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An Analytical Study in Multi Physics and Multi Criteria Shape Optimization

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AN ANALYTICAL STUDY IN MULTI PHYSICS AND MULTI CRITERIA SHAPE OPTIMIZATION

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Abstract. A simple multi-physical system for the potential flow of a fluid through a shroud in which a mechanical component, like a turbine vane, is placed, is modeled mathematically. We then consider a multi criteria shape optimization problem, when the shape of the component is allowed to vary under a certain set of 2nd order Hölder continuous differentiable transformations of a baseline shape with boundary of the same continuity class. As objective functions, we consider a simple loss model for the fluid dynamical efficiency and the probability of failure of the component due to repeated application of loads that stem from the fluid's static pressure. For this multi-physical system, it is shown that under certain conditions the Pareto front is maximal in the sense that the Pareto front of the feasible set coincides with Pareto front of its closure. We also show that the set of all optimal forms with respect to scalarization techniques deforms continuously (in the Hausdorff metric) with respect to preference parameters.

Key words. Shape optimization, multi criteria optimization, multi physics

AMS subject classifications. 49Q10, 74P10, 90C29

1. Introduction. The design of a mechanical component requires choosing a material and a shape. Often, a component serves a primary objective, but also requires a certain level of endurance. Material damage is caused by the loads that are imposed during service. The quest for an optimal design in the majority of cases therefore is at least a bi-criteria optimization problem and in many cases a multi criteria one [19].

In mechanical engineering, multi criteria optimization often comes along with coupled multi physics simulations. If we take the design of turbine blades as an example, the simulation of external flows and cooling air flows inside a blade have to be combined with a thermal and a mechanical simulation inside the blade [48].

Mathematical optimization is widely used in mechanical engineering, see e.g. [40, 35]. On the other hand, some directions of contemporary mathematical research – like topology optimization (we refer, e.g., to [3, 7]) – were initiated by mechanical engineers [21]. While the given field is interdisciplinary, from the mathematical point of view one would not only like to propose and analyse new optimization algorithms, but also to understand the existence and the properties of optimal solutions. While for mono-criteria optimization such a framework has been established [14, 23, 30, 3, 17], a general framework for multi criteria optimization is still missing, see however [30, 18, 15] for numerical studies addressing the topic.

Component life models from materials science are used to judge the mechanical integrity of a component after a certain number of load cycles, see e.g. [5]. In recent times, such models have been extended by a probabilistic component [22, 31, 27, 42, 28, 44, 34, 4], which makes it possible to compute shape derivatives and gradients [12, 45, 26, 8] and therefore place component reliability in the context of shape optimization. However, as remarked in [27], the probability of failure as a objective functional requires more regular solutions as provided by the usual weak theory based on H^1 Sobolev spaces [20]. As we find here, this is also the case for simplistic models of fluid dynamical efficiency. As in previous works [27, 9], we therefore apply a framework based on Hölder continuous classical solution spaces and extend it to multi criteria

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optimization.

Within this general framework, we prove existence of Pareto optimal designs, see also [18] for a related result in a different setting requiring less regularity. Here, however, we show how to use the graph compactness property [30] along with the lower semi-continuity of all objective functionals to prove certain maximality properties of the non dominated feasible points: Namely that the Pareto front in the set of feasible points [19] coincides with the Pareto front of the closure of the feasible points. Put in other words, each dominated design is also dominated by at least one Pareto optimal design.

We give a simplistic multi physical system as an example that fits the general framework. This mathematical model couples a potential flow with structural mechanics and is motivated from gas turbine engineering. We define two (rather singular) objective functionals, namely a aerodynamic loss based on the theory boundary layers [41] and furthermore the probability of failure after a certain number of load cycles [27, 34]. Each of these models includes nonlinear functions that depend on second derivatives of the solution after restiction to the boundary of the underlying PDE's domains. For this system, we prove that the assumptions of the general framework are fulfilled and we conclude that a maximal Pareto front exists in this case.

Multi criteria optimization relates to preferences of a decision maker [19, 35]. Here, we are interested in continuity properties of Pareto optimal shapes, when the preference is expressed by a parameter in a merit function, which, e.g., cloud be the weights in a weighted sum approach. The stability of the optimal solutions to such scalarization techniques in dependence of a parameter is already investigated in the literature, see e.g. [6, 29, 46, 47], for finite dimensional and infinite dimensional spaces. Here we show that our general framework is indeed suitable to prove certain continuity properties of the arg min sets of scalarized multi criteria optimization problems in the Hausdorff distance as a function of the scalarization – or preference – parameter. Such structural properties of the Pareto front for the first time are applied in the context of shape optimization.

Our paper is organised as follows: We introduce the physical systems which underlie the multi criteria shape optimization problem we consider in Section 2. Afterwards, in Section 3 we describe our framework for multi criteria shape optimization. By deriving uniform bounds for the solution spaces of the physical systems in Section 4, we prove the well posedness of the shape optimization problem. Up to here we considered optimality in terms Pareto optimality. In Section 5 we apply scalaraization techniques to transform the problem into an univariate shape problem and investigate the dependency of the optimal shapes on the specific used technique. In Section 6 we give a resume and an outlook on future research direction. Some technical details on Hölder functions and solutions of elliptic partial differential equations can be found in Appendix 6.

2. A Simple Multi Physics System. We intend to optimize the shape of some component, e.g. a turbine vane, in terms of reliability and efficiency. Reliability depends on surface and volume forces acting on the component. In our setting, the component lies in a shroud and within the shroud a fluid is flowing past the component. Due to static pressure the fluid imposes a surface force on the component. Hence it is indispensable to include the fluid flow field into the optimization process. At the same time, the component leads to frictional loss in the fluid that diminishes the efficiency of the design.

In the following we describe a simple model which approximates the fluid flow in

a simple way as potential flow and model frictional loss via a post processing step to the solution that is based on a simple model for the boundary layer. We also consider the effect of the fluid's mechanical loads to the component. As the static pressure takes the role of a boundary condition for the partial differential equation of linear elasticity, the internal stress fields of the component depend on the flow field, too. The component's fatigue life that results in the probability of failure, i.e. the formation of a fatigue crack, as a second objective functional.

2.1. Potential Flow Equation. As component we consider a compact domain $\Omega \subset \mathbb{R}^3$ with $C^{k,\alpha}$ boundary that is partially contained in some larger compact domain $D \subset \mathbb{R}^3$ representing the shroud with $C^{k,\alpha}$ boundary as well. We assume that $D \backslash \mathring{\Omega}$ is simply connected and also has $C^{k,\alpha}$ boundary and that there exist a ball B_ϵ with $\epsilon > 0$ such that $B_\epsilon \subset \Omega \backslash D$. The shroud D has an inlet and outlet where the fluid flows in and out, respectively. At the remaining boundary part the fluid cannot leak. In this work we consider an incompressible and rotation free perfect fluid in a steady state. The assumption of zero shearing stresses in a perfect fluid – or zero viscosity – simplifies the equation of motion so that potential theory can be applied. The resulting solution still provides reasonable approximations to many actual flows. The viscous forces are limited to a thin layer of fluid adjacent to the surface and therefore in favor of simplicity we leave these effects out since they have little effect on the general flow pattern[1].

A fundamental condition is that no fluid can be created or destroyed within the shroud D. The equation of continuity express this condition. Consider a three dimensional velocity field $v(x)$ on $D \subseteq \mathbb{R}^3$, then the continuity equation is given by

$$\nabla \cdot v = 0.$$

If a velocity field v is rotation free, $\nabla \times v = 0$, then there exist a velocity potential or flow potential ϕ such that

$$v = \nabla \phi.$$

Hence our assumptions give us a velocity potential ϕ that satisfies the Laplace equation

$$\Delta \phi = \nabla \cdot \nabla \phi = 0.$$

Let n be the the outward normal of the boundary ∂D. By applying suited Neumann boundary conditions g that correspond with our assumptions for a conserved flow through the inlet and outlet of the shroud, we get the potential flow equation

$$(2.1) \quad \begin{aligned} \nabla v = \Delta \phi &= 0 && \text{in } D \backslash \Omega \\ v_n = \frac{\partial \phi}{\partial n} &= g && \text{on } \partial D \backslash \partial \Omega \\ v_n = \frac{\partial \phi}{\partial n} &= 0 && \text{on } D \cap \partial \Omega. \end{aligned}$$

Here we assume that g is only non-zero in the inlet and outlet regions and is continued to be zero on the upper and lower wall of the shroud. Therefore, no discontinuities occur where $\partial \Omega$ meets ∂D.

The existence of a solution to the potential flow equation is secured by the following lemma. It also gives a Schauder estimate that leads to a uniform bound for the solution space we investigate in subsection 4.1. This uniform bound is crucial for

---

[1] unless the local effects make the flow separate from the surface
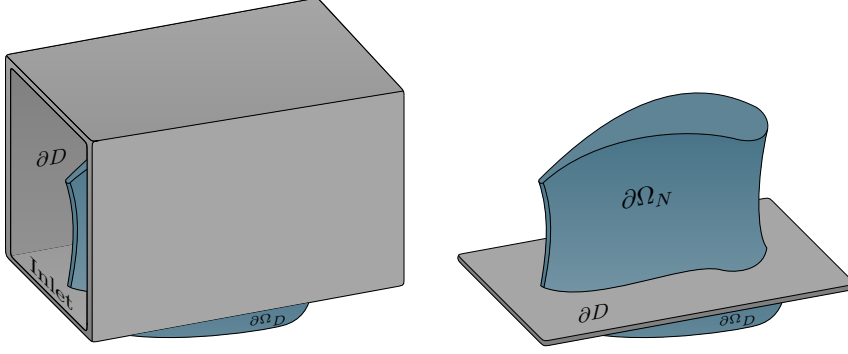
Fig. 1. *A turbine blade Ω within a shroud D.*

the existences of solutions to the multi criteria optimization problem we consider in this work and which we introduce in section 3.

LEMMA 2.1 (Schauder Estimate for Flow Potentials). *Consider us to be in the situation described above. I.e. we consider the potential flow equation (2.1). Let $\Omega \subset D \subset \mathbb{R}^3$ be compact domains with $C^{k,\alpha}$-boundaries and $g \in C^{1,\alpha}(\bar{D}, \mathbb{R}^3)$ with $k \in \mathbb{N}$. Then if $k \geqslant 2$ and $\int_{\partial D} g \, dA = 0$, the potential flow problem given by*

$$\begin{aligned} \Delta\phi &= 0 & in \ D\backslash\Omega \\ \tfrac{\partial\phi}{\partial n} &= g & on \ \partial D\backslash\partial\Omega \\ \tfrac{\partial\phi}{\partial n} &= 0 & on \ D\cap\partial\Omega. \end{aligned}$$

possess a solution $\phi \in C^{2,\alpha}(\overline{D\backslash\Omega}, \mathbb{R}^3)$. To obtain uniqueness we fix $u = 0$ at some point $x_0 \in D\backslash\Omega$. This solution verifies

$$(2.2) \qquad \|\phi\|_{C^{2,\alpha}(D\backslash\Omega)} \leqslant C\left(\|\phi\|_{C^{0,\alpha}(D\backslash\Omega)} + \|g\|_{C^{1,\alpha}(\partial D\backslash\partial\Omega)}\right),$$

with constant $C = C(\Omega)$.

Proof. [37] proves the existence of the solution and [25] provides the Schauder Estimate. □

2.2. Elasticity Equation. One of the most crucial demands on the component Ω is the reliability. Fatique failure is the most appearing type of failure for e.g. gas turbines where the event of failure for a component as e.g. a blade or vain is the appearance of the first crack. For this purpose, we consider the elasticity equation which models the deformation of a component under given surface and volume forces and allows us to calculate the stress fields that drive crack formation.

Let n be the outward normal of the boundary $\partial\Omega$ and let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ be a partition where $\partial\Omega_D$ is clamped, and on $\partial\Omega_N$ a force surface density $g_{|\partial\Omega_N}$ is imposed. Then according to [20] the mixed problem of linear isotropic elasticity, or

the elasticity equation, is described by

$$
\begin{array}{ll}
\nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\
\sigma(u) = \lambda(\nabla \cdot u)I + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega_D \\
\sigma(u) \cdot n = g_{\mathrm{s}} & \text{on } \partial\Omega_N
\end{array}
\tag{2.3}
$$

Here $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients and $u : \Omega \to \mathbb{R}^3$ is the displacement field on Ω. I is the identity on $\mathbb{R}^3$. The linearized strain rate tensor $\epsilon(u) : \Omega \to \mathbb{R}^{3\times3}$ is defined as $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$. Approximate numerical solutions can be computed by a finite element approach (see e.g. [20] or [32]).

We set $\partial\Omega_D = \partial B_\epsilon$ and $\partial\Omega_N = \partial\Omega$. The potential equation (2.1) gives the velocity field at the component's boundary $\partial\Omega$. Assuming that the total energy density, also denoted stagnation pressure p_{st} is constant at the inlet, we can calculate the static pressure p_{s} from Bernoulli's law

$$
p_{\mathrm{st}} = \frac{1}{2}\rho|\nabla\phi|^2 + p_{\mathrm{s}}.
\tag{2.4}
$$

We consider the static pressure p_{st} as surface load on the component Ω. Therefore the surface load g_{st} is given by

$$
g_{\mathrm{s}} = -p_{\mathrm{s}} = \left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{st}}\right)n.
$$

This yields as boundary condition on Ω_N for (2.3)

$$
\sigma(u) \cdot n = g_{\mathrm{st}} \quad \Leftrightarrow \quad \sigma(u) \cdot n = \left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{s}}\right)n.
\tag{2.5}
$$

Hence, the displacement vector u to the elasticity equation not only depends on the shape Ω but also on the solution ϕ to the potential equation (2.1).

The following lemma provides existence and uniqueness of solutions u along with a Schauder estimate.

LEMMA 2.2 (Schauder estimate for displacement fields, [8]). *Consider the elasticity equation (2.3). Let $\Omega \subset \mathbb{R}^3$ be a compact domain with $C^{k,\alpha}$-boundaries. As volume load we consider $f \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and as surface load $g_s \in C^{k+1,\alpha}(\partial\overline{\Omega}_N, \mathbb{R}^3)$, $k \in \mathbb{N}_0$. Then, the disjoint displacement-traction problem given by*

$$
\begin{array}{ll}
\nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\
\sigma(u) = \lambda(\nabla \cdot u)I + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega_D \\
\sigma(u) \cdot n = g_s & \text{on } \partial\Omega_N
\end{array}
$$

has a unique solution $u \in C^{k+2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ which satisfies

$$
\|u\|_{C^{k+2,\alpha}(\Omega)} \leqslant C\left(\|f\|_{C^{k,\alpha}(\Omega)} + \|g_{st}\|_{C^{k+1,\alpha}(\partial\Omega_N)} + \|u\|_{C^{0,\alpha}(\Omega)}\right),
\tag{2.6}
$$

with constant $C(\Omega) > 0$.

2.3. Optimal Reliability and Efficiency. Low cycle fatigue (LCF) driven surface crack initiation is particularly important for the reliability of highly loaded engineering parts as turbine components[33, 43]. The design of such engineering

parts therefore requires a model that is capable of accurately quantify risk levels for LCF crack initiation, crack growth and ultimate failure. Here we refer to the model introduced in [34] that models the statistical size effect but also includes the notch support factor by using stress gradients arising from the elasticity equation (2.3):

$$(2.7) \qquad J_R(\Omega, u_\Omega) := \int_{\partial\Omega \cap \bar{D}} \left(\frac{1}{N_{\det}(\nabla u, \nabla^2 u_\Omega(x))} \right)^m dA.$$

Ω represents the shape of the component, u_Ω is the displacement field and the solution to the elasticity equation on Ω, $N_{\det}$ is the deterministic number of life cycles at each point of the surface of Ω and m is the Weibull shape parameter. The proability of failure (PoF) after t load cycles is then given as $PoF(t) = 1 - e^{-t^m J_R(\Omega, u_\Omega)}$. Minimizing the probability of failure thus clearly is equivalent to minimizing $J_R(\Omega, u_\Omega)$.

For a detailed discussion including experimental validation we refer to [36]. We can apply this model as cost functional in order to optimize the component Ω w.r.t reliability.

An another condition for the component is the efficiency that is connected with the viscosity of the fluid flowing through the shroud. Viscosity is a measure which describes the internal friction of a moving fluid. In a laminar fluid the effect of viscosity is limited to a thin layer near the surface of the component. The fluid does not slip along the surface, but adheres to it. In the case of potential flow, there is a transition from zero velocity at the surface to the full velocity which is present at a certain distance from the surface. The layer where this transition takes place is called the boundary layer or frictional layer. The thickness of the boundary layer is not constant but (roughly) proportional to the square root of the kinematic viscosity ν and is growing from the leading edge, the location where the fluid first impinge on the surface of the component. Friction of the fluid on the surface leads to energy dissipation. A coefficient for the inflicted local wall shear stress is given by

$$(2.8) \qquad \tau_w(x) = \frac{0.322 \cdot \mu |v|^{\frac{3}{2}}}{\sqrt{\nu \cdot \operatorname{dist}_{LE}(x)}}$$

where μ is the viscosity and dist_{LE} the distance to the leading edge along the component's surface $\partial\Omega$. For a detailed introduction to boundary layer theory one can see e.g. [13, 41]. With this coefficient one can derive an estimate for the loss of power due to friction given by

$$(2.9) \qquad J_E(\Omega, \phi_\Omega) := \int_{\partial\Omega \cap \bar{D}} |v| \tau_w \, dA.$$

If we want to use this measure as cost functional in the setting for shape optimization we introduce in the next chapter, we have to apply additional assumptions on our shapes, i.e. a fixed and unique leading edge for all shapes in the shape space. This however changes very little in our general analysis.

For the multi physics and multi criteria shape optimization we have presented, we realize that objective functionals contain boundary integrals of second order derivatives of the solutions of second order elliptic BVPs. This can only be realised if one considers regular shapes and strong solutions.

3. A Multi Criteria Optimization Problem. We are interested in shapes with sufficiently smooth boundary given by hemisphere transformations from Hölder spaces. In the first subsection we outline this approach including conditions for existence of optimal shapes. Then we show that the above multi physics optimization fits in this general framework.

3.1. General Definitions. We denote a family of admissible shapes with $\tilde{\mathcal{O}}$ and for every shape $\Omega \in \tilde{\mathcal{O}}$ we denote with $V_1(\Omega), \ldots, V_n(\Omega)$, $n \in \mathbb{N}$ state spaces of real valued functions on Ω. Consider a sequence of shapes $(\Omega_n)_{n \in \mathbb{N}}$ in $\tilde{\mathcal{O}}$ and let $\Omega \in \tilde{\mathcal{O}}$. The convergence of Ω_m against Ω is denoted with $\Omega_m \xrightarrow{\tilde{\mathcal{O}}} \Omega$ as $m \to \infty$. For a sequence of functions $(\boldsymbol{y}_m)_{m \in \mathbb{N}}$ with $\boldsymbol{y}_m \in \times_{i=1}^n V_i(\Omega_m)$ for all $m \in \mathbb{N}$ we denote the convergence against some $\boldsymbol{y} \in \times_{i=1}^n V_i(\Omega)$ with $\boldsymbol{y}_m \rightsquigarrow \boldsymbol{y}$ as $m \to \infty$. We assume that for every $\Omega \in \tilde{\mathcal{O}}$ one can solve uniquely a given set of state problems, eg. PDEs or a variational inequalities. By associating the corresponding unique solutions $v_{i,\Omega} \in V_i(\Omega)$ with $\Omega \in \tilde{\mathcal{O}}$, one obtains the map $v_i : \Omega \mapsto v_{i,\Omega} \in V_i(\Omega)$. Let $\mathcal{O}$ be a subfamily of $\tilde{\mathcal{O}}$, then $\mathcal{G} = \{(\Omega, \boldsymbol{v}_\Omega) \,|\, \Omega \in \mathcal{O}\}$ is called the graph of the mapping $\boldsymbol{v} := (v_1, \ldots, v_n)$. A cost functional J on $\tilde{\mathcal{O}}$ is given by a map $J : (\Omega, \boldsymbol{y}) \mapsto J(\Omega, \boldsymbol{y}) \in \mathbb{R}$, where $\Omega \in \tilde{\mathcal{O}}$ and $\boldsymbol{y} \in \times_{i=1}^n V_i(\Omega)$. Then a vector of l cost functionals is defined by $\boldsymbol{J} := (J_1, \ldots, J_l)$ and the image of $\mathcal{O}$ (or $\mathcal{G}$) under $\boldsymbol{J}$ is denoted with $\mathcal{Y} \subset \mathbb{R}^l$.

DEFINITION 3.1 (Pareto optimality). *Consider a subfamily $\mathcal{O}$ of $\tilde{\mathcal{O}}$ with corresponding Graph $\mathcal{G}$ to given state spaces $\boldsymbol{V} = (V_1, \ldots, V_n)$. A point $(\Omega^*, \boldsymbol{v}^*) \in \mathcal{G}$ is called Pareto optimal w.r.t. cost functionals $\boldsymbol{J} = (J_1, \ldots, J_l)$ if there is no $(\Omega, \boldsymbol{v}) \in \mathcal{G}$ such that $J_k(\Omega, \boldsymbol{v}) \leqslant J_k(\Omega^*, \boldsymbol{v}^*)$ for $1 \leqslant k \leqslant l$ and $J_i(\Omega, \boldsymbol{v}) < J_i(\Omega, \boldsymbol{v})$ for some $i \in \{1, \ldots, l\}$. The associated value $\boldsymbol{J}(\Omega^*, \boldsymbol{v}^*)$ is called nondominated.*

Now let $\mathcal{Y} := \boldsymbol{J}(\mathcal{G})$. With the concept of Pareto optimality described we can define $\mathcal{Y}_N := \{\boldsymbol{J}(\Omega, \boldsymbol{v}) \in \mathcal{Y} \,|\, \boldsymbol{J}(\Omega, \boldsymbol{v})$ is nondominated in $\mathcal{Y}\}$, i.e. the corresponding Pareto front by which obviously lies on the boundary of $\mathcal{Y}$.

DEFINITION 3.2 (Multi criteria shape optimization problem). *Consider a subfamily $\mathcal{O}$ of $\tilde{\mathcal{O}}$ and for every $\Omega \in \mathcal{O}$ let $\boldsymbol{v}_\Omega = (v_{1,\Omega}, \ldots, v_{n,\Omega})$ be the unique solutions to given state problems on Ω. Let $\boldsymbol{J} = (J_1, \ldots, J_l)$ be cost functionals. We define an optimal shape design problem by*

(3.1)
$$\begin{cases} \textit{Find } \Omega^* \in \mathcal{O} \textit{ such that} \\ (\Omega^*, \boldsymbol{v}_{\Omega^*}) \textit{ is Pareto optimal w.r.t. } \boldsymbol{J}. \end{cases}$$

The next theorem gives us conditions for the existence of a solution to a optimal shape design problem. In the next section we will define our shape optimization problem and use this theorem to proof the existence of a solution to it.

THEOREM 3.3. *Let $\tilde{\mathcal{O}}$ be a family of admissible domains and $\mathcal{O}$ a subfamily. Consider cost functionals $\boldsymbol{J} = (J_1, \ldots, J_l)$ on $\tilde{\mathcal{O}}$ and assume for each $\Omega \in \tilde{\mathcal{O}}$ we have state problems with state spaces $\boldsymbol{V}(\Omega) = (V_1(\Omega, \ldots, V_n(\Omega))$ such that each state problem has a unique solution $v_{k,\Omega} \in V_k(\Omega)$, $1 \leqslant k \leqslant n$. When the following both assumptions hold true*

(i) *Compactness of $\mathcal{G} = \{(\Omega, \boldsymbol{v}_\Omega) \,|\, \Omega \in \mathcal{O}\}$:*
    Every sequence $(\Omega_m, \boldsymbol{v}_{\Omega_m})_{m \in \mathbb{N}}$ has a subsequence $(\Omega_{m_k}, \boldsymbol{v}_{\Omega_{m_k}})_{k \in \mathbb{N}}$ that satisfies

$$\Omega_{m_k} \xrightarrow{\tilde{\mathcal{O}}} \Omega, \quad k \to \infty$$
$$\boldsymbol{v}_{\Omega_{m_k}} \rightsquigarrow \boldsymbol{v}_\Omega, \quad k \to \infty,$$

    for some $(\Omega, \boldsymbol{v}_\Omega) \in \mathcal{G}$,

(ii) *Lower semicontinuity of J_k:*
    Let $(\Omega_m)_{m \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{O}}$ and $(\boldsymbol{y}_m)_{m \in \mathbb{N}}$ be a sequence such that $\boldsymbol{y}_m \in$

$\boldsymbol{V}(\Omega_m)$ *for all* $m \in \mathbb{N}$. *Consider some elements* Ω, $\boldsymbol{y}$ *in* $\tilde{\mathcal{O}}$ *and* $\boldsymbol{V}(\Omega)$, *respectively. Then*

$$\left.\begin{array}{ll} \Omega_m \xrightarrow{\tilde{\mathcal{O}}} \Omega, & m \to \infty \\ \boldsymbol{y}_m \rightsquigarrow \boldsymbol{y}, & m \to \infty \end{array}\right\} \Longrightarrow \liminf_{n\to\infty} J_k(\Omega_n, \boldsymbol{y}_n) \geqslant J_k(\Omega, \boldsymbol{y}),$$

for all $1 \leqslant k \leqslant l$.
Then the multi criteria shape design problem (3.1) possesses at least one solution and the Pareto front covers all nondominated points in $\bar{\mathcal{Y}}$, *e.g.* $\mathcal{Y}_N = \bar{\mathcal{Y}}_N$, *the set of non dominated points in the closure of* $\mathcal{Y}$.

Proof. First, we proof the existence of an optimal shape. In Theorem 2.10 in [30] it is proven that in this setting a lower semicontinuous cost functional possesses at least one minimal solution. We apply this without loss of generality on cost functional J_1 and minimize it on $\mathcal{G}$. Due the compactness of $\mathcal{G}$ and the lower semicontinuity of J_1 the resulting set of arguments of the minimum $\arg\min_{(\Omega, \boldsymbol{v}_\Omega) \in \mathcal{G}} J_1$ is also compact. Hence we can apply Theorem 2.10 on the next cost functional J_2 and minimize it on $\arg\min_{(\Omega, \boldsymbol{v}_\Omega) \in \mathcal{G}} J_1$. We continue this procedure until we minimized every cost functional and obtain by this at least one Pareto optimal solution.

For the second assertion, we recall that $\mathcal{Y}_N$ lies on the boundary of $\mathcal{Y}$ and it follows directly that $\mathcal{Y}_N \subseteq \bar{\mathcal{Y}}_N$. Conversely let $\mathbf{J}(\Omega^*, \boldsymbol{v}^*) \in \bar{\mathcal{Y}}_N$. Consider a sequence $(\mathbf{J}(\Omega_n, \boldsymbol{v}_n))_{n \in \mathbb{N}}$ with $\mathbf{J}(\Omega_n, \boldsymbol{v}_n) \to \mathbf{J}(\Omega^*, \boldsymbol{v}^*)$ as $n \to \infty$. We assume that the corresponding sequence of shapes $(\Omega_n)_{n \in \mathbb{N}}$ converge to some $\Omega \in \mathcal{O}$ as well (since $\mathcal{G}$ is compact we can always go to subsequences). Due to the lower semicontinuity of $\boldsymbol{J}$ we have

$$J_i(\Omega, \boldsymbol{v}_\Omega) \leqslant \lim_{n\to\infty} J_i(\Omega_n, \boldsymbol{v}_{\Omega_n}) = J_i(\Omega^*, \boldsymbol{v}^*) \qquad \text{for all } 1 \leqslant i \leqslant l.$$

The Pareto optimality of $\mathbf{J}(\Omega^*, \boldsymbol{v}^*)$ gives that $\mathbf{J}(\Omega, \boldsymbol{v}) = \mathbf{J}(\Omega^*, \boldsymbol{v}^*)$ and since $\mathbf{J}(\Omega, \boldsymbol{v}) \in \mathcal{Y}$ it follows that $\mathbf{J}(\Omega^*, \boldsymbol{v}^*) \in \mathcal{Y}$ and therefore $\bar{\mathcal{Y}}_N \subseteq \mathcal{Y}_N$. □

3.2. Multi Physics Shape Optimization. In the previous subsection we introduced a general framework of multi criteria shape optimization. We now state a class of shape optimization problems that includes the multi physics shape optimization problem given by the coupled potential and elasticity equation as introduced in section 2. As we will see in section 4, multi criteria shape optimization problems from this class fulfill the required assumptions of Theorem 3.3 to ensure us the existence of the Pareto front.

We consider shapes with Hölder continuous boundaries. This assumptions ensures strong regularity for the solutions of the physical problems in this setting which enables us to deal with cost functionals defined on the boundaries of the shapes containing first and second derivatives as motivated in subsection 2.3. In the following, $C^{k,\alpha}$ stands for the real valued functions with k-th derivatives being Hölder continuous with exponent α, see the appendix 6.

DEFINITION 3.4. *Let* Ω, Ω' *be bounded domains in* $\mathbb{R}^d$.
(i) *A* $C^{k,\alpha}$-*diffeomorphism on* Ω *is a bijective mapping* $f : \Omega \to \Omega'$ *such that* $f \in \left[C^{k,\alpha}(\Omega)\right]^d$ *and* $f^{-1} \in \left[C^{k,\alpha}(\Omega')\right]^d$.
(ii) *The set of* $C^{k,\alpha}$-*diffeomorphisms is denoted by* $\mathcal{D}^{k,\alpha}(\Omega, \Omega')$ *or* $\mathcal{D}^{k,\alpha}(\Omega)$ *if* $f : \Omega \to \Omega$.

DEFINITION 3.5. *Consider a bounded domain* $\Omega \subset \mathbb{R}^d$. *The boundary of* Ω *is of class* $C^{k,\alpha}$, $0 \leqslant \alpha \leqslant 1$, *if at each point* $x_0 \in \partial\Omega$ *there is a ball* $B = B(x_0)$ *and a*

$C^{k,\alpha}$-diffeomorphism T of B onto $G \subset \mathbb{R}^d$ such that:

$$(i)\ T(B \cap \Omega) \subset \mathbb{R}^d_+; \quad (ii)\ T(B \cap \partial\Omega) \subset \partial\mathbb{R}^d_+;$$

We shall say that the diffeomorphism T straightens the boundary near x_0 and call it hemisphere transform. Note that by this definition Ω is of class $C^{k,\alpha}$ if each point of $\partial\Omega$ has a neighbourhood in which $\partial\Omega$ is the graph of a $C^{k,\alpha}$ function of $d-1$ of the coordinates $x_1, \ldots, x_n$. The converse is true if $k \geqslant 1$.

DEFINITION 3.6. Let $K > 0$ be a positive constant and $\Omega_0 \subset \Omega^{ext} \subset \mathbb{R}^3$ be compact $C^{k,\alpha}$ domains. The elements of the set

$$U^{ad}_{k,\alpha}(\Omega^{ext}) := \left\{ \psi \in \mathcal{D}^{k,\alpha}(\overline{\Omega^{ext}}) \,\big|\, \psi|_{\overline{\Omega \setminus D}} = id,\ \|\psi\|_{[C^{k,\alpha}(\Omega^{ext})]^3} \leqslant K, \right.$$
$$\left. \|\psi^{-1}\|_{[C^{k,\alpha}(\Omega^{ext})]^3} \leqslant K \right\}$$

are called design-variables. These design variables induce, in a natural way the set of admissible shapes

$$\mathcal{O}_{k,\alpha}(\Omega_0, \Omega^{ext}) := \left\{ \psi(\Omega_0) \,|\, \psi \in U^{ad}_{k,\alpha}(\Omega^{ext}) \right\}$$

assigned to Ω_0. Note that due to the Hölder continuity, every $\Omega \in \mathcal{O}_{k,\alpha}$ is compact.

On the space of admissible domains we can define the Hausdorff distance as metric. Note that in general the Hausdorff distance is no metric because the identity of indiscernibles is not given. Here however, the compactness of the shapes ensure us this property.

DEFINITION 3.7. For two non-empty subsets Ω, Ω' of a metric space (M, d) we define their Hausdorff distance by

$$d_H(\Omega, \Omega') := \max\{\sup_{x \in \Omega} \inf_{y \in \Omega'} d(x - y),\ \sup_{y \in \Omega'} \inf_{x \in \Omega} d(x - y)\}$$

If we equip the set $F(M)$ of all closed subsets of a metric space (M, d) with the Hausdorff distance then we obtain another metric space. Since the shapes in $\mathcal{O}_{k,\alpha}$ are compact, the Hausdorff distance defines a metric on $\mathcal{O}_{k,\alpha}$. By the following Lemma, we see in chapter 4, that $(\mathcal{O}_{k,\alpha}, d_H)$ is additionally compact.

THEOREM 3.8 (Blaschke's Selection Theorem [39]). Let (M, d) be a metric space where M is a compact subset of a Banach space B. Then the set $F(M)$ of all closed subsets of M is compact w.r.t. the Hausdorff distance d_H.

From now on we write $\boldsymbol{J}(\Omega, \boldsymbol{v}) = (J_1(\Omega, \boldsymbol{v}), \ldots, J_l(\Omega, \boldsymbol{v}))$, where $\boldsymbol{v} = (v_1, \ldots, v_n)$ and $\boldsymbol{J} = (J_1 \ldots, J_l)$. Additionally we use the notation $\nabla \boldsymbol{v} = (\nabla v_1, \ldots, \nabla v_n)$.

DEFINITION 3.9 (Local Cost Functionals). Let $\mathcal{O} \subset \mathcal{P}(\mathbb{R}^3)$ be a shape space with state spaces $V_1(\Omega), \ldots, V_n(\Omega)$, $\Omega \in \mathcal{O}$ and graph $\mathcal{G} := \{(\Omega, \boldsymbol{v}) \,|\, \Omega \in \mathcal{O}\}$. We assume that $V_i(\Omega) \subseteq C^k(\Omega, \mathbb{R}^3)$ for all $1 \leqslant i \leqslant n$, then the local cost functional on $\mathcal{G}$ is given by

$$(3.2) \quad \begin{aligned} J(\Omega, \boldsymbol{v}) := &\int_\Omega \mathcal{F}_{vol}(x, \boldsymbol{v}, \nabla \boldsymbol{v}, \ldots, \nabla^k \boldsymbol{v})\, dx \\ &+ \int_{\partial\Omega} \mathcal{F}_{sur}(x, \boldsymbol{v}, \nabla \boldsymbol{v}, \ldots, \nabla^k \boldsymbol{v})\, dA, \end{aligned}$$

where $\mathcal{F}_{vol}$, $\mathcal{F}_{vol} : \mathbb{R}^d \to \bar{\mathbb{R}}_+$ and $d = 3 + n \sum_{j=0}^{k} 3^{j+1} = 3 + \frac{3n}{2}(3^{k+1} - 1)$. We denote the volume integral and surface integral with

$$J_{vol}(\Omega, \boldsymbol{v}) := \int_{\Omega} \mathcal{F}_{vol}(x, \boldsymbol{v}, \nabla \boldsymbol{v}, \dots, \nabla^k \boldsymbol{v}) \, dx,$$

$$J_{sur}(\Omega, \boldsymbol{v}) := \int_{\partial\Omega} \mathcal{F}_{sur}(x, \boldsymbol{v}, \nabla \boldsymbol{v}, \dots, \nabla^k \boldsymbol{v}) \, dA.$$

DEFINITION 3.10 (Multi Physics Shape Optimization Problem). *Let* $(\mathcal{O}_{k,\alpha}, d_H)$ *be the space of admissible shapes and let* $J_1, \dots, J_l$ *be local cost functionals on the Graph* $\mathcal{G} := \{(\Omega, u_\Omega, \phi_\Omega) \mid \Omega \in \mathcal{O}_{k,\alpha}, u_\Omega$ *solves* (2.3) *on* Ω, ϕ_Ω *solves* (2.1) *on* $\Omega\}$. *The multi physics shape optimization problem is given by:*

(3.3)
$$\begin{cases} Find \ \Omega^* \in \mathcal{O}_{k,\alpha} \ such \ that \\ (\Omega^*, u_{\Omega^*}, \phi_{\Omega^*}) \ is \ Pareto \ optimal \ w.r.t. \ \boldsymbol{J}. \end{cases}$$

Before this section ends we note that our choice of state problems here is only exemplary. In the next section we see that we can include an arbitrary amount of physical models in this multi physics shape optimization problem as long as they provide unique solution with sufficient regularity and a compact solution space on the shape space $\mathcal{O}_{k,\alpha}$.

4. Existence of Pareto Optimal Shapes.

4.1. Uniform Bounds for Solution Spaces. In this section the approach outlined in Thoerem 3.3 will be followed in order to show the existence of an optimal shape for the multi physics shape optimization problem. This approach includes the compactness of the Graph $\mathcal{G}$ which requires bounded solution spaces on $\mathcal{O}_{k,\alpha}$. In this subsection we derive such uniform bounds based on the Schauder estimates given in section 2.

LEMMA 4.1. *Let* ϕ *be a unique solution to* (2.1) *with* $k \geqslant 2$. *Then there exist a constant* $K > 0$ *such that for all* $\Omega \in \mathcal{O}_{k,\alpha}$ *we have*

$$\|\phi\|_{C^{2,\alpha}(D\setminus\Omega)} \leqslant K,$$

Proof. This estimate is based on Lemma 2.1:

$$\|\phi\|_{C^{2,\alpha}(D\setminus\Omega)} \leqslant C \left(\|\phi\|_{C^{0,\alpha}(D\setminus\Omega)} + \|g\|_{C^{1,\alpha}(\partial D\setminus\partial\Omega)} \right),$$

where the constant $C > 0$ depends on the shape $\Omega \in \mathcal{O}_{k,\alpha}$. First we outline that the constant C can be chosen uniformly overpronlem the shapes Ω in $\mathcal{O}_{k,\alpha}$. A full proof is provided in [25] (or [9]). In order to proof estimate (2.2) one straightens the boundary $\partial\Omega$ piecewise with hemisphere transforms. The dependence of the constant C is through the ellipticity of the differential operator and hence depends on the bounds of the hemisphere transform that is used to straightens the boundary. Let T be such a hemisphere transform for Ω_0. For every shape $\psi(\Omega_0) \in \mathcal{O}_{k,\alpha}$ we can construct a hemisphere transform by pulling $\psi(\Omega_0)$ back to Ω_0 and apply T afterwards, i.e. $T_{\psi(\Omega_0)} := T \circ \psi^{-1}$. Due to the definition of design variables ψ is uniformly bounded and thus one can show that $T_{\psi(\Omega_0)}$ is uniformly bounded in $\mathcal{O}_{k,\alpha}$ as well (see e.g. [9]).

Next we note that $\|g\|_{C^{1,\alpha}(\partial D\setminus\partial\Omega)}$ is obviously uniformly bounded by $\|g\|_{C^{1,\alpha}(\partial D)}$.

We further estimate $\|\phi\|_{C^{0,\alpha}(D\setminus\Omega)}$. One can see (e.g. [27]) that for every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that

$$\|\phi\|_{C^{0,\alpha}(D\setminus\Omega)} \leqslant \epsilon\|\phi\|_{C^{1,\alpha}(D\setminus\Omega)} + C(\epsilon)\int_{D\setminus\Omega} |\phi|\,dx.$$

We choose $\epsilon < 1/C$ and get

$$\|\phi\|_{C^{2,\alpha}(D\setminus\Omega)} \leqslant C\left(\|\phi\|_{C^{0,\alpha}(D\setminus\Omega)} + \|g\|_{C^{1,\alpha}(\partial D)}\right),$$

$$\leqslant \frac{1}{1-\epsilon C}\left(C(\epsilon)\int_{D\setminus\Omega} \phi\,dx + C\|g\|_{C^{1,\alpha}(\partial D)}\right)$$

$$\leqslant \frac{1}{1-\epsilon C}\left(C(\epsilon)\|\phi\|_{H^1(D\setminus\Omega)} + C\|g\|_{C^{1,\alpha}(\partial D)}\right)$$

One can easily verify the a-priori estimate $\|\phi\|_{H^1(D\setminus\Omega)} \leqslant C_p|D|\|g\|_{C^{1,\alpha}(\partial D)}$ holds for a constant $C_p > 0$. This yields

$$\|\phi\|_{C^{2,\alpha}(D\setminus\Omega)} \leqslant \frac{1}{1-\epsilon C}\left(C(\epsilon)\|\phi\|_{H^1(D\setminus\Omega)} + C\|g\|_{C^{1,\alpha}(\partial D)}\right)$$

$$\leqslant \frac{C(\epsilon)C_p|D| + C}{1-\epsilon C}\|g\|_{C^{1,\alpha}(\partial D)} := K$$

$\square$

We recall that the elasticity equation (2.3) describes the surface force g_{st} by the static pressure p_{st} that the fluid exerts on the component Ω (see (2.5)). Hence in our framework the surface force g_{st} is given by Bernoulli's equation and we have

$$(4.1) \qquad \sigma(u)\cdot n = g_{\mathrm{st}} \quad \Leftrightarrow \quad \sigma(u)\cdot n = \left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{s}}\right)\cdot n.$$

The solution u of the elasticity equation not only depends on the shape $\Omega \in \mathcal{O}_{k,\alpha}$ but on the solution ϕ of potential equation (2.1) as well. We can derive a uniform bound for u as we have for ϕ from estimate (2.6) which already provides an uniform bound in $\mathcal{O}_{k,\alpha}$ if the surface load g_{st} is independent of ϕ. However, in our framework g_{st} depends on ϕ and thus we have to further estimate the surface force g_{st} in $\mathcal{O}_{k,\alpha}$.

LEMMA 4.2. *Let u be the unique solution to (2.3). Then there exist a constant $K > 0$ such that for all $\Omega \in \mathcal{O}_{k,\alpha}$ we have*

$$\|u\|_{C^{2,\alpha}(\Omega)} \leqslant K,$$

with $k \in \mathbb{N}_0$.

Proof. Consider estimate (2.6):

$$(4.2) \qquad \|u\|_{C^{2,\alpha}(\Omega)} \leqslant C\left(\|f\|_{C^{0,\alpha}(\Omega)} + \|g_{\mathrm{st}}\|_{C^{1,\alpha}(\partial\Omega)} + \|u\|_{C^{0,\alpha}(\Omega)}\right)$$

In [8] it is proven that the constant C is uniform in $\mathcal{O}_{k,\alpha}$. Further $\|f\|_{C^{0,\alpha}(\Omega)}$ is bounded by $\|f\|_{C^{0,\alpha}(\Omega^{\mathrm{ext}})}$ and $\|g_{\mathrm{st}}\|_{C^{1,\alpha}(\partial\Omega)}$ depends on the potential ϕ in terms of (2.5). For $\|g_{\mathrm{st}}\|_{C^{1,\alpha}(\partial\Omega)}$ we can estimate

$$\|g_{\mathrm{st}}\|_{C^{1,\alpha}(\partial\Omega)} = \left\|\left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{s}}\right)\cdot n\right\|_{C^{1,\alpha}(\partial\Omega)}$$

$$\leqslant p_{\mathrm{s}} + \rho(1 + \|\nabla\phi\|_\infty)\|\nabla\phi\|_{C^{1,\alpha}(\partial\Omega)}.$$

Equation (2.1) models an incompressible fluid and therefore the fluid density ρ is constant. Since p_{st} is also constant and Lemma 4.1 gives that the potential ϕ is uniformly bounded in $\mathcal{O}_{k,\alpha}$, we get that g_{st} is uniformly bounded as well.

Now for $\epsilon > 0$ one can estimate

$$\|u\|_{C^{0,\alpha}(\Omega)} \leqslant \epsilon \|u\|_{C^{1,\alpha}(\Omega)} + C(\epsilon) \int_{\Omega} |u| \, dx.$$

with constant $C(\epsilon) > 0$. Applying this with $\epsilon < 1/C$ on Lemma 4.2 and estimating $\int_{\Omega} |\phi| \, dx \leqslant \|\phi\|_{H^1(\Omega)}$ yields

$$\|u\|_{C^{2,\alpha}(\Omega)} \leqslant \tilde{C} \left(\|f\|_{C^{0,\alpha}(\Omega^{\text{ext}})} + \|g_{\text{st}}\|_{C^{1,\alpha}(\partial\Omega)} + \|u\|_{H^1(\Omega)} \right)$$

Let $V_{DN} = \{v \in [H^1(\Omega)]^3 \, | \, v = 0 \text{ a.e. on } \partial\Omega_D\}$, consider the weak formulation of (2.3):

$$\int_{\Omega} \text{tr}(\sigma(u)\epsilon(v)) \, dx \int_{\Omega} fv \, dx + \int_{\Omega_N} g_{\text{st}} v \, dA \quad \forall v \in V_{DN},$$

where $\sigma_{i,j} = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{k,l}$ and $\epsilon_{i,j} = 1/2(\partial_j x_i + \partial_i x_j)$. One can see that for all $v \in V_{DN}$ we have

$$B_{\Omega}(v,v) = \int_{\Omega} \text{tr}(\sigma(u)\epsilon(v)) \, dx \geqslant q\|\epsilon(v)\|_{L^2(\Omega)}^2,$$

with a constant $q > 0$. Since f and g_{st} are uniform bounded,

$$|L_{\Omega}(v)| = \left| \int_{\Omega} fv \, dx + \int_{\partial\Omega_N} g_{\text{st}} v \, dA \right| \leqslant C\|v\|_{H^1(\Omega)}$$

for constant C that is uniform in $\mathcal{O}_{k,\alpha}$. Korn's second inequality (.3) then implies

$$q\|\epsilon(u)\|_{H^0(\Omega)}^2 \leqslant C\|u\|_{H^1(\Omega)} \leqslant C_K \|\epsilon(u)\|_{H^0(\Omega)}$$

$$\Rightarrow \|u\|_{H^1(\Omega)} \leqslant \frac{C_K}{qC}$$

Using the same arguments as [38] develops for for the local epigraph parametrization, the constant C_K is uniform with respect to a class of domains possessing a uniform cone property. Therefore the previous inequality is uniform in $\mathcal{O}$ and the assertion is proven. □

4.2. Pareto Optimality. In order to prove the existence of an optimal shape to the multi physics shape optimization problem (3.3) we want to make use of Theorem 3.3. Therefore we show that the local cost functionals from Definition 3.9 are lower semicontinuous -we even show that they are continuous- and that the graph from Definition 3.10 is compact. The continuity is given and discussed in Lemma 4.9. We start this section by proving the compactness of the graph. We denote with $\mathcal{P}_{k,\alpha} := \{\phi_{\Omega} \, | \, \phi_{\Omega} \text{ solves } (2.1) \text{ with } \Omega \in \mathcal{O}_{k,\alpha}\}$ and $\mathcal{E}_{k,\alpha} := \{u_{\Omega} \, | \, u_{\Omega} \text{ solves } (2.3) \text{ with } \Omega \in \mathcal{O}_{k,\alpha}\}$ the spaces of solutions to (2.1) and (2.3) on admissible shapes respectively. We equipped these spaces with the metric that is induced by the Hölder norm. The solutions in these spaces are defined on different and distinct domains and therefore are not comparable w.r.t. $\|\cdot\|_{C^{k,\alpha}}$. We give our solution to this problem in the following first definition of this section:

DEFINITION 4.3 ($C^{k,\alpha}$-convergence of functions with varying domains). *Recalling the set $\mathcal{O}_{k,\alpha}$ and Ω^{ext} from Definition 3.6 let $p_\Omega : [C^{k,\alpha}(\Omega)]^{3n} \to [C_0^{k,\alpha}(\Omega^{ext})]^{3n}$ be the extension operator that can be derived from Lemma .2. For $\boldsymbol{v} \in [C^{k,\alpha}(\Omega)]^{3n}$ set $\boldsymbol{v}^{ext} = p_\Omega \boldsymbol{v}$. For $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}_{k,\alpha}$, $\Omega \in \mathcal{O}_{k,\alpha}$ and $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ with $\boldsymbol{v}_n \in [C^{k,\alpha}(\Omega_n)]^{3n}$, $n \in \mathbb{N}$, the expression $\boldsymbol{v}_n \rightsquigarrow \boldsymbol{v}$ as $n \to \infty$ is defined by $\boldsymbol{v}_n^{ext} \to \boldsymbol{v}^{ext}$ in $[C_0^{k,\alpha}(\Omega^{ext})]^{3n}$.*

Remark 4.4. Obviously, in the same way as above, we can extend a Hölder continuous functions p on $D\backslash\Omega$ to the whole domain D for all $\Omega \in \mathcal{O}_{k,\alpha}$.

With this definition of convergence, we can proof the compactness of $\mathcal{G}$. We show that the metric spaces $(\mathcal{O}_{k,\alpha}, d_H)$, $(\mathcal{P}_{k,\alpha}, \|\cdot\|_{C^{k,\alpha'}})$, and $(\mathcal{E}_{k,\alpha}, \|\cdot\|_{C^{k,\alpha'}})$ are each compact where $0 < \alpha' < \alpha < 1$ and that $\mathcal{G}$ is a closed subset of $\mathcal{O}_{k,\alpha} \times \mathcal{P}_{k,\alpha} \times \mathcal{E}_{k,\alpha}$.

LEMMA 4.5. *The space of admissible shapes $\mathcal{O}_{k,\alpha}(\Omega_0, \Omega^{ext})$ equipped with the Hausdorff distance d_H is a compact metric space.*

Proof. We proof that $\mathcal{O}_{k,\alpha}$ is sequentially compact. First we show that the space of design variable $U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$ is compact. Then the compactness of $\mathcal{O}_{k,\alpha}(\Omega_0, \Omega^{\mathrm{ext}})$ follows out of it. Due to its definition $U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$ is a bounded subspace of $C^{k,\alpha}$ and therefore precompact in $C^{k,\alpha'}(\bar{\Omega}^{\mathrm{ext}})$ for any $0 < \alpha' < \alpha$ (see Lemma .4). Hence, for every sequence $(\psi_n)_{n\in\mathbb{N}} \subset U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$ there exists a convergent subsequence. For the compactness of $U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$ it remains to show that the limit of $(\psi_n)_{n\in\mathbb{N}}$ lies in $U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$. Since $U_{k,\alpha}^{\mathrm{ad}}(\Omega^{\mathrm{ext}})$ is precompact in $C^{k,\alpha'}$ and $C^{k,\alpha'}$ is a Banach space, that sequence has a subsequence $(\psi_{n_l})_{l\in\mathbb{N}}$ with $\psi_{n_l} \to \psi$ and $\psi_{n_l}^{-1} \to \psi^{-1}$ in $\|\cdot\|_{C^{k,\alpha'}}$ for some ψ. First we note that since $\|\psi_{n_l}\|_{C^{k,\alpha}(\Omega^{\mathrm{ext}})} \leqslant K$ we have for any $\gamma \in \mathbb{N}^3$ with $0 \leqslant |\gamma| \leqslant k$

$$\left| \frac{\partial^{|\gamma|}\psi_{n_l}(x)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi_{n_l}(y)}{\partial^\gamma x} \right| \leqslant \left(K - \max_{|\gamma|=k} \left\| \frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^\gamma} \right\|_\infty \right) |x-y|^\alpha$$

and

$$\max_{|\gamma|=k} \left[\frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^\gamma} \right]_{0,\alpha} \leqslant K - \max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^\gamma} \right\|_\infty,$$

$$\max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^\gamma} \right\|_\infty \longrightarrow \max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi}{\partial x^\gamma} \right\|_\infty \leqslant K.$$

We apply these estimates to show that $\psi \in C^{k,\alpha}$ and $\|\psi\|_{C^{k,\alpha}(\Omega^{\mathrm{ext}})} \leqslant K$:

$$
\begin{aligned}
\left| \frac{\partial^{|\gamma|}\psi(x)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi(y)}{\partial^\gamma x} \right| &\leqslant \left| \frac{\partial^{|\gamma|}\psi(x)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi_{n_l}(x)}{\partial^\gamma x} \right| + \left| \frac{\partial^{|\gamma|}\psi_{n_l}(x)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi_{n_l}(y)}{\partial^\gamma x} \right| \\
&\quad + \left| \frac{\partial^{|\gamma|}\psi_{n_l}(y)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi(y)}{\partial^\gamma x} \right| \\
&\leqslant \left| \frac{\partial^{|\gamma|}\psi(x)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi_{n_l}(x)}{\partial^\gamma x} \right| \\
&\quad + \left(K - \max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^\gamma} \right\|_\infty \right) |x-y|^\alpha \\
&\quad + \left| \frac{\partial^{|\gamma|}\psi_{n_l}(y)}{\partial^\gamma x} - \frac{\partial^{|\gamma|}\psi(y)}{\partial^\gamma x} \right|
\end{aligned}
$$

For the second term we used the Hölder continuity of ψ_{n_l}. The first and third term converge to zero since $\|\psi_{n_l}\|_{C^{k,\alpha}} \to \|\psi\|_{C^{k,\alpha}}$. Overall we get

$$
\left| \frac{\partial^{|\gamma|}\psi(x)}{\partial^{\gamma}x} - \frac{\partial^{|\gamma|}\psi(y)}{\partial^{\gamma}x} \right| \leqslant \left| \frac{\partial^{|\gamma|}\psi(x)}{\partial^{\gamma}x} - \frac{\partial^{|\gamma|}\psi_{n_l}(x)}{\partial^{\gamma}x} \right|
$$
$$
+ \left(K - \max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi_{n_l}}{\partial x^{\gamma}} \right\|_{\infty} \right) |x-y|^{\alpha}
$$
$$
+ \left| \frac{\partial^{|\gamma|}\psi_{n_l}(y)}{\partial^{\gamma}x} - \frac{\partial^{|\gamma|}\psi(y)}{\partial^{\gamma}x} \right|
$$
$$
\longrightarrow \left(K - \max_{|\gamma|\leqslant k} \left\| \frac{\partial^{|\gamma|}\psi}{\partial x^{\gamma}} \right\|_{\infty} \right) |x-y|^{\alpha}.
$$

This gives $\psi \in C^{k,\alpha}$ and $\|\psi\|_{C^{k,\alpha}(\Omega^{\text{ext}})} \leqslant K$, hence $\psi \in U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$. Therefore $U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$ is closed and with that compact.

We make use of the compactness of $U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$ w.r.t. $\|\cdot\|_{C^{k,\alpha'}(\Omega^{\text{ext}})}$ to show the compactness of $(\mathcal{O}_{k,\alpha}, d_{\mathcal{O}})$. Consider a sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}_{k,\alpha}$. Due to the definition of $\mathcal{O}_{k,\alpha}$ there exists a corresponding sequence $(\psi_n)_{n\in\mathbb{N}} \subset U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$ with $\psi_n(\Omega_0) = \Omega_n$ for all $n \in \mathbb{N}$. Since $U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$ is compact, there exists a subsequence $(\psi_{n_l})_{l\in\mathbb{N}}$ that converge to some $\psi \in U_{k,\alpha}^{\text{ad}}(\Omega^{\text{ext}})$ in $\|\cdot\|_{C^{k,\alpha'}}$. We show that the corresponding subsequence of shapes $(\Omega_{n_l}) = \psi_{n_l}(\Omega_0)$ converge to $\Omega = \psi(\Omega_0)$ by using the convergence of $\psi_{n_l} \to \psi$ in $\|\cdot\|_{C^{k,\alpha'}}$:

$$
d_{\mathcal{O}}(\Omega_{n_l}, \Omega) = \max\{ \sup_{x\in\Omega_{n_l}} \inf_{y\in\Omega}|x-y|, \ \sup_{y\in\Omega} \inf_{x\in\Omega_{n_l}} |x-y|\}
$$
$$
= \max\{ \sup_{x\in\Omega_0} \inf_{y\in\Omega_0} |\psi_{n_l}(x) - \psi(y)|, \ \sup_{y\in\Omega_0} \inf_{x\in\Omega_0} |\psi_{n_l}(x) - \psi(y)|\}
$$
$$
\leqslant \max\{ \sup_{x\in\Omega_0} |\psi_{n_l}(x) - \psi(x)|, \ \sup_{y\in\Omega_0} |\psi_{n_l}(y) - \psi(y)|\}
$$
$$
\xrightarrow[l\to\infty]{} 0.
$$

Hence, each sequence in $\mathcal{O}_{k,\alpha}$ has a convergent subsequence that converge in $\mathcal{O}_{k,\alpha}$ w.r.t the Hausdorff distance. Therefore $(\mathcal{O}_{k,\alpha}, d_{\mathcal{O}})$ is sequentially compact. $\qquad\square$

LEMMA 4.6. *Let $0 < \alpha' < \alpha < 1$ and $k \geqslant 2$. Then the solution space $\mathcal{P}_{k,\alpha}$ is compact in $C^{2,\alpha'}(\Omega^{ext})$.*

Proof. First Lemma 2.1 gives that $\mathcal{P}_{k,\alpha} \subset C^{2,\alpha'}$. Consider the space of extensions $\mathcal{P}_{k,\alpha}^{\text{ext}}$ consisting the extensions from Definition 4.3 of the solutions $\phi_\Omega \in \mathcal{P}_{k,\alpha}$. We denote the extension from $\phi_\Omega \in \mathcal{P}_{k,\alpha}$ on D with ϕ_Ω^{ext}. With Lemma 4.1 and (.1) this extension holds

$$
\|\phi_\Omega^{\text{ext}}\|_{C^{2,\alpha}(D)} \leqslant C\|\phi\|_{C^{2,\alpha}(D\backslash\Omega)} \leqslant CK,
$$

where K is uniform in $\mathcal{O}_{k,\alpha}$. In [9] it is shown that the constant C can also be chosen uniformly w.r.t. $\mathcal{O}_{k,\alpha}$ which yields an uniform bound for ϕ_Ω^{ext}. Hence, $\mathcal{P}_{k,\alpha}^{\text{ext}}$ is a bounded subset of $C^{2,\alpha}(\Omega^{\text{ext}})$ and therefore precompact in $C^{2,\alpha'}(\Omega^{\text{ext}})$ (see Lemma .4). Since $C^{k,\alpha'}$ is a Banach space it remains to show that $\mathcal{P}_{k,\alpha}^{\text{ext}}$ is closed.

For this consider a convergent sequence $(\phi_{\Omega_n}^{\text{ext}})_{n\in\mathbb{N}} \subset \mathcal{P}_{k,\alpha}^{\text{ext}}$ with limit ϕ and corresponding shapes $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}_{k,\alpha}$. As $\mathcal{O}_{k,\alpha}$ is compact we can find a subsequence of shapes $(\Omega_{n_l})_{l\in\mathbb{N}}$ that converge against some $\Omega \in \mathcal{O}_{k,\alpha}$. Now consider the corresponding subsequence of solutions $(\phi_{\Omega_{n_l}}^{\text{ext}})_{l\in\mathbb{N}}$. This subsequence also converge against ϕ and since we have seen in the proof of Lemma 4.5 that $\phi \in C^{2,\alpha}$ and the convergent is in $\|\cdot\|_{C^{2,\alpha'}}$, ϕ is the extension to a solution ϕ_Ω for (2.1) and therefore lies in $\mathcal{P}_{k,\alpha}^{\text{ext}}$. □

LEMMA 4.7. *Let $0 < \alpha' < \alpha < 1$ and $k \in \mathbb{N}_0$. The solution space $\mathcal{E}_{k,\alpha}$ is compact in $C^{2,\alpha'}(\Omega^{ext})$.*

Proof. The proof follows the exact same arguments as in Lemma 4.6 and therefore is omitted. □

LEMMA 4.8. *Consider the multi physics shape optimization problem (3.3) with boundary regularity $k \geqslant 2$. Then the Graph $\mathcal{G}$ is compact w.r.t. the corresponding maximum product metric.*

Proof. Lemma 4.5, Lemma 4.6 and Lemma 4.7 are implying that $\mathcal{O}_{k,\alpha} \times \mathcal{P}_{k,\alpha} \times \mathcal{E}_{k,\alpha}$ is compact. Let $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}_{k,\alpha}$ and $\Omega \in \mathcal{O}_{k,\alpha}$ with $\Omega_n \to \Omega$ in d_H. Then one can see that due to the compactness of $\mathcal{O}_{k,\alpha} \times \mathcal{P}_{k,\alpha} \times \mathcal{E}_{k,\alpha}$, $\phi_{\Omega_n}^{\text{ext}} \to \phi^{\text{ext}}$ and $u_{\Omega_n}^{\text{ext}} \to u^{\text{ext}}$ in $\|\cdot\|_{C^{2,\alpha}}$ with $\phi_{|\Omega}^{\text{ext}}$ solves (2.1) on Ω and $u_{|\Omega}^{\text{ext}}$ solves (2.3) on Ω. Hence $\mathcal{G}$ is a closed subspace of a compact metric space and therefore compact as well. □

LEMMA 4.9 (Continuity of Local Cost Functionals [27]). *Let $\mathcal{F}_{vol}, \mathcal{F}_{sur} \in C^0(\mathbb{R}^d)$ (with d as in Definition 3.9 with $r = 3$) and let $\mathcal{O}_{k,\alpha}$ only consists C^0-admissible shapes. For Ω and $\boldsymbol{v} \in [C^k(\overline{\Omega})]^{3n}$ consider the volume integral $J_{vol}(\Omega, \boldsymbol{v})$ and the surface integral $J_{sur}(\Omega, \boldsymbol{v})$*

Let $\Omega_n \subset \mathcal{O}_{k,\alpha}$ with $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$ as $n \to \infty$ and let $(\boldsymbol{v}_n)_{n\in\mathbb{N}} \subset [C^k(\overline{\Omega}_n)]^{3n}$ be a sequence with $\boldsymbol{v}_n \rightsquigarrow \boldsymbol{v}$ as $n \to \infty$ for some $\boldsymbol{v} \in [C^k(\overline{\Omega}_n)]^{3n}$. Then
  (i) $J_{vol}(\Omega_n, \boldsymbol{v}_n) \longrightarrow J_{vol}(\Omega, \boldsymbol{v})$ as $n \to \infty$.
  (ii) If the set $\mathcal{O}_{k,\alpha}$ only consists of C^1-admissible shapes one obtains
    $J_{sur}(\Omega_n, \boldsymbol{v}_n) \longrightarrow J_{sur}(\Omega, \boldsymbol{v})$ as $n \to \infty$ as well.

Proof. (i) First, we apply the characteristic function on the volume integral and obtain

$$J_{\text{vol}}(\Omega_n, \boldsymbol{v}_n) := \int_{\Omega^{\text{ext}}} \chi_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, \boldsymbol{v}_n^{\text{ext}}, \nabla \boldsymbol{v}_n^{\text{ext}}, \ldots, \nabla^k \boldsymbol{v}_n^{\text{ext}}) \, dx.$$

Because of $\mathcal{F}_{\text{vol}} \in C^0(\mathbb{R})$ and $\boldsymbol{v}_n \rightsquigarrow \boldsymbol{v}$ as $n \to \infty$ there exist a constant $C > 0$ such that $|\chi_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, \boldsymbol{v}_n^{\text{ext}}, \nabla \boldsymbol{v}_n^{\text{ext}}, \ldots, \nabla^k \boldsymbol{v}_n^{\text{ext}})| \leqslant C$ is valid for all $n \in \mathbb{N}$ almost everywhere in Ω_{ext}. Moreover, $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$ and $\boldsymbol{v}_n^{\text{ext}} \to \boldsymbol{v}^{\text{ext}}$ in $[C_0^k(\Omega^{\text{ext}})]^{3n}$ ensure the existence of

$$\lim_{n\to\infty} \chi_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, \boldsymbol{v}_n^{\text{ext}}, \nabla \boldsymbol{v}_n^{\text{ext}}, \ldots, \nabla^k \boldsymbol{v}_n^{\text{ext}})$$
$$= \chi_\Omega \cdot \mathcal{F}_{\text{vol}}(x, \boldsymbol{v}^{\text{ext}}, \nabla \boldsymbol{v}^{\text{ext}}, \ldots, \nabla^k \boldsymbol{v}^{\text{ext}}),$$

for all $x \in \Omega^{\text{ext}}$. The are pointwise and uniformly bounded in Ω^{ext} which let us apply

Lebesgue's dominated convergence theorem:

$$
\begin{aligned}
\lim_{n\to\infty} J_{\mathrm{vol}}(\Omega_n, \boldsymbol{v}_n) &= \lim_{n\to\infty} \int_{\Omega^{\mathrm{ext}}} \chi_{\Omega_n} \cdot \mathcal{F}_{\mathrm{vol}}\left(x, \boldsymbol{v}_n^{\mathrm{ext}}, \nabla \boldsymbol{v}_n^{\mathrm{ext}}, \ldots, \nabla^k \boldsymbol{v}_n^{\mathrm{ext}}\right) dx \\
&= \int_{\Omega^{\mathrm{ext}}} \lim_{n\to\infty} \chi_{\Omega_n} \cdot \mathcal{F}_{\mathrm{vol}}\left(x, \boldsymbol{v}_n^{\mathrm{ext}}, \nabla \boldsymbol{v}_n^{\mathrm{ext}}, \ldots, \nabla^k \boldsymbol{v}_n^{\mathrm{ext}}\right) dx \\
&= \int_{\Omega^{\mathrm{ext}}} \chi_{\Omega} \cdot \mathcal{F}_{\mathrm{vol}}\left(x, \boldsymbol{v}^{\mathrm{ext}}, \nabla \boldsymbol{v}^{\mathrm{ext}}, \ldots, \nabla^k \boldsymbol{v}^{\mathrm{ext}}\right) dx \\
&= J_{\mathrm{vol}}(\Omega, \boldsymbol{v})
\end{aligned}
$$

(ii) The second assertion can be analogously proven as in [9] and we only state the main ideas here.

First we note that every shape $\Omega \in \mathcal{O}_{k,\alpha}$ by its definition can be considered as two-dimensional submanifold and therefore locally embeddable into $\mathbb{R}^2$. Let $A_n^i \subset \partial\Omega$, $1 \leqslant i \leqslant m$ with $\cup_{i=1}^m A_i = \partial\Omega$ and $A_i \cap A_j = \varnothing$ for $i \neq j$. We can find in i and n uniformly bounded chart mappings $h_n^i : A_i^n \to \tilde{A}_i$ with $\tilde{A}_i \subset \mathbb{R}^2$. We use them to straighten the boundary of Ω_n to obtain a volume integral, e.g.

$$
\begin{aligned}
J_{\mathrm{sur}}(\Omega_n, \boldsymbol{v}_n) &= \int_{\partial\Omega_n} \mathcal{F}_{\mathrm{sur}}\left(x, \boldsymbol{v}_n, \nabla \boldsymbol{v}_n, \ldots, \nabla^k \boldsymbol{v}_n\right) dA \\
&= \sum_{i=1}^m \int_{A_n^i} \mathcal{F}_{\mathrm{sur}}\left(x, \boldsymbol{v}_n, \nabla \boldsymbol{v}_n, \ldots, \nabla^k \boldsymbol{v}_n\right) dA \\
&= \sum_{i=1}^m \int_{\tilde{A}^i} \mathcal{F}_{\mathrm{sur}}\left(h_n^i(s), \boldsymbol{v}_n(h_n^i(s)), \nabla \boldsymbol{v}_n(h_n^i(s)), \ldots, \nabla^k \boldsymbol{v}_n(h_n^i(s))\right) \sqrt{g^{h_n^i}(s)}\, ds.
\end{aligned}
$$

With corresponding Gram determinants $g^{h_n^i}$. Due to the fact that the chart mappings h_n^i are uniform bounded and since $\tilde{A}_i$ is independent of n one can see that similarly to (i) we can apply Lebesgue's Theorem here which give us the assertion after. □

Remark 4.10. The continuity assumption of Lemma 4.9 ensures the existence of an integrable majorant for $\mathcal{F}_{\mathrm{vol}}$ and $\mathcal{F}_{\mathrm{sur}}$. Example 2.9 does not fulfil this assumption. However, (2.8) is integrable on compact sets and one can easily find an integrable majorant, by applying the uniform bound of Lemma 4.1.

THEOREM 4.11. *Consider boundary regularity $k \geqslant 2$. Then the multi physics shape optimization problem (3.3) possesses at least one Pareto optimal solution $(\Omega^*, \phi_{\Omega^*}, u_{\Omega^*}) \in \mathcal{G}$ and covers all nondominated points in $\mathcal{Y}$, e.g. $\mathcal{Y}_N = \bar{\mathcal{Y}}_N$*

Proof. Lemma 4.8 provides the compactness of $\mathcal{G}$ and Lemma 4.9 the continuity of the local cost functionals. Then Theorem 3.3 provides the existence of an optimal shape and the closeness of the set of optimal shapes. □

5. Scalarization and Multi Physics Optimization. Scalarizing is the traditional approach to solving a multicriteria optimization problem. This includes formulating a single objective optimization problem that is related to the original Pareto optimality problem by means of a real-valued scalarizing function typically being a function of the objective function, auxiliary scalar or vector variables, and/or scalar or vector parameters. Additionally scalarization techniques sometimes further restrict the feasible set of the problem with new variables or/and restriction functions. In this section we investigate the stability of the parameter-dependent optimal shapes to different types of scalarization techniques with underlying design problem (3.3).

First, let us define the scalarization methods we consider. This involves a certain class of real-valued functions $S_\theta : \mathbb{R}^l \to \mathbb{R}$, referred to as scalaraization function that possibly depends on a parameter θ which lies in a parameter space Θ. The scalarization problem is given by

$$(5.1) \qquad \begin{aligned} &\min S_\theta \left(\mathbf{J}(\Omega, u_\Omega, \phi_\Omega) \right) \\ &\text{subject to } (\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G}_\theta, \end{aligned}$$

where $\mathcal{G}_\theta \subseteq \mathcal{G}$. For the sake of notational convenience, we sometimes identify an element $(\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G}_\theta$ only by its distinct shape Ω. If we assume that $\mathcal{G}_\theta$ is closed and the scalarization $S_\theta(\mathbf{J})$ is lower semicontinuous on $\mathcal{G}_\theta \times \{\theta\}$ then by the results of section 4, (5.1) obviously has an optimal solution for $\theta \in \Theta$. For a fixed $\theta \in \Theta$ we shall denote the space of all optimal shapes to an achievement function problem with $\zeta_\theta = \arg\min_{\Omega \in \mathcal{G}_\theta} S_\theta \left(\mathbf{J}(\Omega, u_\Omega, \phi_\Omega) \right)$. We assume that $\Theta \subset \mathbb{R}^l$ is closed and equip the space $\mathcal{Z} := \{\zeta_\theta \,|\, \theta \in \Theta\}$ with the Hausdorff distance which in this setting defines, due to the closeness of the optimal shapes sets, a metric (see Lemma 5.1 and Lemma 5.2).

In the following we gather some definitions and assertion from chapter 4 of [6]. We define the optimal set mapping $\chi : \Theta \longrightarrow \mathcal{Z}$, the optimal value mapping $\tau : \Theta \longrightarrow \mathbb{R}$ and the graph mapping $G : \Theta \longrightarrow 2^{\mathcal{G}}$ which maps a parameter $\theta \in \Theta$ to the corresponding set of optimal shapes ζ_θ, the corresponding optimal value $\min_{\Omega \in \mathcal{G}_\theta} S_\theta(\mathbf{J})$ and the corresponding graph $\mathcal{G}_\theta$ respectively. With these definitions in hand, we can describe the stability of the optimal shapes for a wide range of scalarization methods. First we state a lemma that shows that $(\mathcal{Z}, d_H)$ is indeed a metric space.

LEMMA 5.1. *The optimal set mapping χ is closed if τ is upper semicontinuous and $S_\theta(\mathbf{J})$ is lower semicontinuous on $\mathcal{G} \times \{\theta\}$.*

COROLLARY 5.2. *If the scalarization function S_θ is lower semicontinuous on $\mathbb{R}^l \times \{\theta\}$ and uniform continuous on $\{r\} \times \Theta$, for $r \in \mathbb{R}^l$, then the Hausdorff distance d_H defines a metric on $\mathcal{Z}$.*

Proof. Due to the continuity of $\mathbf{J}$ (see Lemma 4.9) and the uniform continuity of S_θ on $\{r\} \times \Theta$ the optimal value maping τ is upper semicontinuous and therefore by Lemma 5.1 the optimal set mapping χ is closed. Since d_H defines a metric on $F(\mathcal{G})$ (the set of all closed subsets of $\mathcal{G}$), $(\mathcal{Z}, d_H)$ defines a metric space. $\square$

Since the sclarization solution is not necessarily unique, we need some sort of continuity property of point-to-set mappings in order to discuss the stability of sets of optimal shapes. The literatur discribes serveral definitions which vary considerably in the statement. We investigate the stabilty according to Hausdorff and Berge (for Berge see [6]) which in this setting are equivalent.

DEFINITION 5.3 (Upper semicontiniuity according to Hausdorff). *Let (Θ, d_Θ) and (X, d_x) be metric spaces. A point-to-set mapping of Θ into X is a function Γ that assigns a subset $\Gamma(\theta)$ of X to each element $\theta \in \Theta$. This function is called upper semicontinuous in θ^*, if for each sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ with $\theta_n \longrightarrow \theta^*$, for $n \to \infty$, we have*

$$(5.2) \qquad \sup_{x \in T(\theta_n)} \inf_{x' \in T(\theta^*)} d_X(x, x') \longrightarrow 0.$$

Γ is called upper semicontinuous if Γ is upper semicontinuous in each $\theta \in \Theta$. For this type of continuity we simply write u.s.c.-H.

The next Theorem states stability conditions for scalarization function problems.

THEOREM 5.4 ([6]).   *Assume that G is u.s.c.-H at θ^* and $G(\theta^*)$ is compact. Further let τ be upper semicontinuous at θ^* and $S_{\theta*}$ lower semicontinuous on $G(\theta^*) \times \{\theta^*\}$. Then the optimal set mapping χ is u.s.c.-H at θ^*.*

The following two corollaries demonstrate continuity properties of shapes under change of preferences for two commonly used scalarization techniques. In particular, the results apply to the shape optimization problem introduced in section 3.2.

COROLLARY 5.5 (Weighted   Sum   Method).   *Consider   cost   functionals $\mathbf{J} = (J_1, \ldots, J_l)$ and let $\Theta \subset \mathbb{R}^l$ be a closed subset. Then the weighted sum scalaraization method which is given by*

$$\min \sum_{i=1}^{l} \theta_i J_i((\Omega, u_\Omega, \phi_\Omega))$$
$$subject\ to\ (\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G},$$

fulfils all conditions of Theorem 5.4 due to the compactness of $\mathcal{G}$ (see Lemma 4.8) and the continuity of $\mathbf{J}$ (see Lemma 4.9).

COROLLARY 5.6 (ϵ-Constraint   Method).   *Consider   cost   functionals $\mathbf{J} = (J_1, \ldots, J_l)$. We optimize cost functional J_j and constrain the other functionals by $J_i \leqslant \epsilon_i \in \mathbb{R}$, for $1 \leqslant i \leqslant n$ and $i \neq j$. If each ϵ_i converges monotonically decreasing to some ϵ_i^* then the ϵ-Constraint Method*

$$\min J_j((\Omega, u_\Omega, \phi_\Omega))$$
$$subject\ to\ J_i \leqslant \epsilon_i,$$

fulfils all conditions of Theorem 5.4.

Proof. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_l)$ and $\mathcal{G}_\epsilon = \{\Omega \in \mathcal{G} \mid J_i(\Omega) \leqslant \epsilon_i, i \neq j\}$. Th u.s.c.-H of G is given due to the continuity of $\mathbf{J}$. The continuity of J_j, the u.s.c.-H of G and the fact that $\mathcal{G}_\epsilon \subseteq \mathcal{G}_{\epsilon'}$ for all $\epsilon^* \leqslant \epsilon' \leqslant \epsilon$ gives that $\tau(\epsilon)$ converge continuously against $\tau(\epsilon^*)$ for $\epsilon \searrow \epsilon^*$. Hence the optimal sets $\chi(\epsilon)$ converge against $\chi(\epsilon^*)$ for $\epsilon \searrow \epsilon^*$ in the sense of u.s.c.-H.   □

Remark 5.7. Whenever the scalarized problem (5.1) has a unique solution $\zeta_\theta = \{\Omega_\theta\}$ for all θ in some neighborhood of $\theta \in \Theta$, $\Omega_{\theta_n} \longrightarrow \Omega_\theta$ in Hausdorff distance (for subsets in $\mathbb{R}^d$), if $\theta_n \to \theta$.

6.   Conclusions. In this work we extended the well known framework of for the existence of optimal solutions in shape optimization [14, 30] to a multi criteria a setting. We formulated conditions for the existence and completeness of Pareto optimal points. Multiple criteria in design are often related to simulations that include different domains of physics. We presented a coupled fluid-dynamic and mechanical system which is motivated by gas turbine design and fits to the given framework. The objective functions in this case are given by fluid losses and mechanical durability expressed by the probability of failure under low cycle fatigue. Both objectives require classical solutions to the underlying partial differential equations and therefore can only be formulated on sufficiently regular shapes, such that elliptic regularity theory applies [1, 2, 25]. We presented a formulation of the family of admissible shapes that implied the existence such classical solutions and thereby provided a non trivial example for the general framework.

In multi criteria optimization [19], the Pareto front contains points which are optimal with respect to different preferences of a decision maker. An interesting point is, if a small variation of the preference also leads to a small variation in the design.

This question however is ill-posed if Pareto optimal solutions need not to be unique. We therefore presented a study where we went over to the sets of Pareto optimal shapes for a given preference and studied the variation of these sets in the Hausdorff-metric. In this setting, certain continuity properties in the preference parameters were derived.

It will be of interest to develop multi-criteria shape optimization also from an algorithmic standpoint using the theory of shape derivatives and gradient based optimization – see e.g. [10, 18] for some first steps in that direction. For a rigorous analysis of numerical schemes of shape optimization, it will be of interest if (a) the optima of the discretized problem are close to the optima of the continuous problem and (b) if the same holds for shape gradients for non optimal solutions, as e.g. used in multi-criteria descent algorithms. In particular, this should be true for the objective values of discretized and continuous solutions, respectively. Potentially, iso-geometric finite elements [16, 24, 49] could be a useful numerical tool to not spoil the domain regularity that is built into our framework by the need of $C^{k,\alpha}$-classical solutions needed for the evaluation of the objectives in multi-criteria shape optimization problems like the one presented here.

Appendix.

DEFINITION .1 (Hölder continuity). *Let $U \subseteq \mathbb{R}^d$ be open. A function $f : U \to \mathbb{R}$ is called Hölder continuous if there exist non-negative real constants $C, \alpha > 0$, such that*

$$|f(x) - f(y)| \leqslant C|x - y|^\alpha.$$

With $C^{k,\alpha}(U)$ we denote the space consisting of function on U having Hölder continuous derivatives up to order k with exponent α. If the function f and its derivatives up to order k are bounded on the closure of U, we can assign the norm

$$\|f\|_{C^{k,\alpha}(\bar{U})} = \max_{|\gamma| \leqslant k} \left\|\frac{\partial^{|\gamma|} f}{\partial x^\gamma}\right\|_\infty + \max_{|\gamma| = k} \left[\frac{\partial^{|\gamma|} f}{\partial x^\gamma}\right]_{0,\alpha},$$

where,

$$\|f\|_\infty = \sup_{x \in U} |f(x)|$$

$$[f]_{0,\alpha} = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We note that $(C^{k,\alpha}(U), \|\cdot\|_{C^{k,\alpha}(U)})$ is a Banach space. A vector field $f : U \to \mathbb{R}^n$ is Hölder continuous with exponent $\alpha > 0$, if each component of f is Hölder continuous with exponent α.

LEMMA .2 ([25], Lemma 6.37). *Let Ω be a $C^{k,\alpha}$ domain in $\mathbb{R}^d$ (with $k \geqslant 1$) and let Ω' be an open set containing $\bar{\Omega}$. Suppose $u \in C^{k,\alpha}(\bar{\Omega})$. Then there exist a function $w \in C_0^{k,\alpha}(\Omega')$ such that $w = u$ and*

$$(.1) \qquad \|w\|_{C^{k,\alpha}(\Omega')} \leqslant C\|u\|_{C^{k,\alpha}(\Omega)},$$

where $C = C(k, \Omega, \Omega')$.

LEMMA .3 (Korn's Second Inequality, [11]). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with piecewise smooth boundary. In addition, suppose $\Gamma_0 \subset \partial\Omega$ has positive two dimensional measure. Then there exist a positive number $c' = c'(\Omega, \Gamma_0)$ such that*

$$\int_\Omega \epsilon(v) \, : \, \epsilon(v) \, dx \geqslant c' \|v\|_{H^1(\Omega)} \quad \forall v \in H_\Gamma^1(\Omega).$$

Here $H_\Gamma^1(\Omega)$ is the closure of $\{v \in [C^\infty(\Omega)]^3 \,|\, v(x) = 0 \text{ for } x \in \Gamma_0\}$ w.r.t. the $\|\cdot\|_{H^1(\Omega)}$-norm.

LEMMA .4 ([25], Lemma 6.36). *Let Ω be a $C^{k,\alpha}$ domain in $\mathbb{R}^d$ (with $k \geqslant 1$) and let S be a bounded set in $C^{k,\alpha}$. Then S is precompact in $C^{j,\beta}(\bar{\Omega})$ if $j + \beta \leqslant k + \alpha$.*

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