x_n|^2}{(1 + |s_n|^2)} < \infty\}$. Hence, we can identify $B \in \mathcal{L}(\mathbb{C}, \ell^2_{-1})$ with a sequence $(b_n) \in \ell^2_{-1}$. Let $(b_n)_{n \in \mathbb{N}} = (\sqrt{2^n})_{n \in \mathbb{N}}$. Then it holds $(b_n)_{n \in \mathbb{N}} \in \ell^2_{-1}$. The Carleson measure criterion of Weiss [Wei88] implies the admissibility of B . Exact controllability in finite time follows from [JZ01b, Theorem 2].

References

- [AI95] S. A. Avdonin and S. A. Ivanov. *Families of Exponentials*. Cambridge University Press, Cambridge, 1995.
- [AJ14] B. Augner and B. Jacob. Stability and stabilization of infinite-dimensional linear port-Hamiltonian systems. *Evol. Equ. Control Theory*, 3:207–229, 2014.
- [Aug16] B. Augner. *Stabilisation of Infinite-dimensional Port-Hamiltonian System*. PhD thesis, University of Wuppertal, 2016.
- [Cur84] R. F. Curtain. Spectral systems. *Internat. J. Control*, 39(4):657–666, 1984.
- [CZ95] R. F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.
- [CZ20] R. F. Curtain and H. Zwart. *Introduction to Infinite-Dimensional Systems Theory*, volume 71 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 2020.
- [DMSB09] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, editors. *Modeling and Control of Complex Physical Systems*. Springer-Verlag, Berlin, 2009.
- [DS71] N. Dunford and J. T. Schwartz. *Linear Operators. Part III: Spectral Operators*. Interscience Publishers [John Wiley & Sons, Inc.], New York, 1971.
- [EMvdS07] D. Eberard, B. M. Maschke, and A. J. van der Schaft. An extension of Hamiltonian systems to the thermodynamic phase space: towards a geometry of nonreversible processes. *Rep. Math. Phys.*, 60(2):175–198, 2007.
- [EN00] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

- [GGK90] I. Gohberg, S. Goldberg, and M. A. Kaashoek. *Classes of Linear Operators. Vol. I*, volume 49 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1990.
- [GW19] B.-Z. Guo and J.-M. Wang. *Control of Wave and Beam PDEs*. Communications and Control Engineering Series. Springer, Cham, 2019. The Riesz Basis Approach.
- [GX04] B.-Z. Guo and G.-Q. Xu. Riesz bases and exact controllability of C_0 -groups with one-dimensional input operators. *Systems Control Lett.*, 52(3-4):221–232, 2004.
- [GZ01] B. Guo and H. Zwart. Riesz spectral systems. *MEMORANDUM*, No. 1594, 2001.
- [HP18] J.-P. Humaloja and L. Paunonen. Robust regulation of infinite-dimensional port-Hamiltonian systems. *IEEE Trans. Automat. Control*, 63(5):1480–1486, 2018.
- [JK19a] B. Jacob and J. T. Kaiser. On exact controllability of infinite-dimensional linear port-Hamiltonian systems. *IEEE Control Systems Letters*, 3(3):661–666, 2019.
- [JK19b] B. Jacob and J. T. Kaiser. Well-posedness of systems of 1-D hyperbolic partial differential equations. *J. Evol. Equ.*, 19(1):91–109, 2019.
- [JMZ15] B. Jacob, K. Morris, and H. Zwart. C_0 -semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain. *J. Evol. Equ.*, 15(2):493–502, 2015.
- [JZ99] B. Jacob and H. Zwart. Equivalent conditions for stabilizability of infinite-dimensional systems with admissible control operators. *SIAM J. Control Optim.*, 37(5):1419–1455, 1999.
- [JZ01a] B. Jacob and H. Zwart. Exact observability of diagonal systems with a finite-dimensional output operator. *Systems Control Lett.*, 43(2):101–109, 2001.
- [JZ01b] B. Jacob and H. Zwart. Exact observability of diagonal systems with a one-dimensional output operator. *Int. J. Appl. Math. Comput. Sci.*, 11(6):1277–1283, 2001.
- [JZ12] B. Jacob and H. J. Zwart. *Linear port-Hamiltonian Systems on Infinite-Dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2012. Linear Operators and Linear Systems.
- [JZ18] B. Jacob and H. Zwart. An operator theoretic approach to infinite-dimensional control systems. *GAMM-Mitt.*, 41(4):e201800010, 14, 2018.

- [LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control Optim.*, 44:1864–1892, 2005.
- [LW83] J.-C. Louis and D. Wexler. On exact controllability in Hilbert spaces. *J. Differential Equations*, 49(2):258–269, 1983.
- [RLGMZ14] H. Ramírez, Y. Le Gorrec, A. Macchelli, and H. Zwart. Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback. *IEEE Trans. Automat. Control*, 59(10):2849–2855, 2014.
- [RW94] D. L. Russell and G. Weiss. A general necessary condition for exact observability. *SIAM J. Control Optim.*, 32(1):1–23, 1994.
- [RZLG17] H. Ramírez, H. Zwart, and Y. Le Gorrec. Stabilization of infinite dimensional port-Hamiltonian systems by nonlinear dynamic boundary control. *Automatica J. IFAC*, 85:61–69, 2017.
- [Sta05] O. Staffans. *Well-Posed Linear Systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.
- [SZ18] J. Schmid and H. Zwart. Stabilization of port-Hamiltonian systems by nonlinear boundary control in the presence of disturbances. <https://arxiv.org/abs/1804.10598v2>, 2018.
- [Tre00a] C. Tretter. Spectral problems for systems of differential equations $y' + A_0 y = \lambda A_1 y$ with λ -polynomial boundary conditions. *Math. Nachr.*, 214:129–172, 2000.
- [Tre00b] C. Tretter. Linear operator pencils $A - \lambda B$ with discrete spectrum. *Integral Equations Operator Theory*, 37(3):357–373, 2000.
- [TW09] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.
- [TW14] M. Tucsnak and G. Weiss. Well-posed systems - the LTI case and beyond. *Automatica J. IFAC*, 50(7):1757–1779, 2014.
- [vdS06] A. van der Schaft. Port-Hamiltonian systems: an introductory survey. In *International Congress of Mathematicians. Vol. III*, pages 1339–1365. Eur. Math. Soc., Zürich, 2006.
- [Vil07] J. Villegas. *A Port-Hamiltonian Approach to Distributed Parameter Systems*. PhD thesis, University of Twente, Netherlands, 2007.
- [Was87] W. Wasow. *Asymptotic Expansions for Ordinary Differential Equations*. Dover Publications, Inc., New York, 1987. Reprint of the 1976 edition.

- [Wei88] G. Weiss. Admissibility of input elements for diagonal semigroups on l^2 . *Systems Control Lett.*, 10(1):79–82, 1988.
- [XF02] G.-Q. Xu and D.-X. Feng. The Riesz basis property of a Timoshenko beam with boundary feedback and application. *IMA J. Appl. Math.*, 67(4):357–370, 2002.
- [XG03] G.-Q. Xu and B.-Z. Guo. Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation. *SIAM J. Control Optim.*, 42(3):966–984, 2003.
- [XW11] C.-Z. Xu and G. Weiss. Eigenvalues and eigenvectors of semigroup generators obtained from diagonal generators by feedback. *Commun. Inf. Syst.*, 11(1):71–104, 2011.
- [ZLMV10] H. Zwart, Y. Le Gorrec, B. Maschke, and J. Villegas. Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain. *ESAIM Control Optim. Calc. Var.*, 16(4):1077–1093, 2010.
- [Zwa10] H. Zwart. Riesz basis for strongly continuous groups. *J. Differential Equations*, 249(10):2397–2408, 2010.