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Pricing Basket Default Swaps Using Quasi-Analytic Techniques

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Abstract

This research work is based on the concept of the one-factor copula model together with the discrete Fourier transform, which is applied to reduce the dimensionality problems associated with the basket default swap pricing. We employ the Gaussian, the student-t and the Clayton one-factor copula to estimate the conditional probability of default. Incorporating the Fourier transform together with the distribution function of a counting process, we derive the quasi-analytical expression for the computation of the swap payment legs. We compute the conditional characteristic function for the corresponding portfolio loss distribution using the fast Fourier transform. Then, employ numerical integration with the aid of the inverse fast Fourier transform to retrieve the distribution function or the unconditional characteristic function. Our results show that in the absence of the trending simulation method, a semi-analytic method which involves the applications of the discrete Fourier transform can be utilized to price the basket credit default swaps.

Keywords: Discrete Fourier Transform, Fast Fourier Transform, Copulas, Convolution, Basket default swaps, Characteristics Function, Probability Distributions.

1 Introduction

Originally, the credit default swaps (CDS) market was created to equip financial institutions, such as banks, with the avenue of allocating, reducing, diversifying their credit risk exposure beyond customer base, as well as freeing up regulatory capital efficiently and affectively. Nowadays, CDS trading have become an indispensable tool which propels the market of credit derivatives, and as such, its market has been described as one of the most significant innovations in the financial markets over the past two decades. The banks, with their tremendous trading activities are seen to be the largest dealer in the market of credit derivatives. They provide liquidity by their willingness to assume the risk responsibility on their trading booklets, which they in turn, seek to hedge. The CDS have been in existence as early as 1990s, and its usage climaxed in the early 2000s prior to the 2007-2008 financial crises. Despite the huge difference

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between its gross notional amount before and after the crises, the CDS trade continues to assume a crucial avenue and platform for trading credit risk. CDS are contracts that allow the seller to pay a compensation fee to the buyer in the event of a default or other stipulated credit events in the contract such as bankruptcy, payment failure, restructuring, obligation default, repudiation or obligation acceleration.

Rule (2001) defined a CDS as "'state-dependent' but 'outcome-independent'" contract, implying that, CDS is fully dependent on whether or not a credit event has occurred and not necessarily on whether the buyer suffers loss or not. The common reference entities which exist in a CDS market ranges from single firms, to a portfolio of firms, or even to a large collection of firms which comprises an index. The CDS focus on a single entity's credit risk, and basket default swaps, on the other hand, are bilateral financial contracts that payoff upon default or multiple defaults amongst a portfolio of entities. These multi-name derivatives contracts are advantageous because they offer the investors some appealing opportunities of leveraging the spread premium, and of using one single contract to hedge a portfolio of contingent claims, such as bonds or loans. Thus, the need for obtaining a series of contracts for single securities has been averted, and also, the improvement of the relative risk-return account as compared to other equivalent credit investment tools will be made viable. They are aimed at transferring credit risk of multiple reference entities, and are generally classified into first-to-default (F2D), n2D, n-out-of-m-to-default and all-to-default.

Generally, valuing the nth-to-default (n2D) basket swaps involves computing the probability of default associated with the default times. This approach can be done in two folds: First, the Monte-Carlo simulations of random scenarios from the corresponding joint probability distribution functions. Our recent paper focused on this concept, and we were able to model the default times using copula models such as Clayton, Gumbel, Frank, Gaussian and the student-t copulas, in connection with the Monte-Carlo simulation techniques [30]. These simulated values were in turn used in the valuation of the n2D basket swaps. The next fold is the quasi-analytic techniques, which is the main highlight of this article. As an objective of this research, we will use the concept of the discrete Fourier transform (DFT) to obtain the probability distribution of default times analytically. Thus, we will utilize the one-factor model, in conjunction with the concept of DFT to obtain semi-analytic computation, and in turn, directly applied to value the swaps. Most of research works which deal with semi-tractable methods of pricing PCDs have been channelled to the CDO tranches and fewer attentions for the n2D basket swaps [21]. Thus, the contributions of this research are presented as follows:

- Empirical and statistical analysis of CDS spread data of some selected highly rated firms.
- Estimating the dependent defaults without simulations and pricing BCDS under the Gaussian and Clayton copula via the FFT and DFT.

This research is structured as follows: Section 1 introduces the topic and in section 2, we offer some literary studies on the financial applications of the Fourier transform, as well as in the valuation of basket default swaps. Section 3 explains the idea of pricing BDS and the concept of dependency on the use of copula models. Section 4 investigates the notion of the discrete and fast Fourier transform and then links the applications of the discrete Fourier transform (DFT) to the evaluation of the premium legs and the default legs of the BDS valuation. Section 5 discusses the numerical results obtained in the course of the valuation, whereas, section 6 concludes the study.

2 Literature study

In the field of quantitative risk analysis, insurance and financial engineering, one of the significant challenges lies in how dependent default events can be modelled and analysed over a specified period of time. Factor models are often used to describe correlated and dependent defaults, arising from some common systematic factors, which exist among a set of financial entities. However, when the dependence is conditioned on these latent factors, the corresponding defaults are independent. The factor model, in connection with the copula function, is fully utilized in the analytical formulation of the joint distribution function of the default times. This is because the copula function allows for statistical inference to be decoupled into inference from their dependence and inference from their marginals owing to the scanty information which exist on dependence in the area of credit risk [7].

The copula models have been used extensively in finance, especially in contingent claims pricing owing to their indispensable applications. For instance, Li (2000) focused on the issue relating to default correlation and introduced the copula function to model the joint distribution functions of the survival times. He equally proved the equivalence of the typical normal CreditMetrics methods to the use of the copula function and then illustrated numerically with the pricing of credit default swaps, as well as the first-to-default contingent claims. Using the US market as a case study, Das and Geng (2003) were able to utilise the copula models in the simulation of correlated default risk and to assess the corresponding joint distribution. Schönbucher and Schubert (2001) incorporated the dynamic default dependency in the context of default risk models which are intensity-based in order to obtain the survival probabilities and thus, the price values of credit spreads. These concepts were connected to the copula models such as the Gumbel, the Clayton and the Gaussian models.

Under interacting default intensities, the pricing and hedging of portfolio credit derivatives (PCDs), such as the basket credit default swaps (BCDS) and the collateralized debt obligations (CDOs) were also considered by Frey and Backhaus (2008). Here, they explicitly modelled the default contagion which are evident among interacting portfolios and thus, they analyzed the corresponding models through the Markov processes. Ackerer and Vatter (2017) modelled joint dependent defaults, joint dependent losses with the aid of factor copula models, and thus fitted the models to the valuation of credit index tranches. To estimate the loss distribution functions of these contingent claims, they computed the individual losses, discretely fitted on some finite grids and then calibrated some specific models to the tranche prices [1]. The concept of correlations and copula dependence modelling can be found further in the works of [6, 12, 13].

The Monte-Carlo simulations, on the other hand, have attracted much attention in the valuations of the PCDs owing to the fact that they are easily implemented to model some dependent and correlated default risks, as well as in the estimation of the expected loss. However, several analytical approximation techniques like the Parametric approximations, Fast Fourier transform (FFT), the recursion method and "probability bucketing"[18] have been introduced as alternatives to this random simulations because the former thrives best when the tail risk is being considered [23]. The Fourier transform method (FTM) has been described as a numerical technique that offered high computational speed and accuracy in estimating the loss distribution functions of a specified PCDs over a certain period of time. The DFT and its natural algorithm, the FFT, are essential tools employed in the computation of the characteristic functions which are utilised in the pricing of such PCDs.

Fusai and Roncoroni (2007) conducted a comparative study on the BCDS pricing using the

Monte-Carlo methods and two semi-analytic methods, that is, the Hull-White recursive algorithm and the FFT. They found out that the Monte-Carlo method requires exceptionally high number of simulations to ensure convergence, and the simulation method is equally not feasible when modelling a basket of more than 100 entities. For a small heterogeneous set of portfolios, the probability generating function and the FFT were equally utilised by Gregory and Laurent to compute the distribution function of the cumulative defaults [16], [21]. Whereas, Zheng (2006) employed a two-hybrid algorithm in the pricing of BCDS and CDOs by combining an analytical estimation of the loss distribution function and Monte-Carlo simulation, with more reference on large heterogeneous portfolios.

More numerical approaches are seen in the works of Bastide *et al.* (2008) where they conducted a comparative analysis on the pricing of basket default swaps using the Stein method in connection to the factor copula models. They further incorporated the Monte-Carlo techniques, probability generating function method and the recursive-based method proposed by Hull and White [18], to the valuation of the basket default swaps and the CDO tranches. Bielecki *et al.* (2007) equally estimated the dependent defaults and credit migrations under the Markovian model which describes the financial market, and thus were able to price the BCDS, together with credits/loans portfolios. Debuysscher *et al.* (2003) employed the FTM to estimate the default or the loss distribution function by first modelling the risk exposure and the default correlation of the entities. Furthermore, the authors used a factor copula to obtain the Fourier transform of the basket's aggregate loss distribution, and then, inverting the corresponding portfolio Fourier transform to obtain the needed distribution function [10]. Merino and Nyfeler (2002) focused on the numerical computation of portfolio losses experienced by credit portfolios like the CDOs. Here, they used the Poisson approximation techniques, the FFT, and the numerical integrations of the quasi Monte-Carlo simulation methods to decrease the computational complexity, which is paramount in price estimations.

3 Basket Default Swaps

This section focuses on the concept of BDS pricing, with particular attention on the n2D swaps because they are more liquid when compared to the other BDS types. BDS offers considerable advantages to credit investors, even though their liquidity cannot be compared to the highly-traded synthetic CDOs. The former permits the financial practitioners to select, manage and easily monitor smaller credit portfolios (say about 5-10) than the broader portfolio of about 100-150 which are evident in synthetic CDOs. In this context, we shall consider a homogeneous loss portfolio for ease on convenience since they are the framework of the BDS pricing. Such portfolio implies that each entity incurs a similar amount of loss in the event of a credit default, even though they might possess different values for their correlations and their default swaps spread curves.

3.1 Pricing Basket default swap

Consider a portfolio with N reference entities or obligors, having A, as the face value of the contract. Let T be the maturity of the contract, $f(t)$ the instantaneous forward rate at time t, and $B(0,t_i)$ as the discount factor supposing t_i maturity. Define n as the seniority of the basket, such that for $n=1$, we have the F2D, and let the portfolio pay a spread of β at dates $\{t_1,\cdots,t_N\}$. These periodic payments made by the protection buyer to the seller are referred to as the Premium Legs. The swap premium is paid at a frequency of $\Delta_i = t_i - t_{i-1}$, which could be annually for $\Delta = 1$, quarter-annual for $\Delta = \frac{1}{4}$. On the other hand, suppose a default occurs before the maturity time T, the seller will pay $A(1-R)$ to the buyer at maturity, or at the default time τ^n. Thus, the risk-neutral pricing measure for obtaining the value of the spread, having zero accrued premium, is solved by equating the expected values of the premium leg and the default leg. The theorem below gives the fair spread value:

Theorem 3.1. *The risk-neutral pricing for the annualized equilibrium spread of n2D swap is given below:*

$$\beta = \frac{(1-R)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]}{\sum_{i=1}^N \Delta B(0,t_i)[1-F^n(t_i)]},$$

where $F^n(t)$ is the probability distribution function of τ^n, and $f(0,t) = r$ if the instantaneous forward rate equals a constant interest rate.

For proof, see [15]. □

Furthermore, in the event of an accrued premium, the buyer pays off an amount of $\beta\Delta(t_{i-1}-t_i)$ to the seller, and this payment stream is made viable when a credit event occurs between two premium time, that is, between (t_{i-1}, t_i).

3.2 Dependence structure via copula models

In this subsection, we focus on the one-factor Gaussian, Clayton and the Student-t Copula models to estimate the joint probability of default based on the assumption of conditionally independent default times.

Gaussian copula:

Let the dynamics of an entity value i, for $i = 1, \cdots, N$ be denoted by

$$V_i = \rho_i X + \sqrt{1-\rho_i^2}\epsilon_i\,, \tag{3.1}$$

where $\rho_i \in [0,1]$ and $\mathrm{Cov}(\epsilon_i, \epsilon_j) = \rho_i\rho_j$. Here, ϵ_i denote the idiosyncratic factors and X the latent factor, with both X and ϵ_i from a standard normal distribution. Let $\tau_1, \cdots, \tau_N$ denote the default times, such that $\tau_i = F_i^{-1}(V_i)$ and let $F_1(t_1), \cdots, F_N(t_N) = \mathbb{P}(\tau_1 \le t_1), \cdots, \mathbb{P}(\tau_N \le t_N)$ be defined as the marginal distribution of default times. Then, we can have the function $F_i(t_i) = \mathbb{P}(V_i \le u_i)$ and $F(t_1, \cdots, t_N) = \mathbb{P}(V_1 \le u_1, \cdots, V_N \le u_N)$, where the parameter u_i refers to certain threshold at which the creditworthiness of the entity i falls below, supposing the i-th asset defaults. Suppose the value of X is known, then the (risk-neutral) joint conditional probability, $\mathbb{P}(V_i \le u_i|X)$ that the ith entity defaults before the given time t can be computed as:

$$\mathbb{P}(V_i \le u_i|X) = \mathbb{P}\left(\epsilon_i < \frac{\Phi^{-1}(F_i(t_i)) - \rho_i X}{\sqrt{1-\rho_i^2}}\middle| X\right), \tag{3.2}$$

and since ϵ_i are standard normally distributed, the joint conditional probability of default is finally given below as:

$$p_{t_i}^{i|X} = \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i X}{\sqrt{1-\rho_i^2}}\right). \tag{3.3}$$

Integrating out the dependency structure evident on the conditional variable X gives the unconditional probability of default, which can be denoted by $F(t_1, \cdots, t_N)$. Thus, the joint probability distribution is given below as:

$$F(t_1, \cdots, t_N) = \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N} \Phi \left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i x}{\sqrt{1-\rho_i^2}} \right) \right] \phi(x) \mathrm{d}x \,, \tag{3.4}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-x^2}{2}}$ is the standard normal density function.

Clayton copula:

Suppose the single factor X follows a Gamma distribution with parameter $\frac{1}{\theta}$, for $\theta > 0$. The conditional probability of default can be defined as follows:

$$\mathbb{P}(V_i \leq u_i | X) = \exp\left[-x\psi^{-1}(F_i(t_i)) \right] \quad \text{for all } t_1, \cdots, t_N \,, \tag{3.5}$$

where ψ defined as the generator function of the Clayton copula with $\psi^{-1}(k) = k^{-\theta} - 1$ and $\psi(k) = (1+k)^{\frac{-1}{\theta}}$. Integrating out the dependency structure over the Gamma distributed random variable x yields the unconditional probability of default, and thus, the joint probability distribution is given below as [21]:

$$F(t_1, \cdots, t_N) = \int_{0}^{\infty} \left[\prod_{i=1}^{N} \exp\left(-x\psi^{-1}(F_i(t_i)) \right) \right] \phi(x) \mathrm{d}x \,, \tag{3.6}$$

where $\phi(x) = \frac{1}{\Gamma\left(\frac{1}{\theta}\right)} \mathrm{e}^{-x} x^{\frac{(1-\theta)}{\theta}}$ is the probability density function of the Gamma function.

Student-t copula:

The student-t copula has a non-zero structure of tail dependence, and thus, it is more efficient in capturing the extreme joint events which might likely occur, especially in equity data [24]. Let R denote a χ^2 distribution random variable with η degrees of freedom. Also, let X and ϵ_i be standard normally distributed random variables such that R, X and ϵ_i are all independent. Then, we can define the latent variable to be given by:

$$V_i = \sqrt{\frac{\eta}{R}} \left(\rho_i X + \sqrt{1-\rho_i^2} \epsilon_i \right) . \tag{3.7}$$

Furthermore, let $u_i = t_{\eta}^{-1}(F_i(t_i))$ be the same threshold in which the default occurs before time t. The joint conditional probability of default $p_{t_i}^{i|X,R}$ of the ith entity prior to time t_i is given below as

$$\mathbb{P}(V_i \leq u_i | X, R) = \mathbb{P}\left(\epsilon_i \leq \frac{u_i \sqrt{\frac{R}{\eta}} - \rho_i X}{\sqrt{1-\rho_i^2}} \middle| X, R \right) .$$

Thus, we have that

$$p_{t_i}^{i|X,R} = \Phi\left(\frac{\sqrt{\frac{R}{\eta}} t_{\eta}^{-1}(F_i(t_i)) - \rho_i X}{\sqrt{1-\rho_i^2}} \right) . \tag{3.8}$$

Obtaining the unconditional probability of default using the student-t copula involves a double integration of the conditional loss distribution over two distributions, that is, the Gaussian and the Chi-square[1]. Thus, the joint probability distribution can be written below as [24]:

[1]The χ^2 distribution function is only valid for $r \geq 0$, having first and second moments as $\mathbb{E}[r] = \eta$ and $\mathbb{E}[r^2] = 2\eta$

$$F(t_1, \cdots, t_N) = \int_0^\infty \int_{-\infty}^\infty \left[\prod_{i=1}^N \Phi\left(\frac{\sqrt{\frac{r}{\eta}} t_\eta^{-1}(F_i(t_i)) - \rho_i X}{\sqrt{1-\rho_i^2}} \right) \right] \phi(x)\phi_\eta(r)\mathrm{d}x\mathrm{d}r\,, \tag{3.9}$$

where $\phi_\eta(r) = \frac{r^{\frac{\eta}{2}-1}\mathrm{e}^{\frac{-r}{2}}}{2^{\frac{\eta}{2}}\Gamma(\frac{\eta}{2})}$ is the density function of the χ^2 distribution.

This double integral is quite computationally intensive and this posses a notable disadvantage of using the student-t copula over the Gaussian copula. Thus, pricing problems and risk management of large baskets are quite impractical when the student-t copula is being utilized.

4 Discrete and Fast Fourier Transform

The Fourier transform approach or the spectral method is another technique, apart from the usual recursion method, employed in the generation of the conditional and the unconditional loss distribution functions. According to Parodi (2014), the FTM works by mapping the original distribution function into another space in which the convolution operation becomes more analytically tractable, and then transferred back to its original space after the problem has been solved. In a nutshell, the FTM approach is entirely dependent on the successful implementation of the FFT.

Definition 4.1. *Characteristic Function [17]: Let X be a continuous random variable, the characteristic function Ψ of X refereed to as the expected value e^{itX} where $i = \sqrt{-1}$ and $t \in \mathbb{R}$ (the argument of Ψ), is defined as*

$$\Psi_X(t) = \mathbb{E}\left[\mathrm{e}^{itX}\right] = \int_{-\infty}^\infty \mathrm{e}^{itX} f_X(x)\mathrm{d}x\,,$$

where $f_X(x)$ is the cumulative density function of X.

□

Note that if X has a probability density f_X, the characteristic function is equivalent to its Fourier transform having a sign reversal in its complex exponential.

Definition 4.2. *Convolution [15]: Let f and g be measurable functions, the convolution of both functions is defined as*

$$(f \otimes g)(x) \triangleq \int_{-\infty}^\infty f(s)g(x-s)\mathrm{d}s \equiv \int_{-\infty}^\infty f(x-s)g(s)\mathrm{d}s\,.$$

□

Definition 4.3. *Fourier and Inverse Fourier Transform [17]: Let X be a continuous random variable, f, an integrable function such that $\hat{f} : \mathbb{R} \to \mathbb{C}$ and $i = \sqrt{-1}$, the FT and the IFT are defined respectively as*

$$\hat{f}(t) = \int_{-\infty}^\infty \mathrm{e}^{itX} f(x)\mathrm{d}x \qquad \textit{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{e}^{-itk}\hat{f}(k)\mathrm{d}k\,.$$

□

Definition 4.4. *Discrete Fourier and inverse discrete Fourier Transform [10]: Let f be a discrete function such that $f : \{x_n\} \to \{X_k\}$ for $n, k = 0, 1, \cdots, N-1$, the DFT and the IDFT are defined respectively as:*

$$X_k = \sum_{n=0}^{N-1} x_n \mathrm{e}^{\frac{-2\pi i n k}{N}} \qquad \textit{and} \qquad x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathrm{e}^{\frac{2\pi i n k}{N}}\,,$$

□

The Fast Fourier transform, however, is the algorithm for executing the DFT of a sequence of variables. The FFT reduces the computational complexity of the DFT from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log_2 N)$, and this is so noticeable when computing a vast dataset. With regards to the immense difference between the two order of complexity, Press *et al.* (2002) stated that for an $N = 10^6$ points, it is roughly the difference between 30 seconds (CPU time) and about two weeks of CPU time measured on a microsecond cycle time computer.

4.1 Portfolio Loss Distribution

For simplicity, we link this subsection to the one-factor Gaussian copula model and the corresponding default probabilities can be used if another copula is being used. Let $Z(t)$ be the cumulative loss on the portfolio at a specified time t with a discrete distribution, and M_i the notional on each of the entity $i = 1, \cdots, N$. The conditional probability of default $p_t^{j|x}$ for the jth name up to time t, which we have conditioned on a specified Gaussian random variable $X = x$ follows from equation (3.3). For the Clayton and the student-t copula models, the conditional probabilities are given in equations (3.5) and (3.8) respectively. Define the indicator function of the default time counting process for each ith entity as $Q_i(t) = \mathbb{I}_{\{\tau_i \le t\}}$, and let R_i be the ith recovery rate, then

$$Z(t) = \sum_{i=1}^{N} (1 - R_i) M_i Q_i(t) \,.$$

For the valuation of multi-name PCDs likes the CDOs and n2D basket swaps, it is pertinent to either simulate the default times or perform an estimate for the loss distribution over a time range $[0, T]$. To calculate the distribution function of $Z(t)$, we need to compute the characteristic function $\Psi_{Z(t)}(s)$ of the portfolio loss, and a natural choice to this would be the implementation of the FFT.

Furthermore, suppose the σ-algebras $\sigma(R)$ and $\sigma(X, \tau_1, \cdots, \tau_N)$ are independent functions, we calculate the characteristic function of the aggregate loss as follows:

$$\Psi_{Z(t)}(s) = \mathbb{E}[\mathrm{e}^{isZ(t)}] \,.$$

Proposition 4.5. *Let $(\mathcal{F}_t)_{t \ge 0}$ be a given filtration. If Y is a stochastic random variable and $s < t$, then the law of iterated expectation states that:*

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s] \,.$$

For proof, see [5]. □

Conditioning on the common factor X, and from the properties of iterated expectations in Proposition (4.5), we have that

$$\Psi_{Z(t)}(s) = \mathbb{E}[\mathbb{E}[\mathrm{e}^{isZ(t)}|X = x]] = \mathbb{E}\left[\mathbb{E}\left[\prod_{j=1}^{N} \exp\{is(1 - R_j) M_j \mathbb{I}_{\{\tau_i \le t\}}\} \middle| X = x\right]\right] .$$

Since τ_j is conditionally independent on X and from the assumption of independent R_j recovery rates[2], the inner expected value can be written in the form

$$\begin{aligned}\Psi_{Z(t)}(s) =&\mathbb{E}\left[\prod_{j=1}^{N}\exp\{is(1-R)\mathbb{I}_{\{\tau_j\le t\}}\}\middle| X=x\right] = \prod_{j=1}^{N}\mathbb{E}\left[\exp\{is(1-R)\mathbb{I}_{\{\tau_j\le t\}}\}\middle| X=x\right]\\ =&\prod_{j=1}^{N}\left[(1-p_t^{j|X}) + p_t^{j|X}\mathrm{e}^{is(1-R)}\right] = \prod_{j=1}^{N}\left[1+p_t^{j|X}\left(\mathrm{e}^{is(1-R)}-1\right)\right].\end{aligned}$$

The unconditional FT can thus be computed by integrating out the common factor over the Gaussian density as shown below:

$$\Psi_{Z(t)}(s) = \mathbb{E}\left[\prod_{j=1}^{N}\left[1+p_t^{j|X}\left(\mathrm{e}^{is(1-R)}-1\right)\right]\right] = \int_{-\infty}^{\infty}\prod_{j=1}^{N}\left[1+p_t^{j|X}\left(\mathrm{e}^{is(1-R)}-1\right)\right]\phi(x)\mathrm{d}x\,.$$

Finally, the actual portfolio loss distribution of $Z(t)$ can be obtained by inverting the characteristic function with the use of a more suitable Fourier transform function, such as the inverse form of the FFT [3], and this is more efficient when non-homogeneous portfolios are being considered [26].

4.2 Derivation of main theorem

This section will give the main highlight of this research work, which involves the derivation of the basket swap premium legs, thereby presenting the quasi-analytic expression for the computation of the swap pricing premium.

Theorem 4.6. *The fair value for the n2D swap, with zero accrued premium, is given as:*

$$\beta = \frac{(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t)=j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i)=j)\right)\right]}{\sum_{i=1}^{N}\Delta_i B(0,t_i)\sum_{j=0}^{n-1}\mathbb{P}(Z(t)=j)}\,.$$

Proof:

Let $\lambda_i(t)$, also known as hazard rates or the default intensity, be the intensity of the Poisson process. Then the marginal density function F_i of default times τ_i is given below as

$$F_i(t) = \mathbb{P}(\tau_i < t) = 1-\exp\left(-\int_0^t \lambda_i(s)\mathrm{d}s\right), \qquad \text{where } \; i=1,\cdots,N.$$

For τ^n been the time of the nth default, the survival and the default probability distribution functions for the nth default are given respectively as:

$$S^n(t) = \mathbb{P}(\tau^n > t) = \mathbb{P}(Z(t) < n) = \sum_{j=0}^{n-1}\mathbb{P}(Z(t)=j) \tag{4.1}$$

$$F^n(t) = \mathbb{P}(\tau^n \le t) = \mathbb{P}(Z(t) \ge n) = \sum_{j=n}^{N}\mathbb{P}(Z(t)=j)\,. \tag{4.2}$$

The value for the premium leg (PL) for the n2D basket swap can be defined as:

$$PL = A\sum_{i=1}^{N}\beta\Delta_i B(0,t_i)\mathbb{I}_{\{\tau^n > t_i\}}\,.$$

[2]The rate R_j is considered constant here and further assumption of a constant unit notional, i.e., $\sum M_i = 1$

[3]From `jupyter notebook`, `ifft` imported from `scipy.fftpack` package

Taking the expectation values, we have

$$PL = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i)\mathbb{E}[\mathbb{I}_{\{\tau^n > t_i\}}] = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i)\mathbb{P}[Z(t_i) < n]\,,$$

where $\mathbb{P}[Z(t_i) < n]$ is the probability of having less than n defaults.

Thus, the final discounted expected value for the **PL** is given below as

$$PL = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i) \sum_{j=0}^{n-1} \mathbb{P}(Z(t) = j)\,. \tag{4.3}$$

Suppose there exists one or multiple credit events before maturity T, then the protection seller will be obligated to offset the difference between the nominal value and the value for the recovery rate R for the defaulted reference entity. The value for the default leg (DL) can be given as:

$$DL = A(1-R)B(0,\tau^n)\mathbb{I}_{\{\tau^n \leq T\}}\,.$$

Taking the expectation values, we have

$$DL = A(1-R)B(0,\tau^n)\mathbb{E}[\mathbb{I}_{\{\tau^n \leq T\}}] = A(1-R)\int_0^T B(0,t)\mathrm{d}F^n(t)\,.$$

Integration by parts yields

$$DL = A(1-R)\left[F^n(T)B(0,T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]\,.$$

Next, we convert the functions to discrete functions by replacing the distribution functions $F^n(t)$ by the probabilities of the counting processes as shown in equation (4.2), and thus we have

$$DL = A(1-R)\left[B(0,t_N)\mathbb{P}(Z(T) \geq n) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\mathbb{P}(Z(t_i) \geq n)\right]\,.$$

Hence, the value of the **DL** reduces to

$$DL = A(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t) = j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i) = j)\right)\right]\,. \tag{4.4}$$

Equations 4.3 and 4.4 give the discrete forms of the premium legs and the default legs respectively. Hence, the fair spread value (in the absence of the accrued premium) for the n2D swap is such that $\beta \implies \mathbb{E}[PL] = \mathbb{E}[DL] = 0$, so that we have

$$\beta = \frac{(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t) = j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i) = j)\right)\right]}{\sum_{i=1}^{N}\Delta_i B(0,t_i)\sum_{j=0}^{n-1}\mathbb{P}(Z(t) = j)}\,. \tag{4.5}$$

□

Equation (4.5) gives the equilibrium price (quasi-analytic expression) of the fair n2D swap spread, and the calculation is straightforward following from the estimation of the probabilities $\mathbb{P}(Z(t) = j)$. Our approach in this technique first uses the convolution concept to output vectors of probabilities which are conditioned on the given X as a common latent variable[4]. Next, we import the `fft` function from the `scipy.stats` library in `jupyter notebook` to obtain the product of the probability vectors, and the corresponding values are finally inverted back using the `ifft` function. Section 4.3 further describes the methodology of obtaining the distribution function using the FT method.

4.3 Probability of at least n defaults

Let $Z_i(T)$ denote the indicator function that an ith entity defaults before maturity T, that is, $Z_i(T) = \mathbb{I}_{\{\tau_i \leq T\}}$, and let $q_i = (1 - p_i) = \mathbb{P}(\tau_i \leq T)$ be the corresponding probability. If the number of defaults existing before T or the counting process in connection with the number of default is denoted by $L_(T) = \sum_{i=1}^{N} Z_i(t)$, then the probability of having at least n defaults can be written as

$$\mathbb{P}(Z(T) \geq n) = \sum_{k=n}^{N} \mathbb{P}(Z(T) = k)\,.$$

The distribution function of $Z(T)$ can be regarded as a convolution of the distribution functions of the $Z_i(T)$, which can be solved via the Fourier approach[15]. Let the probability vectors of Bernoulli random variables $Z_i(T)$ be defined as follows:

$$\gamma_{ij} = \begin{cases} 1 - q_i & \text{if } j = 0 \\ q_i & \text{if } j = 1 \\ 0 & \text{otherwise}\,, \end{cases}$$

as well as its DFT as

$$\eta_{ij} = \sum_{k=0}^{M-1} \gamma_{ij} \exp\left(\frac{2\pi\sqrt{-1}kj}{M}\right) \qquad \text{for } \; j = 1, \cdots, M\,.$$

Then the corresponding $\mathbb{P}(Z(T) = j)$ is obtained by taking the inverse of the DFT of the product of the FFT values η_{ij}. That is

$$\mathbb{P}(Z(T) = j) = \frac{1}{M} \sum_{k=0}^{M-1} \left[\exp\left(\frac{-2\pi\sqrt{-1}kj}{M}\right) \prod_{i=1}^{N} \gamma_{ij} \right]\,.$$

After this step, it suffices to calculate the unconditional loss distribution and this can be done by integrating the results over the Gaussian factor distribution. Finally, this value is fixed into the n2D basket credit analytics to solve for the premium legs, the default legs and then the corresponding break-even spread.

5 Results and Discussions

This section focuses on the numerical experiments obtained in the course of the BDS valuation. First, we consider some empirical and statistical analysis of some selected CDS spread data and then secondly, we employed the concept of the Gaussian and the Clayton copula, in conjunction with the DFT to out the fair spread values.

[4]For now, we only consider the Gaussian copula. This approach is equally obtainable when the distribution functions are changed to reflect the copula model been considered.

Data Analysis

Our dataset consists of a 5-year monthly CDS spread quotes emanating from the following ten companies: Xerox, Coca Cola, Boeing, IBM, Johnson & Johnson, Oracle, Pepsi, McDonald, Walmart and AT & T, and these shall be denoted by Xe, Co, Bo, IB, JJ, Or, Pe, Mc, Wa, and AT respectively. We conduct some empirical and statistical analysis based on these spread values, with values in the units of basis points (bp). The sample comprises of 132 monthly observations from January 31, 2008, till December 31, 2018, which spans across different credit ratings (from triple AAA to BB rating) and tailored around the Corporate Sector. In Figure 1 below, we made a plot of the corresponding time series, and each of the line in this plot represents the spread of different entities.

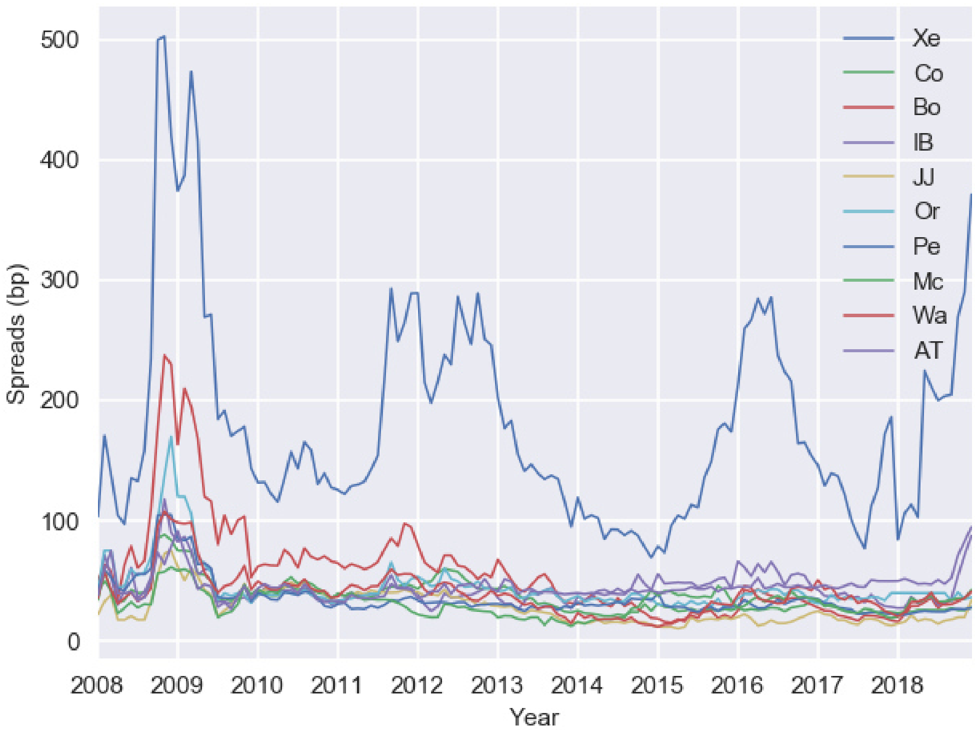


Figure 1: Time series for the 5-Year monthly CDS spreads for ten entities

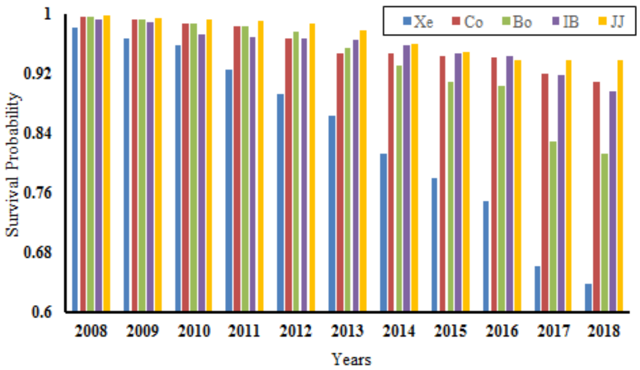
For each of the monthly CDS spread spanning through 11 year time period, we output the yearly observances by calculating the average combined quotes in each year. Based on this, we obtain the mean, standard deviation, skewness and kurtosis for each of the reference entities, and this is shown in Table 1 below. Consider the lower rate firm, the Xerox Holdings Corporation for instance; it is observed that both it's mean and the corresponding standard deviation spreads are significantly higher when compared to other less-risky firms in this paper. Throughout the time period for this CDS swap analysis, the Xerox firm achieved high swap spread in October 2008, November 2008, December 2008 and March 2009, with values 499 bp, 503 bp, 420 bp and 473 bp, respectively. This increment in spread values might be the after effect of the global financial crises of 2007-2008 which rendered many entities downgraded as a result of some perceived credit risks. Furthermore, the firm was able to recuperate during the last eight months of 2014 till the first-quarter of 2015, where it became accountable for smaller spread values having the lowest in December 2014 at 69 bp.

As for the skewness and kurtosis, the values are generally small for all the reference entities. Skewness ascertains the symmetric structure of the distribution about its mean. From Table 1, we observed that fewer reference entities like the Xerox, Coca-Cola, Boeing and AT & T had a majority of their dataset to be positively skewed, whereas the other reference entities had many of their data to be skewed to the left. Thus, negatively skewed datasets are mostly associated with lower credit spreads, and this was in line with the explanations reported by Cremers *et al.* (2008). Kurtosis, on the other hand, measures how light-tailed or heavy-tailed a distribution is in connection to the normal distribution. As can be observed from Table 1, we noticed that the majority of the entities dataset had lighter tails in comparison to their normal distribution tails, especially the Pepsi Corporation which assumed the highest. Whereas, the distribution functions for firms such as Oracle and McDonald had much heavier tails.

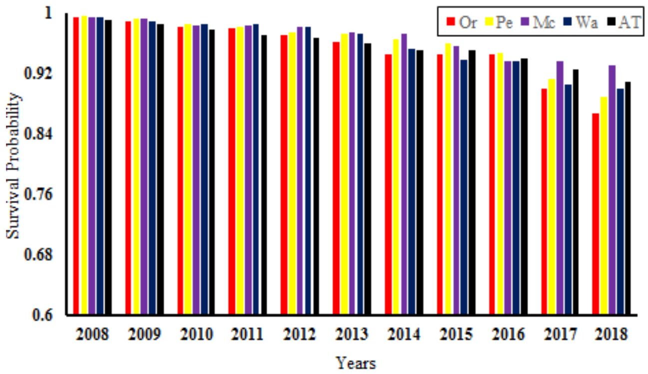
Table 1: Summary statistics for the 5-Year monthly CDS spreads for ten entities

Year	Entities (Mean ($\bar{X}$) and Standard Deviation (SD))																			
	Xe		Co		Bo		IB		JJ		Or		Pe		Mc		Wa		AT	
	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
2008	224.8	156.1	25.4	1.2	30.5	6.7	42.0	19.1	18.7	5.2	38.5	2.4	24.5	1.9	33.0	5.8	32.0	5.9	57.6	16.1
2009	269.5	114.6	28.8	3.2	21.0	3.3	35.6	4.4	17.5	3.7	35.1	2.5	26.3	4.2	25.5	4.2	35.0	8.1	46.8	2.5
2010	138.1	15.3	27.4	2.3	35.9	5.2	56.8	6.7	17.6	3.4	39.5	3.3	31.8	3.1	34.6	3.8	38.8	5.8	45.6	2.3
2011	187.6	69.6	26.3	1.8	19.8	3.5	48.5	3.3	15.3	3.6	30.4	2.3	28.2	3.9	38.6	3.3	21.9	7.6	42.6	1.7
2012	247.4	30.5	23.3	2.8	30.0	5.3	41.8	5.6	15.2	1.1	35.2	2.4	31.8	2.2	22.5	8.5	17.8	2.9	39.9	1.4
2013	146.9	29.4	33.9	7.4	47.1	10.3	34.7	5.9	23.2	4.6	38.8	3.9	28.1	1.8	17.5	2.9	29.0	7.7	42.1	4.1
2014	92.5	13.1	50.2	6.5	62.2	8.4	36.5	5.9	35.7	4.1	48.6	4.9	31.2	1.4	24.8	3.8	41.7	5.8	43.3	4.7
2015	124.4	37.9	40.9	4.1	72.5	13.7	40.9	5.9	39.1	3.1	43.1	9.5	30.6	3.5	34.0	4.4	48.4	7.0	38.2	4.8
2016	228.4	47.7	40.6	3.1	67.4	5.5	38.9	4.2	42.6	4.4	38.1	2.8	36.4	4.5	44.0	5.3	44.5	4.2	41.7	5.4
2017	127.9	31.9	50.1	16.9	125.2	48.3	51.3	21.2	41.6	13.3	63.0	33.9	55.5	21.2	39.8	14.0	63.6	22.7	47.0	19.3
2018	198.5	86.5	51.9	21.0	102.4	71.5	59.6	26.3	34.8	21.4	77.5	39.8	64.5	25.1	39.7	14.2	55.0	28.9	52.2	16.8
	Entities (Kurtosis (Kur) and Skewness (SK))																			
	Xe		Co		Bo		IB		JJ		Or		Pe		Mc		Wa		AT	
	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur	SK	Kur
2008	1.19	-0.30	1.14	-0.47	1.24	0.03	1.06	0.68	1.12	-0.09	1.51	1.66	1.08	-0.56	0.55	-1.59	1.17	-0.43	0.51	-1.52
2009	0.63	-1.20	0.54	-1.28	0.51	-0.69	0.57	-1.30	0.01	-1.01	0.94	-0.74	0.42	-1.54	0.04	-1.20	0.80	-1.04	1.31	1.39
2010	0.54	-0.60	-0.07	-1.09	0.55	-0.81	-0.78	-0.16	-0.85	1.39	-1.29	3.33	-1.03	-0.01	0.05	-0.76	-0.30	0.31	-0.83	-0.78
2011	0.56	-1.68	0.33	-1.49	0.89	-0.68	-0.86	1.61	-1.32	0.86	1.19	1.18	0.35	-1.16	-0.81	0.93	0.05	-1.04	0.59	-0.69
2012	0.06	-0.98	0.31	-1.59	0.77	0.06	-0.82	-0.06	-0.14	-0.63	1.26	1.75	-0.77	1.33	-0.01	-1.25	0.14	-1.00	-0.08	-0.65
2013	0.31	0.28	0.21	-1.14	0.26	-0.34	0.04	-1.04	-0.60	-0.49	-0.14	-1.09	-0.29	-0.59	-0.84	-0.61	-0.45	-0.43	1.71	2.00
2014	0.27	0.66	1.61	3.43	-0.63	0.35	1.07	2.23	0.12	2.11	0.49	0.80	0.28	-1.62	1.36	0.96	-0.03	-0.11	0.96	0.31
2015	0.35	-1.28	1.44	2.39	0.18	-1.40	0.67	0.63	-0.25	-1.99	0.70	-0.21	2.40	7.41	0.72	0.40	0.13	-1.70	-0.13	-1.70
2016	-0.41	-1.24	1.54	3.37	-0.57	0.27	-0.27	-0.51	0.21	-1.08	-0.88	0.20	-0.28	-1.32	-0.58	0.66	-0.18	-1.95	2.52	7.05
2017	0.22	-0.13	1.28	0.90	0.72	-0.10	-0.45	-0.68	0.54	-0.33	0.19	-0.06	0.87	-0.44	0.45	0.41	0.41	-0.10	0.39	-1.63
2018	0.43	-0.18	0.23	0.57	-0.82	1.03	1.69	2.00	2.46	7.21	-1.55	0.95	-0.12	-0.78	-1.04	1.94	-0.51	0.55	1.63	1.47

Under the risk-neutral pricing measure $\mathcal{Q}$, consider a model of exponential default which have intensity λ. The CDS spread is directly proportional to $(1-R)$, with λ as constant and we further assume a standard recovery rate $R = 40\%$. The survival probability of the set of data frames can be obtained using the expression $\exp(-s * t/(1-R))$. In Figures 2a and 2b below, we made plots for the probability of survival of the first five and last five entities.



(a) Survival Probabilities of First 5 Entities



(b) Survival Probabilities of Last 5 Entities

Figure 2: Probability of Survival

As time increased, the probability of survival declined as observed in the plots above. Generally, the firms considered in this research are highly-rated entities by the credit agencies, and as such, the probability that each of them survive still remained high. However, this was in contrast to the Xerox firm, denoted by Xe, as a sharp descent was seen in its tendency to survive as time goes beyond 2018.

Next, we output the correlation matrix for the reference entity pairs on our dataset using Kendall's τ, a rank correlation measure which is based on the concept of concordance and discordance. For $B = (x-\tilde{x})(y-\tilde{y})$ and according to Agresti (2010), Kendall's τ is defined by

$$\tau = \mathbb{P}(B > 0) - \mathbb{P}(B < 0)\,,$$

where the pairs $(x, \tilde{x})$ and $(y, \tilde{y})$ following from the joint distribution function $F(x, y)$ are concordant supposing $x > \tilde{x}$ and $y > \tilde{y}$. Furthermore, for each pairs (x_i, y_i) and (x_j, y_j), let $x_{ij} = \mathrm{sgn}(x_i - x_j)$ and $y_{ij} = \mathrm{sgn}(y_i - y_j)$, then the Kendall's τ is[5]

$$\tau = \frac{2}{N(N-1)} \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{i<j}^{N} x_{ij} y_{ij}\,.$$

We thus implemented `data.corr(method ='kendall')` in ipython notebook, where data is the file, in order to output the correlations with respect to any two data sets. Using Kendall's τ, we presented the summary of the Kendall's τ for our reference entities in Table 2 below:

[5]The sign or the signum function, also referred to as $\mathrm{sgn}(x)$ is defined as

$$\mathrm{sgn}(x) = \frac{\mathrm{d}}{\mathrm{d}x}|x|, \quad x \neq 0 \qquad \text{OR} \qquad \mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 1 \\ 0 & \text{if } x = 0\,. \end{cases}$$

Table 2: Kendall's τ correlation matrices for pairs of entities

Kendall's Tau (τ)	Xe	Co	Bo	IB	JJ	Or	Pe	Mc	Wa	AT
Xe	1.0000	0.4237	0.3986	0.2284	0.3680	0.4428	0.1793	0.1522	0.4645	0.2738
Co	0.4237	1.0000	0.6193	0.1047	0.6395	0.4772	0.4641	0.2490	0.5742	0.0407
Bo	0.3986	0.6193	1.0000	0.0862	0.6430	0.4867	0.4756	0.1760	0.6026	-0.0703
IB	0.2284	0.1047	0.0862	1.0000	0.0780	0.1411	0.2625	0.3698	0.1466	0.2466
JJ	0.3680	0.6395	0.6430	0.0780	1.0000	0.3569	0.3764	0.2766	0.6460	0.0322
Or	0.4428	0.4772	0.4867	0.1411	0.3568	1.0000	0.3322	0.0708	0.4147	0.2160
Pe	0.1793	0.4641	0.4756	0.2625	0.3764	0.3322	1.0000	0.2530	0.4046	-0.0466
Mc	0.1522	0.2490	0.1760	0.3698	0.2766	0.0708	0.2530	1.0000	0.3135	0.1788
Wa	0.4645	0.5742	0.6026	0.1466	0.6460	0.41470	0.4046	0.3135	1.0000	0.1257
AT	0.2738	0.0407	-0.0703	0.2466	0.0322	0.2160	-0.0466	0.1788	0.1257	1.0000

As can be observed, there are strong positive correlations which exist between firms Co, Bo, JJ and Wa. There also exist weak negative correlations between firms Bo, Pe versus AT, whereas the remaining are mostly weak positive correlations.

Gaussian Copula and DFT: In the computation of the n2D basket swap, we first compute the convolution of the corresponding vectors using the DFT techniques, the unconditional probabilities of default, the probabilities of having at least n defaults, and finally, the estimated swap prices. The following parameters $\lambda = 0.01, \rho = 0.35, N = 10, T = 1, \cdots, 5$ are utilised for Tables 3, 4 and Figure 3. Also, additional parameters of $R = 0.4, \Delta t = 1$ (that is, the frequency of the swap premium payment and $\Delta t = 1$ implies annual payment), $r = 0.06$ for Table 5, and the following results were obtained:

Table 3: Gaussian - Convolution of probabilities with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$5.3660\mathcal{E}^{-2}$	$5.3122\mathcal{E}^{-2}$	$5.2460\mathcal{E}^{-2}$	$5.1704\mathcal{E}^{-2}$	$5.0871\mathcal{E}^{-2}$
$3.2922\mathcal{E}^{-4}$	$8.6175\mathcal{E}^{-4}$	$1.5108\mathcal{E}^{-3}$	$2.2430\mathcal{E}^{-3}$	$3.0368\mathcal{E}^{-3}$
$9.0895\mathcal{E}^{-7}$	$6.2907\mathcal{E}^{-6}$	$1.9580\mathcal{E}^{-5}$	$4.3789\mathcal{E}^{-5}$	$8.1580\mathcal{E}^{-5}$
$1.4897\mathcal{E}^{-9}$	$2.7212\mathcal{E}^{-8}$	$1.5037\mathcal{E}^{-7}$	$5.0658\mathcal{E}^{-7}$	$1.2987\mathcal{E}^{-6}$
$1.5967\mathcal{E}^{-12}$	$7.7251\mathcal{E}^{-11}$	$2.6191\mathcal{E}^{-10}$	$3.8459\mathcal{E}^{-9}$	$1.3567\mathcal{E}^{-8}$
$1.1866\mathcal{E}^{-15}$	$1.5038\mathcal{E}^{-13}$	$2.6191\mathcal{E}^{-12}$	$2.0021\mathcal{E}^{-11}$	$9.7188\mathcal{E}^{-11}$
$1.1686\mathcal{E}^{-17}$	$2.0605\mathcal{E}^{-16}$	$6.2899\mathcal{E}^{-15}$	$7.2382\mathcal{E}^{-14}$	$4.8348\mathcal{E}^{-13}$
$-6.2582\mathcal{E}^{-18}$	$-6.4455\mathcal{E}^{-18}$	$1.0328\mathcal{E}^{-17}$	$1.8424\mathcal{E}^{-16}$	$1.6484\mathcal{E}^{-15}$
$-1.8498\mathcal{E}^{-18}$	$7.8333\mathcal{E}^{-19}$	$3.0567\mathcal{E}^{-18}$	$-3.0652\mathcal{E}^{-18}$	$9.5703\mathcal{E}^{-18}$
$-8.7018\mathcal{E}^{-18}$	$-6.0027\mathcal{E}^{-19}$	$8.2505\mathcal{E}^{-18}$	$-7.3395\mathcal{E}^{-18}$	$-4.6489\mathcal{E}^{-18}$
$5.3130\mathcal{E}^{-18}$	$-3.8145\mathcal{E}^{-18}$	$-5.1768\mathcal{E}^{-18}$	$9.8087\mathcal{E}^{-18}$	$8.1739\mathcal{E}^{-19}$

Table 3 outputs the probability vectors of obtaining α-defaults, such that $\alpha = 0, 1, \cdots, N$, which were conditioned on the latent variable X. Note that $A\mathcal{E}^{-a} = A \times 10^{-a}$. The FT approach played a unique role in this result outputs, as both the FFT and the IFFT were employed in the solution of the DFT problem set. Generally, on the positive probability scale, we observed from the table that the conditional probability vectors reduced as N increased and this monotone decrease was altered when the negative scale was reached. For instance, the conditional probability of one default is normally larger when compared to having more than one defaults. With respect to time, the conditional probability vectors decreased with an increase in time if no default was considered, whereas, for defaults greater than 0 and up to 7, the probability values increased steadily with respect to time. For defaults greater than 7, we observed a haphazard movement of the probability vectors which could be as a result of the Gaussian model and from the fact that the latent variable was drawn from the whole real line $\mathbb{R}$.

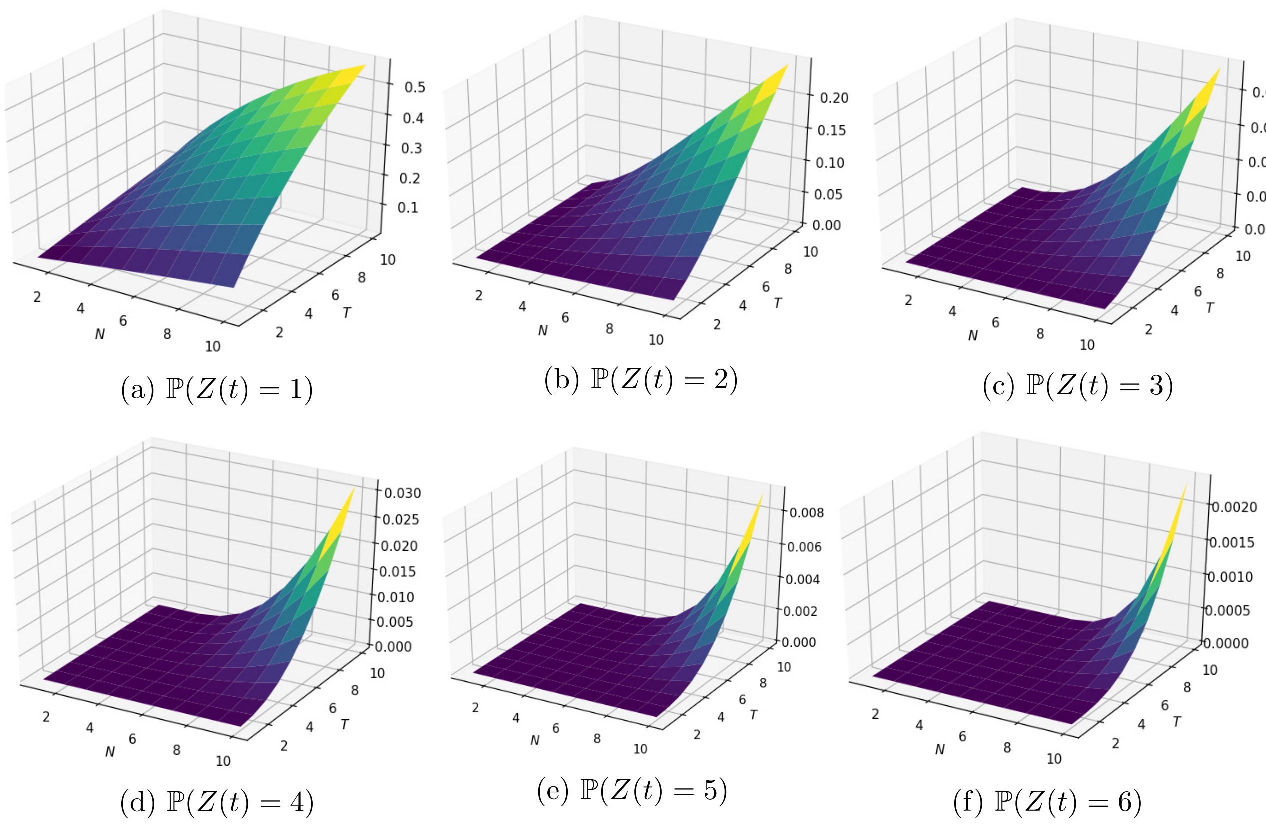
Next, we seek to compute the corresponding unconditional probabilities of default or the marginal probabilities, and these are obtained by numerical integration over the factors distribution, thereby integrating out the dependency property on the conditional variable. In this instance, we truncate the domain of the normal distribution to be $[-7, 7]$, and the following values give the probabilities:

Table 4: Gaussian - Unconditional probabilities of defaults with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$9.0940\mathcal{E}^{-1}$	$8.3114\mathcal{E}^{-1}$	$7.6194\mathcal{E}^{-1}$	$7.0009\mathcal{E}^{-1}$	$6.4445\mathcal{E}^{-1}$
$8.2542\mathcal{E}^{-2}$	$1.4388\mathcal{E}^{-1}$	$1.9103\mathcal{E}^{-1}$	$2.2750\mathcal{E}^{-1}$	$2.5560\mathcal{E}^{-1}$
$7.3046\mathcal{E}^{-3}$	$2.1390\mathcal{E}^{-2}$	$3.8378\mathcal{E}^{-2}$	$5.6549\mathcal{E}^{-2}$	$7.4922\mathcal{E}^{-2}$
$6.8320\mathcal{E}^{-4}$	$3.0885\mathcal{E}^{-3}$	$7.1520\mathcal{E}^{-3}$	$1.2642\mathcal{E}^{-2}$	$1.9307\mathcal{E}^{-2}$
$6.6161\mathcal{E}^{-5}$	$4.3551\mathcal{E}^{-4}$	$1.2609\mathcal{E}^{-3}$	$2.6173\mathcal{E}^{-3}$	$4.5344\mathcal{E}^{-3}$
$6.3904\mathcal{E}^{-6}$	$5.8752\mathcal{E}^{-5}$	$2.0788\mathcal{E}^{-4}$	$4.9902\mathcal{E}^{-4}$	$9.6970\mathcal{E}^{-4}$
$5.8942\mathcal{E}^{-7}$	$7.3413\mathcal{E}^{-6}$	$3.1224\mathcal{E}^{-5}$	$8.5716\mathcal{E}^{-5}$	$1.8529\mathcal{E}^{-4}$
$4.9250\mathcal{E}^{-8}$	$8.1241\mathcal{E}^{-7}$	$4.1028\mathcal{E}^{-6}$	$1.2776\mathcal{E}^{-5}$	$3.0535\mathcal{E}^{-5}$
$3.4646\mathcal{E}^{-9}$	$7.4428\mathcal{E}^{-8}$	$4.4232\mathcal{E}^{-7}$	$1.5530\mathcal{E}^{-6}$	$4.0864\mathcal{E}^{-6}$
$1.8130\mathcal{E}^{-10}$	$5.0100\mathcal{E}^{-9}$	$3.4811\mathcal{E}^{-8}$	$1.3724\mathcal{E}^{-7}$	$3.9641\mathcal{E}^{-7}$
$5.2713\mathcal{E}^{-12}$	$1.8581\mathcal{E}^{-10}$	$1.5029\mathcal{E}^{-9}$	$6.6358\mathcal{E}^{-9}$	$2.1003\mathcal{E}^{-8}$

Table 4 outputs the unconditional probability of having n defaults using numerical integration, and we used the `integrate.quad` function found in the `scipy` library to execute the integral. The probabilities became more uniform upon the removal of the dependency structure. For zero defaults, the unconditional probabilities decreased with an increase in time and increased with an increase in time when the defaults were bigger than zero. Also, a negative correlation was observed between the number of defaults and the probabilities, notwithstanding the time value.

Furthermore, we applied the concept of the counting process to estimate the probabilities of having less than n entities defaulting at a specified time t_i in a given portfolio. This is denoted by $\mathbb{P}(Z(t_i) < n)$ and the following Figure 3 showed the probabilities when the $n = 1, 2, 3, 4, 5, 6$ out of 10 entities. We observed that the probability of having at least n defaults was generally greater than that of the $n + 1$ defaults. This was due to the fact that it took more time for $n + 1$ reference entities to default in comparison to n entity defaulting. The default probability equally increased with an increase in time since there was a greater likelihood of credit events amongst the entities in a given basket.



(a) $\mathbb{P}(Z(t) = 1)$ (b) $\mathbb{P}(Z(t) = 2)$ (c) $\mathbb{P}(Z(t) = 3)$

(d) $\mathbb{P}(Z(t) = 4)$ (e) $\mathbb{P}(Z(t) = 5)$ (f) $\mathbb{P}(Z(t) = 6)$

Figure 3: Gaussian - Probability of n defaults against time T and number of entities N

Having computed the above steps, that is, the conditional probabilities of default with respect to the Gaussian density function, the unconditional probabilities of default, and the probabilities of having at least n entities defaulting, it suffices to incorporate all these in the valuation of the n2D basket swaps.

Table 5: Comparison of n2D BDS prices using FT under the one-factor Gaussian copula

Rank	n2D swap premium via Fourier Transform						
	$\lambda = 0.01$	$\lambda = 0.015$	$\lambda = 0.02$	$\lambda = 0.025$	$\lambda = 0.03$	$\lambda = 0.035$	$\lambda = 0.04$
1	553.1536	826.3604	1105.7815	1393.6095	1691.2408	1999.6699	2319.6655
2	164.6480	285.4171	415.1819	550.7922	690.7409	834.2651	980.9814
3	54.3331	112.7715	183.2751	262.0463	346.6463	435.4700	527.4395
4	17.0634	42.9693	79.2484	123.9775	175.4164	232.1254	292.9536
5	4.8727	14.9355	31.5276	54.5161	83.3601	117.3604	155.7923
6	1.2279	4.5737	11.0928	21.3393	35.5340	53.6560	75.5247
7	0.2632	1.1874	3.3139	7.1201	12.9823	21.1591	31.7966
8	0.0452	0.2466	0.7921	1.9056	3.8195	6.7525	10.8946
9	0.0056	0.0367	0.1357	0.3663	0.8079	1.5590	2.7135
10	0.0004	0.0030	0.0126	0.0383	0.0937	0.1972	0.3719

In Table 5, we output the fair spread values under the Gaussian copula model solved via the DFT approach, and we varied both the default intensity and the seniority of the portfolio. The hazard rates or the intensity defaults are indispensable variables in the valuation of n2D basket swap, as they measure the conditional probabilities of having no earlier default in any given year. The hazard rate increase led to a corresponding decrease in the probability of survival, and when the survival probability was significantly low, the chances of the portfolio having no default became extremely low. Hence this behaviour accounted for the presence of larger spread values and thus, both the spreads and the hazard rates were directly proportional in confirmation with the results of Jabbour *et al.* (2008). The rank, on the other hand, measures the seniority of the defaulting entity, as rank = 1 means the first entity to default in a portfolio of $N = 10$ entities. Thus, as the rank increased, the likelihood of experiencing a default became slimmer and thus giving rise to smaller swap spread.

Clayton Copula and DFT: We compared results using the Clayton copula in connection with the DFT techniques. This copula model arises from an asymmetric Archimedean family, with much dependence clustered in its negative tail than in their positive tail, and very useful for modelling correlated defaults. We employ the same techniques as in the Gaussian copula model, but we now use the main probability distribution (Gamma Distribution) which describes the Clayton copula model. In contrast to the Gaussian correlation structure, here, we use the Clayton copula parameter $\theta \in (0, \infty)$ which measures the dependency level between the given variables. Using a parameter of $\theta = 0.193$, we output the probability vectors defined under the Clayton copula and the following results are given in Table 6:

Table 6: Clayton - Convolution of probabilities with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$6.4801\mathcal{E}^{-3}$	$3.8768\mathcal{E}^{-3}$	$2.4990\mathcal{E}^{-3}$	$1.6756\mathcal{E}^{-3}$	$1.1661\mathcal{E}^{-3}$
$3.8987\mathcal{E}^{-3}$	$4.4991\mathcal{E}^{-3}$	$4.1476\mathcal{E}^{-3}$	$3.5809\mathcal{E}^{-3}$	$3.0145\mathcal{E}^{-3}$
$1.0555\mathcal{E}^{-3}$	$2.3496\mathcal{E}^{-3}$	$3.1090\mathcal{E}^{-3}$	$3.4437\mathcal{E}^{-3}$	$3.5068\mathcal{E}^{-3}$
$1.6935\mathcal{E}^{-4}$	$7.2712\mathcal{E}^{-4}$	$1.3810\mathcal{E}^{-3}$	$1.9625\mathcal{E}^{-3}$	$2.4175\mathcal{E}^{-3}$
$1.7830\mathcal{E}^{-5}$	$1.4767\mathcal{E}^{-4}$	$4.0257\mathcal{E}^{-4}$	$7.3396\mathcal{E}^{-4}$	$1.0937\mathcal{E}^{-3}$
$1.2873\mathcal{E}^{-6}$	$2.0565\mathcal{E}^{-5}$	$8.0469\mathcal{E}^{-5}$	$1.8822\mathcal{E}^{-4}$	$3.3927\mathcal{E}^{-4}$
$6.4541\mathcal{E}^{-8}$	$1.9888\mathcal{E}^{-6}$	$1.1170\mathcal{E}^{-5}$	$3.3520\mathcal{E}^{-5}$	$7.3088\mathcal{E}^{-5}$
$2.2189\mathcal{E}^{-9}$	$1.3189\mathcal{E}^{-7}$	$1.0632\mathcal{E}^{-6}$	$4.0935\mathcal{E}^{-6}$	$1.0797\mathcal{E}^{-5}$
$5.0062\mathcal{E}^{-11}$	$5.7398\mathcal{E}^{-9}$	$6.6414\mathcal{E}^{-8}$	$3.2805\mathcal{E}^{-7}$	$1.0467\mathcal{E}^{-6}$
$6.6932\mathcal{E}^{-13}$	$1.4803\mathcal{E}^{-10}$	$2.4584\mathcal{E}^{-9}$	$1.5579\mathcal{E}^{-8}$	$6.0128\mathcal{E}^{-8}$
$4.0270\mathcal{E}^{-15}$	$1.7179\mathcal{E}^{-12}$	$4.0951\mathcal{E}^{-11}$	$3.3294\mathcal{E}^{-10}$	$1.5544\mathcal{E}^{-9}$

From Table 6, when there was no default, we observed a steady decrease in the values of the conditional probability vectors across time. This monotone decrease was altered when the default increased to one, and for 2 defaults up till 10, the probability vectors increased monotonically. These altered movements were as a result of the random component in the Gamma distribution function of the Clayton copula model. Additionally, the convolution of vectors reduced at time = 1 for all the number of defaults. From time = 2 till 5, we observed that the probability vectors increased at $n = 1$, and then reduced gradually when the number of defaults became greater than one.

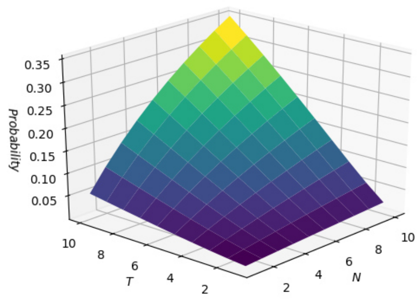
Next, the corresponding unconditional probabilities of default or the marginal probabilities were obtained by numerical integration over the factors distribution (Gamma distribution), thereby integrating out the dependency property on the conditional variable. From equation (3.6), the joint probability distribution function which served as our unconditional probability of default in this instance was obtained by integrating over the positive real line, that is, $\mathbb{R}_{>0} = (0, \infty)$, and the following values give the probabilities:

Table 7: Clayton - Unconditional probabilities of defaults with varying time

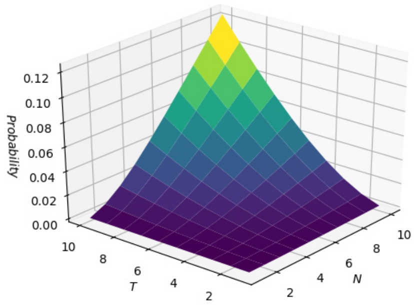
$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$9.7835\mathcal{E}^{-1}$	$9.4475\mathcal{E}^{-1}$	$9.0761\mathcal{E}^{-1}$	$8.6921\mathcal{E}^{-1}$	$8.3062\mathcal{E}^{-1}$
$1.9830\mathcal{E}^{-2}$	$4.8995\mathcal{E}^{-2}$	$7.6774\mathcal{E}^{-2}$	$1.0440\mathcal{E}^{-1}$	$1.3019\mathcal{E}^{-1}$
$1.6222\mathcal{E}^{-3}$	$6.1101\mathcal{E}^{-3}$	$1.2613\mathcal{E}^{-2}$	$2.0569\mathcal{E}^{-2}$	$2.9577\mathcal{E}^{-2}$
$1.7854\mathcal{E}^{-4}$	$9.6272\mathcal{E}^{-4}$	$2.4317\mathcal{E}^{-3}$	$4.5705\mathcal{E}^{-3}$	$7.3400\mathcal{E}^{-3}$
$2.0347\mathcal{E}^{-5}$	$1.5406\mathcal{E}^{-4}$	$4.6942\mathcal{E}^{-4}$	$1.0050\mathcal{E}^{-3}$	$1.7839\mathcal{E}^{-3}$
$2.1372\mathcal{E}^{-6}$	$2.2716\mathcal{E}^{-5}$	$8.3362\mathcal{E}^{-5}$	$2.0285\mathcal{E}^{-4}$	$3.9715\mathcal{E}^{-4}$
$1.9194\mathcal{E}^{-7}$	$2.8810\mathcal{E}^{-6}$	$1.2777\mathcal{E}^{-5}$	$3.5414\mathcal{E}^{-5}$	$7.6567\mathcal{E}^{-5}$
$1.3785\mathcal{E}^{-8}$	$2.9456\mathcal{E}^{-7}$	$1.5869\mathcal{E}^{-6}$	$5.0285\mathcal{E}^{-6}$	$1.2040\mathcal{E}^{-5}$
$7.3106\mathcal{E}^{-10}$	$2.2416\mathcal{E}^{-8}$	$1.4751\mathcal{E}^{-7}$	$5.3669\mathcal{E}^{-7}$	$1.4281\mathcal{E}^{-7}$
$2.5210\mathcal{E}^{-11}$	$1.1175\mathcal{E}^{-9}$	$9.0305\mathcal{E}^{-9}$	$3.7882\mathcal{E}^{-8}$	$1.1241\mathcal{E}^{-7}$
$4.2145\mathcal{E}^{-13}$	$2.7192\mathcal{E}^{-11}$	$2.7113\mathcal{E}^{-10}$	$1.3164\mathcal{E}^{-9}$	$4.3706\mathcal{E}^{-9}$

In Table 7, we output the unconditional probability of defaults. The observed values decreased with an increase in time when there was no default, but increased steadily when defaults occurred in the basket of the entities. Also, the values obtained for the Clayton copula were generally lesser in comparison with the Gaussian copula, but only with the exception of the zero default values.

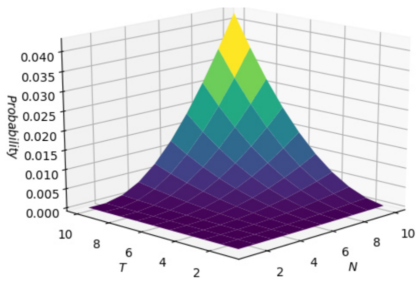
Next, we apply the concept of counting process in connection with the Clayton copula, to compute the probabilities of having less than n entities defaulting at a specified time t_i in a given portfolio. This is denoted by $\mathbb{P}(Z(t_i) < n)$ and the following figures give the probabilities when the $n = 1, 2, 3, 4, 5, 6$ out of 10 entities.



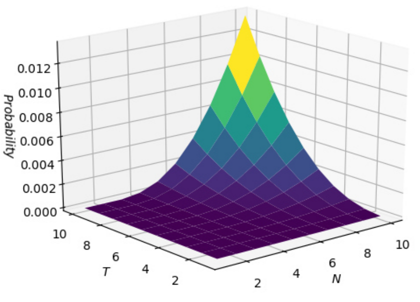
(a) $\mathbb{P}(Z(t) = 1)$



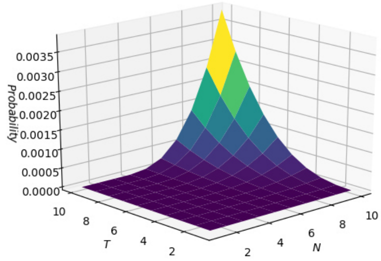
(b) $\mathbb{P}(Z(t) = 2)$



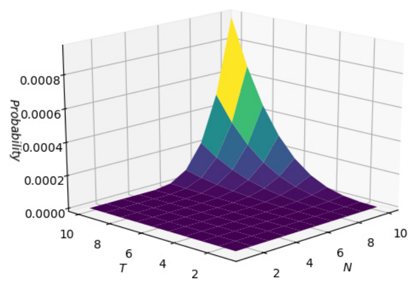
(c) $\mathbb{P}(Z(t) = 3)$



(d) $\mathbb{P}(Z(t) = 4)$



(e) $\mathbb{P}(Z(t) = 5)$



(f) $\mathbb{P}(Z(t) = 6)$

Figure 4: Clayton - Probability of n defaults against time T and number of entities N

Figure 4 showed the results obtained, and we observed that the probabilities became extremely small in comparison with the Gaussian probabilities. This further asserts that as the number of at least n defaults goes up, the corresponding probabilities declines drastically.

Table 8: Comparison of n2D BDS prices using FT under the one-factor Clayton copula

Rank	n2D swap premium via Fourier Transform ($\theta = 0.1938$)						
	$\lambda = 0.01$	$\lambda = 0.015$	$\lambda = 0.02$	$\lambda = 0.025$	$\lambda = 0.03$	$\lambda = 0.035$	$\lambda = 0.04$
1	256.9671	417.5711	587.0253	763.7078	946.7921	1135.8055	1330.4752
2	74.0353	143.4878	225.2093	315.9714	413.6235	516.6302	623.8508
3	23.6040	54.1627	95.1930	145.0698	202.3589	265.8131	334.3513
4	7.2595	19.6489	38.7389	64.4075	96.2804	133.8511	176.5486
5	2.0313	6.4822	14.3657	26.1523	42.0887	62.2469	86.5653
6	0.4933	1.8606	4.6480	9.2969	16.1731	25.5603	37.6588
7	0.0986	0.4416	1.2483	2.7534	5.1971	8.8124	13.8145
8	0.0151	0.0806	0.2591	0.6329	1.3012	2.3762	3.9787
9	0.0016	0.0100	0.0367	0.0998	0.2245	0.4432	0.7955
10	0.0001	0.0006	0.0027	0.0081	0.0200	0.0427	0.0824

Table 8 gives the corresponding swap values with regards to varying intensity default and the rank of the portfolio default. The same characteristics were exhibited by the swap values in comparison with the Gaussian copula, but we observed that the Clayton swap values were significantly lesser than that of the Gaussian swap. This could be as a result of the choice of the discrepancies between the θ-Clayton dependency and the ρ-Gaussian correlation structure. Furthermore, it could result from the the fact that the Clayton aims at modelling the extreme values for tail dependency which makes the values to be restricted in some sense, and finally the varied distribution functions of both copula families could equally account for the different spread values. However, the question still remains on which copula seems to be the best?, and the answer lies in the category of default events we are modelling. For instance, the Clayton copula remains the best in comparison to the Gaussian copula, especially the modelling of some extreme joint events in systematic risk, in the joint tail or the fat-tailed functions. For modelling events in the low tailed distributions or operational risk, or high-stress forecasting under credit risk scenarios, the Gaussian copula model outperforms the Clayton [29, 11, 20].

6 Conclusion

The price estimation of the basket credit default swaps is directly linked to the computation of the joint probability of default, and this research successfully obtained such probability without simulation. Here, we employed the quasi-analytic techniques which involved a combination of the copula modelling and discrete Fourier transform to compute the probabilities of default, thereby connecting it to the price estimation of the swap payment stream. We introduced the concept of one-factor Gaussian, student-t and the Clayton copula in describing the probability distribution functions. This concept further led to some semi-analytic expressions for the conditional and the unconditional portfolio loss distribution functions, and the corresponding expressions were solved efficiently with numerical integration via the discrete Fourier transform approach and its inverse. Data analysis with the inclusion of statistical and empirical analysis were equally conducted on the CDS spread quotes of 10 entities in order to estimate the correlation structures and the chances of survival of the basket entities.

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