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Spontaneous wave formation in stochastic self-driven particle systems

Martin Friesen , Hanno Gottschalk , Barbara Rüdiger*, and Antoine Tordeux†

Abstract. Waves and oscillations are commonly observed in the dynamics of systems self-driven agents such as pedestrians or vehicles. Interestingly, many factors may perturb the stability of homogeneous solutions, leading to the spontaneous formation of waves related as stop-and-go waves or phantom jam in the literature. In this article, we demonstrate that even a minimal additive stochastic noise in stable first-order dynamics can describe stop-and-go phenomena. The noise is not a classic white one, but a colored noise described by a Gaussian Ornstein-Uhlenbeck process. It turns out that the joint dynamics of particles and noises forms again a (Gaussian) Ornstein-Uhlenbeck process whose characteristics can be explicitly expressed in terms of parameters of the model. We analyze its stability and characterize the presence of waves by calculating the correlation and autocorrelation functions of the distance spacing between the particles. The autocorrelation of the noise induces collective oscillation in the system and spontaneous emergence of waves. While the correlation and autocorrelation functions are complex-valued and difficult to analyze and interpret, we show that these functions become real-valued in the continuum limit when the system size is infinite. Finally, we propose consistent statistical estimations of the model parameters and compare experimental trajectories of single-file pedestrian motions to simulation results of the calibrated stochastic model.

Key words. Self-driven particle system, stop-and-go wave, stability analysis, autocorrelation, interacting particle system, Markovian process

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1. Introduction. The emergence of collective behaviors is frequently observed in the dynamics of self-driven agents interacting locally. Examples are collective motions and self-organization including the formation of patterns, structures, waves, or clusters in bacterial colonies, animal aggregations, or pedestrian dynamics [11, 10, 40, 22, 21]. Spontaneous formation of stop-and-go waves in congested road traffic flows or pedestrian streams is a typical example of self-organization. Indeed, stop-and-go phenomena, also related as phantom jam, jamiton, or self-sustained waves in the literature [26, 35, 17], currently occur in traffic, pedestrian or again bicycle flows. Even in experiments with periodic boundary conditions where the infrastructure cannot explain their presence stop-and-go waves emerge, the flows of vehicles or pedestrians tending to stream jerky instead of streaming homogeneously [36, 41].

Road traffic flows are modeled thanks to microscopic, mesoscopic or macroscopic models (see [13, 8] for reviews). Microscopic approaches describe the individual trajectories of vehicles with car-following models, agent-based approaches, or cellular automata. Mesoscopic

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models are gas-kinetic approaches describing PDF for the speeds and the vehicle positions, while macroscopic models are partial differential equations for aggregated performances. The spontaneous formation of stop-and-go waves is generally explained by means of instability of the homogeneous solutions and phase transition. See, e.g., the references [4, 25, 5, 16] for microscopic models, [24, 6, 9] for mesoscopic models, or [15, 14, 18] for macroscopic ones. Macroscopic hyperbolic continuum, such as Korteweg-de Vries, modified Korteweg-de Vries or time-dependent Ginzburg-Landau soliton equations, are derived from microscopic car-following models [28, 29, 7, 3]. Generally speaking, the models are second order differential equations defined by relaxation processes for the speed, the distance spacing of the flow density. The stability of the homogeneous can break down due to inertia effects, i.e. when the relaxations times (i.e. the inertia) exceed critical thresholds [31, 30]. In the unstable case, the solutions can be periodic, quasi-periodic, limit cycles or even chaotic dynamics describing stationary stop-and-go waves [37].

Recent results have pointed out that spontaneous stop-and-go waves can also simply emerge from stochastic effects [19, 38]. Stochastic effects have different roles in the dynamics of self-driven systems. Generally speaking, the introduction of white noises tends to increase the disorder and prevent self-organization [39, 20], while coloured noises can generate complex structures and patterns [2, 12]. Stochastic effects and notions of noise are one of the main emphases of pedestrian or traffic modeling approaches. In most of the cases, the noises added to the dynamics are white noises [23, 37, 11, 20]. In this article, we demonstrate that the introduction of a colored noise, namely a truncated Brownian noise provided by the Ornstein-Uhlenbeck process, induces the spontaneous formation of stop-and-go waves in stable dynamics of the first order. In contrast to classical inertial deterministic modelling approaches, no instability neither as phase transition phenomena are observed. The waves are due to non-linear stochastic effects. They are highlighted by analysing the correlation and autocorrelation of the spacing difference describing oscillations and traffic waves propagating at characterised speeds given by the Rankine–Hugoniot formula. The stochastic model has been introduced to describe pedestrian dynamics in [38]. It is defined in the next section. The model is solved in Sec. 3, while its stability is analyzed in Sec. 4. The covariance and autocovariance are calculated for the finite system with periodic boundary conditions in Sec. 5, and at the limit of an infinite system in Sec. 6. Finally, we demonstrate convergence property of statistical estimates for the model parameters in Sec. 7.

2. Stochastic following model. We consider N particles on a system of length L with periodic boundary conditions. We denote in the following as $(x_n(t))_{n=1,\dots,N}$ the curvilinear positions of the particles $n = 1, \dots, N$ at time $t \geq 0$ (see Fig. 1) and suppose that the particles are initially ordered by their index, i.e.

$$x_1(0) \leq x_2(0) \leq ... \leq x_N(0) \leq L + x_1(0).$$

In the following model, the speed of a particle is a deterministic optimal velocity $V : s \mapsto V(s)$ depending on the spacing s, coupled to an additive stochastic noise. We consider in the rest of the paper congested traffic states and the affine optimal velocity function

$$V(s) = \lambda(s - \ell),$$

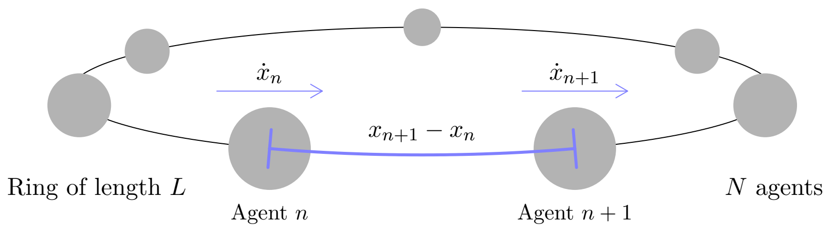

Figure 1. *Scheme of the system with periodic boundary conditions. x_n is the curvilinear position while $\Delta x_n = x_{n+1} - x_n$ is the spacing and $\dot{x}_n$ the speed of the particle with number n.*

with $\lambda > 0$ the inverse of the equilibrium time gap between the particles and $\ell \geq 0$ their length. The time evolution of the particle with number $n = 1, \ldots, N$ is supposed to follow the stochastic ordinary differential equation

$$\dot{x}_n(t) = \lambda(\Delta x_n(t) - \ell) + \xi_n(t), \qquad t \geq 0, \tag{1}$$

where $(\xi_n(t))_{t\geq 0}$ denotes the noise and the spacing between the particles are

$$\begin{cases} \Delta x_n(t) &= x_{n+1}(t) - x_n(t), \qquad n = 1, \ldots, N-1, \\ \Delta x_N(t) &= L + x_1(t) - x_N(t). \end{cases} \tag{2}$$

We suppose that the noise is given by independent Ornstein-Uhlenbeck processes, i.e.

$$d\xi_n(t) = -\beta \xi_n(t)\, dt + \sigma\, dW_n(t), \tag{3}$$

where $W_n(t)$, $n = 1, \ldots, N$, are independent Wiener processes, $\beta > 0$ denotes the relaxation rate and $\sigma \in \mathbb{R}$ the amplitude of the noise, respectively. Applying the Itô formula to $K_n(t) = e^{\beta t}\xi_n(t)$ one finds that each $\xi_n(t)$ is given by

$$\xi_n(t) = e^{-\beta t}\xi_n(0) + \sigma \int_0^t e^{\beta(s-t)}\, dW_n(s). \tag{4}$$

Instead of (1), we analyse the spacing difference of $x_n(t)$ to the homogeneous solution $x_n^H(t)$, i.e.

$$y_n(t) = \Delta x_n(t) - \Delta x_n^H(t) \tag{5}$$

where the homogeneous solution is given by

$$\begin{cases} x_n^H(t) = x_n^H(0) + t\lambda(L/N - \ell), \\ \Delta x_n^H(t) = L/N \end{cases} .$$

Representation (5) has the advantage that it allows us to study the effects of noise such as stop-and-go waves. We have for all $n = 1, \dots, N$

$$\dot{y}_n(t) = \lambda\big(y_{n+1}(t) - y_n(t)\big) + \xi_{n+1}(t) - \xi_n(t).$$

This equation can be expressed by the system of stochastic ordinary differential equations

$$\dot{Y}(t) = \lambda A Y(t) + A\Xi(t), \tag{6}$$

where $Y(t) = \big[y_1(t), y_2(t), \dots, y_N(t)\big]^\top \in \mathbb{R}^N$, $\Xi(t) = \big[\xi_1(t), \xi_2(t), \dots, \xi_N(t)\big]^\top \in \mathbb{R}^N$ and

$$A = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & -1 \end{bmatrix} \in \mathrm{M}_{N\times N}.$$

Let us stress that the processes $[x_1(t), \dots, x_n(t)]^\top$ obtained from (1) as well as $Y(t)$ obtained from (6) both take values in $\mathbb{R}^N$, i.e. they are not measured on the (periodic) ring but on an infinite lane and by assuming, as given in (2), that the spacing of the vehicle N is $\Delta x_N(t) = L + x_1(t) - x_N(t) + L$.

3. Solving the model. Rewriting (6) into the differential form

$$dY(t) = \big(\lambda A Y(t) + A\Xi(t)\big)\, dt \tag{7}$$

shows that the noise $\Xi(t)$ enters in the definition of $Y(t)$ as an additional random drift parameter. Hence $Y(t)$ cannot be a Markov process in its own. To overcome this difficulty we enlarge the state space from $\mathbb{R}^N$ to $\mathbb{R}^N \times \mathbb{R}^N$ by also taking the evolution of the noise Ξ into account. In this way $Z := (Y, \Xi)$ becomes a Markov process with state space $\mathbb{R}^N \times \mathbb{R}^N$.

Indeed, using (3) combined with (7) we find that $Z(t) = (Y(t), \Xi(t))$ solves the system of stochastic differential equations

$$dZ(t) = BZ(t)\, dt + G\, dW(t), \qquad Z(0) = (Y(0), \Xi(0)), \tag{8}$$

where $W(t) = (W_n(t))_{n=1,\dots,2N}$ is a family of independent Wiener processes and the $N \times N$ matrices B, G are given by

$$B = \begin{pmatrix} \lambda A & A \\ 0 & -\beta 1_N \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 \\ 0 & \sigma 1_N \end{pmatrix},$$

where 1_N denotes the identity matrix acting on $\mathbb{R}^N$.

The particular form of (8) shows that Z is a $2N$-dimensional Ornstein-Uhlenbeck process and hence is given by

$$Z(t) = e^{tB} Z(0) + \int_0^t e^{(t-s)B} G\, dW(s). \tag{9}$$

Following the general theory of Ornstein-Uhlenbeck processes (see, e.g., [34, 1]) we find that Z is a Feller process with Markov generator

$$Lf(z) = \sum_{k=1}^{2N} (Bz)_j \frac{\partial f(z)}{\partial z_j} + \frac{1}{2} \sum_{k,j=1}^{2N} (GG^\top)_{kj} \frac{\partial^2 f(z)}{\partial z_k \partial z_j}.$$

Moreover, it is a Gaussian process whose characteristic function is, for $z, p \in \mathbb{R}^{2N}$, given by

$$\begin{aligned} \mathbb{E}[e^{i\langle p, Z(t)\rangle} \mid Z(0) = z] &= \exp\left(i\langle z, e^{tB^\top} p\rangle - \frac{1}{2} \int_0^t \langle e^{sB^\top} p, GG^\top e^{sB^\top} p\rangle \, ds \right) \\ &= \exp\left(i\langle \mu_z(t), p\rangle - \frac{1}{2} \langle p, \Sigma(t) p\rangle \right) \end{aligned} \tag{10}$$

where its expectation $\mu_z(t)$ and covariance operator $\Sigma(t)$ are given by

$$\mu_z(t) = e^{tB} z, \qquad \Sigma(t) = \int_0^t e^{sB} GG^\top e^{sB^\top} \, ds.$$

More generally one can also compute its covariance structure at different times.

Lemma 1. *For $t, s \geq 0$ it holds*

$$\operatorname{cov}(Z(t), Z(s)) = e^{tB} \int_0^{\min\{t,s\}} e^{-uB} GG^\top e^{-uB^\top} \, du e^{sB^\top}.$$

Proof. Denote by $\mathbb{1}_{\{u \leq t\}} = \begin{cases} 1, & u \leq t \\ 0, & u > t \end{cases}$ the indicator function on the set $\{u \leq t\}$. Using (9) we find that

$$\begin{aligned} \operatorname{cov}(Z(t), Z(s)) &= \operatorname{cov}\left(e^{tB} \int_0^\infty \mathbb{1}_{\{u \leq t\}} e^{-uB} G \, dW(u), \; e^{sB} \int_0^\infty \mathbb{1}_{\{v \leq s\}} e^{-vB} G \, dW(v) \right) \\ &= \mathbb{E}\left[e^{tB} \int_0^\infty \mathbb{1}_{\{u \leq t\}} e^{-uB} G \, dW(u) \left(\int_0^\infty e^{-vB} G \, dW(v) \right)^\top e^{sB^\top} \right] \\ &= e^{tB} \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{1}_{\{u \leq t\}} \mathbb{1}_{\{v \leq s\}} e^{-uB} G \, dW(u) \, dW^\top(v) G^\top e^{-vB^\top} \right] e^{sB^\top} \\ &= e^{tB} \int_0^\infty \mathbb{1}_{\{u \leq t\}} \mathbb{1}_{\{u \leq s\}} e^{-uB} G^\top \, du e^{sB^\top} \\ &= e^{tB} \int_0^{\min\{t,s\}} e^{-uB} GG^\top e^{-uB^\top} \, du e^{sB^\top}, \end{aligned}$$

where we have used that $dW_j(u) \, dW_k(v) = \delta_{kj} \delta(u - v) \, du \, dv$ which yields

$$dW(u) \, dW(v)^\top = (\, dW_j(u) \, dW_k(v))_{j,k \in \{1,\dots,2N\}} = 1_{2N} \delta(u - v) \, du \, dv$$

with 1_{2N} denoting the identity matrix acting on $\mathbb{R}^{2N}$. ∎

As Z is a Gaussian process, it is completely characterized by its expectation and covariance structure. Based on the formulas of this section we can express all desired (statistical) quantities in terms of the characteristic function and hence its mean and covariance structure.

4. Stability analysis. In this section we investigate the long-time behaviour of the mean $\mathbb{E}[X(t)]$, the limiting distribution $X(\infty)$, and finally invariant measures for the Markovian dynamics. Such analysis crucially relies on the spectra of A and B which are, therefore, investigated first.

Proposition 2. *The matrix A is diagonalizable with eigenvalues*

$$\omega_k = \gamma_k - 1, \qquad \gamma_k = e^{2\pi i \frac{k}{N}}, \qquad k = 0, \dots, N-1,$$

and corresponding eigenvectors

$$\mathbf{u}_k = \begin{bmatrix} \gamma_k^0 & \gamma_k^1 & \dots & \gamma_k^{N-1} \end{bmatrix}^\top, \qquad k = 0, \dots N-1. \tag{11}$$

The coefficients of the matrix exponential e^{At} are given by

$$e^{At}(n,m) = \frac{1}{N}\sum_{k=0}^{N-1} \gamma_k^{n-m} e^{\omega_k t}, \qquad 1 \le n, m \le N \tag{12}$$

and it holds for each $y \in \mathbb{R}^N$

$$\left\| e^{At}y - \left(\frac{1}{N}\sum_{k=1}^{N} y_k\right)\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}\right\|_N \le \sqrt{N}\|y\|_N e^{-2\sin\left(\frac{\pi}{N}\right)^2 t}, \qquad t \ge 0, \tag{13}$$

where $\|y\|_N^2 = \sum_{n=1}^N |y_n|^2$ denotes the euclidean norm on $\mathbb{R}^N$.

Proof. The matrix A is invariant by circular permutation, i.e. $\theta(A) = A$ with

$$\theta : (a_{n,m}, 1 \le n, m \le N) \mapsto (a_{n+1,m+1}, 1 \le n, m \le N),$$

and $(a_{n,m})_{n,m}$ being the coefficients of A and by assuming that $a_{N+1,m} = a_{1,m}$, $a_{n,N+1} = a_{n,1}$ and $a_{N+1,N+1} = a_{1,1}$. Likewise, let $\theta(\mathbf{u}) = (u_{j+1})_{j=1,\dots,N}$ with $u_{N+1} := u_1$. If $\mathbf{u} = \begin{bmatrix} u_1 & \dots & u_N \end{bmatrix}^\top$ is an eigenvector of A associated to the eigenvalue ω, i.e. $A\mathbf{u} = \omega\mathbf{u}$, then

$$\omega\theta(\mathbf{u}) = \theta(\omega\mathbf{u}) = \theta(A\mathbf{u}) = \theta(A)\theta(\mathbf{u}) = A\theta(\mathbf{u}),$$

i.e. $\theta(\mathbf{u})$ is also an eigenvector with eigenvalue ω. Iterating this procedure yields $u_1 = \gamma u_2 = \dots = \gamma^N u_1$ and $\gamma = \sqrt[N]{1}$ is a N-th root of unity. The eigenvectors of A are then given by (11). Using $A\mathbf{u}_k = \omega_k \mathbf{u}_k$, the eigenvalues are precisely $\omega_k = \gamma_k - 1$. The N eigenvalues $(\omega_0, \dots, \omega_{N-1})$ are distinct, therefore A is diagonalisable and $A = PDP^{-1}$, with

$$P(n,m) = \gamma_{m-1}^{n-1}, \qquad P^{-1}(n,m) = \frac{1}{N}\gamma_{n-1}^{1-m}, \qquad D = \operatorname{diag}(\omega_0, \dots, \omega_{N-1}).$$

Using $e^{At} = Pe^{Dt}P^{-1}$ gives (12). Moreover, using the particular form of the eigenvalues it is not difficult to show that

$$e^{At}(n,m) \underset{t\to\infty}{\rightarrow} \frac{\gamma_0^{n-m}}{N} = \frac{1}{N} \qquad \text{for all } 1 \le n, m \le N.$$

From this we could deduce a similar estimate to (13) where the constant $\sqrt{N}$ is replaced by N. In order to prove the stronger estimate (13), we first set $\mathbf{v}_0 = N^{-1/2}\mathbf{u}_0$ and then find by the Graham-Schmidt procedure an orthonormal basis $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$ of $\mathbb{R}^N$ such that $A\mathbf{v}_n = \omega_n \mathbf{v}_n$, $n = 0, \ldots, N-1$. For

$$y = \sum_{n=0}^{N-1} \langle y, \mathbf{v}_n \rangle \mathbf{v}_n$$

we obtain

$$e^{At} y = \sum_{n=0}^{N-1} \langle y, \mathbf{v}_n \rangle e^{\omega_n t} \mathbf{v}_n$$

and hence

$$\begin{aligned}
\left\| e^{At} y - \langle y, \mathbf{v}_0 \rangle \mathbf{v}_0 \right\|_N &\le \sum_{n=1}^{N} |\langle y, \mathbf{v}_n \rangle| e^{\Re(\omega_n) t} \\
&\le \sqrt{N} e^{-2\sin\left(\frac{\pi}{N}\right)^2 t} \left(\sum_{n=1}^{N-1} |\langle y, \mathbf{v}_n \rangle|^2 \right)^{1/2} \\
&\le \sqrt{N} \|y\|_N e^{-2\sin\left(\frac{\pi}{N}\right)^2 t},
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality and

$$\Re(\omega_n) = \cos\left(\frac{2\pi n}{N}\right) - 1 = -2\sin\left(\frac{\pi n}{N}\right) \le -2\sin\left(\frac{\pi}{N}\right), \qquad n = 1, \ldots, N-1.$$

The assertion follows from the identity

$$\langle y, \mathbf{v}_0 \rangle \mathbf{v}_0 = \frac{1}{N} \langle y, \mathbf{u}_0 \rangle \mathbf{u}_0 = \left(\frac{1}{N} \sum_{n=1}^{N} y_n \right) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

■

Next we continue with the analysis of the spectrum for B.

Proposition 3. *The matrix B has eigenvalues*

$$(\lambda\omega_0, \ldots, \lambda\omega_{N-1}, -\beta, \ldots, -\beta) \tag{14}$$

and corresponding eigenvectors

$$\left(\begin{bmatrix} \mathbf{u}_0 \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} \mathbf{u}_{N-1} \\ 0 \end{bmatrix}, \begin{bmatrix} -(\beta + \lambda A)^{-1} A e_1 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} -(\beta + \lambda A)^{-1} A e_N \\ e_N \end{bmatrix} \right), \tag{15}$$

where $e_1, \ldots, e_N \in \mathbb{R}^N$ denote the canonical basis vectors in $\mathbb{R}^N$. In particular B is diagonalisable and for each $z \in \mathbb{R}^{2N}$

$$\left\| e^{Bt} z - \left(\frac{1}{N} \sum_{n=1}^{N} z_n \right) \begin{bmatrix} \mathbf{u}_0 \\ 0 \end{bmatrix} \right\|_{2N} \le \sqrt{2N} \|z\|_{2N} e^{-\delta t}, \qquad t \ge 0, \tag{16}$$

where $\delta = \min\{\beta, 2\sin(\pi/N)^2\} > 0$ and $\mathbf{u}_0 = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^\top \in \mathbb{R}^N$.

Proof. The characteristic equation for B is

$$0 = \det\left(\begin{bmatrix} w1_N & 0 \\ 0 & w1_N \end{bmatrix} - \begin{bmatrix} \lambda A & A \\ 0 & -\beta 1_N \end{bmatrix}\right) = \det(w1_N - \lambda A)\det(w1_N + \beta 1_N),$$

whose solutions in $w \in \mathbb{C}$ are exactly (14). Let $\begin{bmatrix} y & \xi \end{bmatrix}^\top \in \mathbb{R}^{2N}$ be an eigenvector for the eigenvalue $\lambda\omega_k$, then

$$\lambda\omega_k \begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} \lambda A & A \\ 0 & -\beta 1_N \end{bmatrix} \begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} \lambda A y + A\xi \\ -\beta\xi \end{bmatrix}.$$

Hence $\xi = 0$ and $y = \mathbf{u}_k$. Similarly, let $\begin{bmatrix} y & \xi \end{bmatrix}^\top \in \mathbb{R}^{2N}$ be an eigenvector for the eigenvalue $-\beta$, then

$$-\beta \begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} \lambda A y + A\xi \\ -\beta\xi \end{bmatrix}.$$

Hence ξ is arbitrary while y satisfies $(\beta 1_N + \lambda A)y = -A\xi$. Choosing $\xi \in \{e_1, \dots, e_N\}$ shows that the eigenvectors are given by (15) and that the corresponding eigenspaces span $\mathbb{R}^{2N}$, i.e. B is diagonalisable. Concerning assertion (16) we proceed similarly to (13). Let $\mathbf{v}_1, \dots, \mathbf{v}_{2N}$ be an orthonormal basis of eigenvectors of B with $\mathbf{v}_1 = N^{-1/2}\begin{bmatrix} \mathbf{u}_0 & 0 \end{bmatrix}^\top$, and denote by $\varrho_1, \dots \varrho_{2N}$ the corresponding eigenvalues with $\varrho_n = \lambda\omega_{n-1}$, $n = 1, \dots, N$, while $\varrho_n = -\beta$ for $n = N+1, \dots, 2N$. For

$$z = \sum_{n=1}^{2N} \langle z, \mathbf{v}_n \rangle \mathbf{v}_n,$$

we obtain

$$e^{Bt} z = \sum_{n=1}^{2N} \langle z, \mathbf{v}_n \rangle e^{\varrho_n t} \mathbf{v}_n$$

and hence

$$\begin{aligned} \left\| e^{Bt} z - \langle z, \mathbf{v}_1 \rangle \mathbf{v}_1 \right\|_{2N} &\le \sum_{n=2}^{2N} |\langle z, \mathbf{v}_n \rangle| e^{\Re(\varrho_n)t} \\ &\le e^{-\delta t}\sqrt{2N} \left(\sum_{n=2}^{2N} |\langle z, \mathbf{v}_n \rangle|^2 \right)^{1/2} \\ &\le \sqrt{2N} \|z\|_{2N} e^{-\delta t}, \end{aligned}$$

where we have used the Cauchy-Schwartz inequality and

$$\Re(\varrho_n) \le -\delta, \qquad n = 2, \dots, 2N.$$

Since $\langle z, v_1 \rangle \mathbf{v}_1 = \left(\frac{1}{N} \sum_{n=1}^N z_n \right) \begin{bmatrix} \mathbf{u}_0 & 0 \end{bmatrix}^\top$, the assertion is proved. ■

Next we study the asymptotic behaviour of $Z(t)$ as $t \to \infty$.

Theorem 4. *It holds $Z(t) \to_{t\to\infty} Z(\infty)$ in law, where $Z(\infty)$ is a Gaussian random variable on $\mathbb{R}^{2N}$ with mean zero and covariance matrix*

$$\Sigma(\infty) = \int_0^\infty e^{tB} G G^\top e^{tB^\top} \, dt.$$

Proof. Using the characterization of convergence in law by characteristic functions (that is Lévy's continuity Theorem), it suffices to show that $\Sigma(\infty)$ is well-defined and that

$$\lim_{t\to\infty} \mathbb{E}[e^{i\langle p, Z(t)\rangle}] = \exp\left(-\frac{1}{2}\langle p, \Sigma(\infty) p\rangle\right), \qquad \forall p \in \mathbb{R}^{2N}. \tag{17}$$

Note that $\Sigma(\infty)$ is well-defined, if

$$\int_0^\infty \left| \langle p, e^{Bt} G G^\top e^{B^\top t} q\rangle \right| dt < \infty, \qquad \forall p, q \in \mathbb{R}^{2N}. \tag{18}$$

Estimating first the scalar product and then the integral by Cauchy-Schwartz we arrive at

$$\begin{aligned}\int_0^\infty \left| \langle p, e^{Bt} G G^\top e^{B^\top t} q\rangle \right| dt &\le \int_0^\infty \|G^\top e^{B^\top t} p\|_{2N} \|G^\top e^{B^\top t} q\|_{2N} \, dt \\ &\le \left(\int_0^\infty \|G^\top e^{B^\top t} p\|_{2N}^2 \, dt\right)^{1/2} \left(\int_0^\infty \|G^\top e^{B^\top t} q\|_{2N}^2 \, dt\right)^{1/2}.\end{aligned}$$

In order to show that these integrals are finite we first estimate $e^{Bt}G$ in the Frobenius norm $\|\cdot\|_{\mathrm{F}}$ of a $2N \times 2N$ matrix. Indeed, for each $p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^\top \in \mathbb{R}^{2N}$ we find $Gp = \begin{bmatrix} 0 & \sigma p_2 \end{bmatrix}^\top$ and hence from (16) applied to $z = Gp$

$$\|e^{Bt} G p\|_{2N} \le \sqrt{2N} \|Gp\|_{2N} e^{-\delta t} \le \sqrt{2N} \|G\|_{\mathrm{F}} \|p\|_{2N} e^{-\delta t},$$

i.e. $\|e^{Bt} G\|_{\mathrm{F}} \le \sqrt{2N} \|G\|_{\mathrm{F}} e^{-\delta t}$. From this we obtain

$$\|G^\top e^{B^\top t} p\|_{2N} \le \|G^\top e^{B^\top t}\|_{\mathrm{F}} \|p\|_{2N} = \|e^{Bt} G\|_{\mathrm{F}} \|p\|_{2N} \le \sqrt{2N} \|G\|_{\mathrm{F}} e^{-\delta t} \|p\|_{2N},$$

which shows that (18) is satisfied.

We proceed to prove (17). Using regular conditional distributions combined with (10) we find that

$$\begin{aligned}\mathbb{E}[e^{i\langle p, Z(t)\rangle}] &= \int_{\mathbb{R}^{2N}} \mathbb{E}[e^{i\langle p, Z(t)\rangle} \mid Z(0) = z] \mathbb{P}[Z(0) \in dz] \\ &= e^{-\frac{1}{2}\langle p, \Sigma(t) p\rangle} \int_{\mathbb{R}^{2N}} e^{i\langle e^{Bt} z, p\rangle} \mathbb{P}[Z(0) \in dz].\end{aligned}$$

Using (18) we conclude that $\Sigma(t) \to \Sigma(\infty)$ as $t \to \infty$. Using (16) we find

$$e^{Bt} z \longrightarrow \left(\frac{1}{N} \sum_{n=1}^N z_n\right) \left\langle p, \begin{bmatrix} \mathbf{u}_0 \\ 0 \end{bmatrix} \right\rangle = 0$$

for $z \in Q = \{w \in \mathbb{R}^{2N} \mid \sum_{n=1}^{N} w_n = 0\}$. Then observing that

$$\begin{aligned}
\sum_{n=1}^{N} Z_n(0) &= \sum_{n=1}^{N} y_n(0) \\
&= \sum_{n=1}^{N} \left(\Delta x_n(0) - \Delta x_n^H(0)\right) \\
&= L + x_1(0) - x_N(0) + \sum_{n=1}^{N-1} \left(x_{n+1}(0) - x_n(0)\right) - \sum_{n=1}^{N} \Delta x_n^H(0) \\
&= L - L = 0
\end{aligned}$$

we find that $Z(0)$ belongs to Q a.s. and hence

$$\int_{\mathbb{R}^{2N}} e^{i\langle e^{Bt}z,p\rangle} \mathbb{P}[Z(0) \in dz] = \int_Q e^{i\langle e^{Bt}z,p\rangle} \mathbb{P}[Z(0) \in dz] \longrightarrow 1, \qquad t \to \infty.$$

This proves (17) and hence the assertion. ■

This result shows that $\mathbb{E}[Z(t)] \longrightarrow 0$ as $t \to \infty$, i.e. the whole dynamics tends asymptotically (in the mean) to the homogeneous solution. Since $\Sigma(\infty) \neq 0$ the limiting law $Z(\infty)$ is non-trivial and describes Gaussian fluctuations around the homogeneous solution. Note that this law is also the unique invariant distribution for the process (at least when restricted to the phyically interesting configurations satisfying $\sum_{n=1}^{N} z_n = 0$). As a consequence of previous result we find for the first component Y

$$\mathbb{E}[Y(t)] \longrightarrow 0 \quad \text{and} \quad Y(t) \xrightarrow{d} Y(\infty), \quad \text{as} \quad t \to \infty,$$

where $Y(\infty)$ is a Gaussian random variable $\mathbb{R}^N$ with covariance structure

$$\langle k, \Sigma_Y(\infty)p\rangle = \int_0^\infty \left\langle G^\top e^{B^\top s} \begin{bmatrix} k \\ 0 \end{bmatrix}, G^\top e^{B^\top s} \begin{bmatrix} p \\ 0 \end{bmatrix} \right\rangle ds.$$

We close this section with a precise formula for $\mathbb{E}[Y(t)]$, while the values for $\Sigma_Y(\infty)$ will be computed in the next section.

Theorem 5. *One has*

$$\mathbb{E}[Y(t)] = e^{\lambda A t}\mathbb{E}[Y(0)] + (\beta + \lambda A)^{-1} \left(e^{-\beta t} - e^{\lambda A t}\right) A\mathbb{E}[\xi(0)].$$

Proof. To simplify notation we let $\overline{Y}(t) = \mathbb{E}[Y(t)]$ and similarly $\overline{\xi}(t) = \mathbb{E}[\xi(t)]$. Taking expectations in (6) gives

$$\overline{Y}(t) = \lambda A \overline{Y}(t) - A\overline{\xi}(t).$$

Using (4) so that $\overline{\xi}(t) = e^{-\beta t}\overline{\xi}(0)$ gives

$$\begin{aligned}\overline{Y}(t) &= e^{\lambda At}\overline{Y}(0) + \int_0^t e^{\lambda A(t-s)}e^{-\beta s}A\overline{\xi}(s)\,ds\\ &= e^{\lambda At}\overline{Y}(0) + e^{\lambda At}\int_0^t e^{-(\beta+\lambda A)s}A\overline{\xi}(0)\,ds\\ &= e^{\lambda At}\overline{Y}(0) + e^{\lambda At}(\beta+\lambda A)^{-1}(e^{-(\beta+\lambda A)t} - 1_N)A\overline{\xi}(0)\\ &= e^{\lambda At}\overline{Y}(0) + (\beta+\lambda A)^{-1}(e^{-\beta t} - e^{\lambda At})A\overline{\xi}(0),\end{aligned}$$

which proves the assertion. ∎

5. Covariance and autocovariance.

Writing

$$Y(t) = e^{\lambda At}K(t),$$

with $K(t)$ a vector of size N, we obtain using Eq. (6) $K'(t) = e^{-\lambda At}A\xi(t)$. One gets by integrating on $[0,t]$

$$K(t) = C + \textstyle\int_0^t e^{-\lambda Au}A\xi(u)\,du.$$

Here $C = K(0) = Y(0)$ and we obtain

$$Y(t) = e^{\lambda At}K(t) = e^{\lambda At}Y(0) + \int_0^t e^{\lambda A(t-u)}A\xi(u)\,du, \tag{19}$$

or again, using the explicit solution $\xi_n(t) = e^{-\beta t}\xi_n(0) + \sigma\int_0^t e^{\beta(u-t)}\,dW_n(u)$ for the Ornstein-Uhlenbeck processes,

$$Y(t) = e^{\lambda At}Y(0) + R_0(t) + \sigma R(t),$$

with

$$R_0(t) = \int_0^t e^{\lambda A(t-u)}Ae^{-\beta u}\,du\;\xi(0),$$

and

$$R(t) = \int_0^t e^{\lambda A(t-u)}A\int_0^u e^{\beta(s-u)}\,dW(s)\,du,$$

$W(t) = (W_1(t),\dots,W_N(t))^\top$ being a vector of independent Wiener processes.

We have

$$\begin{aligned}R_0(t) &= \textstyle\int_0^t e^{-(\lambda A+I_N\beta)u}\,du\; e^{\lambda At}A\xi(0)\\ &= [\lambda A + I_N\beta]^{-1}\big(I_N - e^{-\beta t}e^{-\lambda At}\big)e^{\lambda At}A\xi(0)\\ &= [\lambda A + I_N\beta]^{-1}\big(e^{\lambda At}A\xi(0) - e^{-\beta t}\xi(0)\big) \;\to\; (0,\dots,0) \;\text{ as }\; t\to\infty,\end{aligned}$$

since $e^{\lambda At}A$ and $e^{-\beta t}\xi(0)$ tends to 0 as $t\to\infty$, while $[\lambda A + I_N\beta]X = 0$ implies $X = 0$ for all $\lambda, \beta > 0$.

We denote respectively in the following $\mathrm{cov}_j(0)$ and $\mathrm{cov}_0(\tau)$ the asymptotic covariance and autocovariance of the spacing difference of the particles

$$\mathrm{cov}(y_n(t), y_{n+j}(t)) \underset{t\to\infty}{\to} \mathrm{cov}_j(0),$$

and

$$\mathrm{cov}(y_n(t), y_n(t+\tau)) \underset{t\to\infty}{\to} \mathrm{cov}_0(\tau).$$

Theorem 6. *The asymptotic covariance of the spacing difference to the spacing difference of the particle $n+j$ ahead is for any particle $n = 1, \ldots, N$,*

$$\mathrm{cov}_j(0) = \frac{\sigma^2}{2\beta N} \sum_{k=1}^{N-1} \frac{\gamma_k^j}{\lambda - \beta - \lambda\gamma_k} \left(\frac{(1-\gamma_k)^2}{\lambda - (\lambda+\beta)\gamma_k} - \frac{2\beta}{\lambda(\lambda + \beta - \lambda\gamma_k)} \right), \tag{20}$$

while the asymptotic autocovariance at time $\tau \geq 0$ is

$$\mathrm{cov}_0(\tau) = \frac{\sigma^2}{2\beta N} \sum_{k=1}^{N-1} \frac{1}{\lambda - \beta - \lambda\gamma_k} \left(\frac{e^{-\beta\tau}(1-\gamma_k)^2}{\lambda - (\lambda+\beta)\gamma_k} - \frac{2\beta e^{-\lambda(1-\gamma_k)\tau}}{\lambda(\lambda + \beta - \lambda\gamma_k)} \right), \tag{21}$$

with $\gamma_k = e^{2\pi i \frac{k}{N}}$ the N-roots of unity.

Proof. The autocovariance of the one-dimensional Ornstein-Uhlenbeck is

$$\mathrm{cov}(\xi_n(t), \xi_n(s)) = \frac{\sigma^2}{2\beta} e^{-\beta(t+s)} \left(e^{2\beta \min\{t,s\}} - 1 \right), \tag{22}$$

Using Eq. (19) by assuming $Y(0) = \xi(0) = (0, \ldots, 0)^\top$ in order to simplify the calculation and by remarking that $A + A^\top = -AA^\top$, the covariance of the process is

$$\mathrm{cov}(Y(t), Y(s)) = \mathbb{E}\left(\int_0^t Ae^{\lambda A(t-u)}\xi(u)\, du \int_0^s \xi^\top(v) e^{\lambda A^\top(s-v)} A^\top\, dv \right)$$

$$= Ae^{\lambda At} \int_0^t \int_0^s e^{-\lambda Au} e^{-\lambda A^\top v} \mathrm{cov}(\xi(u), \xi(v))\, dv\, du\; e^{\lambda A^\top s} A^\top$$

$$= \tfrac{\sigma^2}{2\beta} Ae^{\lambda At} \int_0^t \int_0^s e^{-(\lambda A + \beta I_N)u} e^{-(\lambda A^\top + \beta I_N)v} \left(e^{2\beta \min\{u,v\}} - 1 \right) dv\, du\; e^{\lambda A^\top s} A^\top$$

$$\begin{aligned} &= \sigma^2 \big[\underbrace{Ae^{\lambda At}}_{\to 0} - \underbrace{Ae^{-\beta t}}_{\to 0} \big] [\lambda A + \beta I_N]^{-1} \left[\lambda^2 (A^\top)^2 - \beta^2 I_N \right]^{-1} \underbrace{e^{\lambda A^\top s} A^\top}_{\to 0} \\ &+ \frac{\sigma^2}{\lambda} \big[\underbrace{e^{\lambda AA^\top t}}_{\to (1/N)_{N^2}} - I_N \big] e^{\lambda A^\top (s-t)} \left[\lambda^2 (A^\top)^2 - \beta^2 I_N \right]^{-1} \\ &- \frac{\sigma^2}{2\beta} \Big[\big[\underbrace{e^{-\beta s} Ae^{\lambda At}}_{\to 0} - e^{-\beta(s-t)} A \big] [\lambda A - \beta]^{-1} \\ &- \big[\underbrace{e^{-\beta s} Ae^{\lambda At}}_{\to 0} - \underbrace{e^{-\beta(t+s)} A}_{\to 0} \big] [\lambda A + \beta]^{-1} \Big] \left[\lambda A^\top + \beta \right]^{-1} A^\top. \end{aligned} \tag{23}$$

The calculation details are provided in Appendix 1. We obtain asymptotically if $s = t + \tau$ with $\tau \geq 0$,

$$\begin{aligned}\lim_{t\to\infty} \operatorname{cov}(Y(t), Y(t+\tau)) &= \frac{\sigma^2}{\lambda}\big[(1/N)_{N^2} - I_N\big] e^{\lambda A^\top \tau}\Big[\lambda^2\big(A^\top\big)^2 - \beta^2 I_N\Big]^{-1} \\ &+ \frac{\sigma^2}{2\beta} e^{-\beta\tau} A\,[\lambda A - \beta I_N]^{-1}\Big[\lambda A^\top + \beta I_N\Big]^{-1} A^\top,\end{aligned}$$

with $(1/N)_{N^2}$ the $N \times N$ matrix with coefficients $1/N$ everywhere. Developing the matrix, one gets for any particle $n = 1, \ldots, N$, the asymptotic covariance of the spacing difference to the spacing difference of the particle $n + j$ ahead

$$\operatorname{cov}_j(0) = \frac{\sigma^2}{2\beta N} \sum_{k=1}^{N-1} \frac{\gamma_k^j}{\lambda - \beta - \lambda\gamma_k}\left(\frac{(1-\gamma_k)^2}{\lambda - (\lambda+\beta)\gamma_k} - \frac{2\beta}{\lambda(\lambda + \beta - \lambda\gamma_k)}\right),$$

while the asymptotic autocovariance at time $\tau \geq 0$ is

$$\operatorname{cov}_0(\tau) = \frac{\sigma^2}{2\beta N} \sum_{k=1}^{N-1} \frac{1}{\lambda - \beta - \lambda\gamma_k}\left(\frac{e^{-\beta\tau}(1-\gamma_k)^2}{\lambda - (\lambda+\beta)\gamma_k} - \frac{2\beta e^{-\lambda(1-\gamma_k)\tau}}{\lambda(\lambda + \beta - \lambda\gamma_k)}\right),$$

with $\gamma_k = e^{2\pi i \frac{k}{N}}$. ■

Corollary 7. *The correlation and autocorrelation*

$$\operatorname{cor}_j(\tau) = \frac{\operatorname{cov}_j(\tau)}{\operatorname{cov}_0(0)}$$

do not depend on the parameter σ.

The correlation and autocorrelation of the spacing difference are not zero (see Fig. 2). Indeed, a single wave propagates backward in the system, the correlation with the neighbor being decreasing with the first half of the predecessors and increasing the second (see Fig. 2, left panel). In adequacy with the LWR theory [33, 27], the waves propagate backward in the system at the speed $v_w = -\lambda\ell$ provided by the Rankine–Hugoniot formula while the particles travel in average at the speed $v = \lambda(L/N - \ell)$. Therefore, the period of the waves is $P = L/(v - v_w) = N/\lambda$ (see Fig. 2, right panel).

6. Covariance and autocovariance for the infinite system. The covariance and autocovariance Eqs. (20) and (21) at the limit $N, L \to \infty$ with L/N constant are the Riemann integrals

$$\operatorname{cov}_j^\infty(\tau) = \frac{\sigma^2}{2\beta}\int_0^1 F(e^{2\pi i t})\,dt = \frac{\sigma^2}{2\beta}\frac{1}{2\pi i}\int_{|z|=1} \frac{F(z)}{z}\,dz \tag{24}$$

with

$$F(z) = \frac{1}{\lambda - \beta - \lambda z}\left(\frac{z^j e^{-\beta\tau}(1-z)^2}{\lambda - (\lambda+\beta)z} - \frac{z^j e^{\lambda(z-1)\tau} 2\beta}{\lambda(\lambda+\beta-\lambda z)}\right).$$

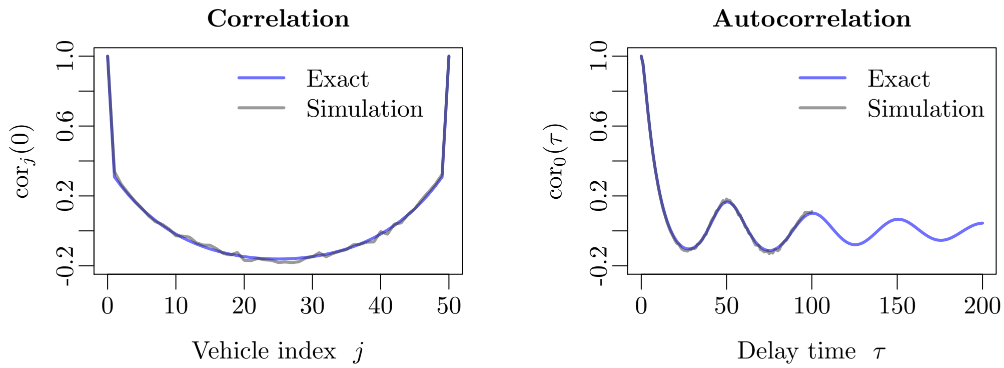

Figure 2. *Empirical and exact correlation and autocorrelation (see Eqs. (20) and (21)) for a system with $N = 50$ particles in stationary state. $\lambda = 1$ and $\beta = 0.1$. The empirical results are measured after 1e5 units of simulation time.*

Theorem 8. *The asymptotic correlation and autocorrelation of the spacing difference in stationary state are respectively at the limit $N, L \to$ with L/N constant*

$$\mathrm{cor}_j^\infty(0) = \frac{1}{2}\left(\frac{\lambda}{\lambda+\beta}\right)^j, \quad j > 0, \tag{25}$$

and

$$\mathrm{cor}_0^\infty(\tau) = \frac{\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}}{\lambda - \beta}, \quad \tau \geq 0. \tag{26}$$

Proof. We decompose the function $F(z)/z$ in simple elements to calculate the asymptotic autocovariance Eq. (24)

$$\begin{aligned}\frac{F(z)}{z} = \frac{z^j e^{-\beta\tau}}{\lambda}&\left(\frac{1}{(\lambda-\beta)z} - \frac{1}{(\lambda-\beta)\left(z - \frac{\lambda-\beta}{\lambda}\right)} + \frac{1}{(\lambda+\beta)\left(z - \frac{\lambda}{\lambda+\beta}\right)}\right)\\ &- \frac{z^j e^{\lambda(z-1)\tau}}{\lambda}\left(\frac{2\beta}{(\lambda^2-\beta^2)z} - \frac{1}{(\lambda-\beta)\left(z-\frac{\lambda-\beta}{\lambda}\right)} + \frac{1}{(\lambda+\beta)\left(z - \frac{\lambda+\beta}{\lambda}\right)}\right).\end{aligned}$$

Using the Cauchy formula

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{z^j}{z-\xi}\,dz = \begin{cases}\xi^j, & |\xi| < 1\\ 0, & |\xi| > 1,\end{cases}$$

we obtain after calculations detailed in Appendix 2

$$\mathrm{cov}_j^\infty(0) = \begin{cases}\dfrac{\sigma^2}{\lambda\beta(\lambda+\beta)}, & j = 0,\\[2ex] \dfrac{\sigma^2\lambda^{j-1}}{2\beta(\lambda+\beta)^{j+1}}, & j > 0,\end{cases}$$

Proceeding in the same way we find for the autocovariance Eq. (21) at the limit $N, L \to \infty$

$$\mathrm{cov}_0^\infty(\tau) = \frac{\sigma^2}{\lambda\beta(\lambda^2 - \beta^2)} \left(\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}\right).$$

The asymptotic variance of the distance spacing is $\mathrm{cov}_0^\infty(0) = \frac{\sigma^2}{\lambda\beta(\lambda+\beta)}$, while the asymptotic correlation and autocorrelation are then respectively (see Fig. 3)

$$\mathrm{cor}_j^\infty(0) = \frac{1}{2}\left(\frac{\lambda}{\lambda+\beta}\right)^j, \quad j > 0,$$

and

$$\mathrm{cor}_0^\infty(\tau) = \frac{\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}}{\lambda - \beta}, \quad \tau \geq 0.$$

∎

In Fig. (3), the correlation and autocorrelation for the spacing difference in stationary state are plotted for $N = 50$, $N = 100$, $N = 200$ and at the limit $N, L \to \infty$ with L/N constant for $\lambda = 1$ and $\beta = 0.1$. The wave period is $P = N/\lambda = 50$ for $N = 50$, while it is $P = 100$ and $P = 200$ for $N = 100$ and $N = 200$ and is infinite at the limit $N, L \to \infty$.

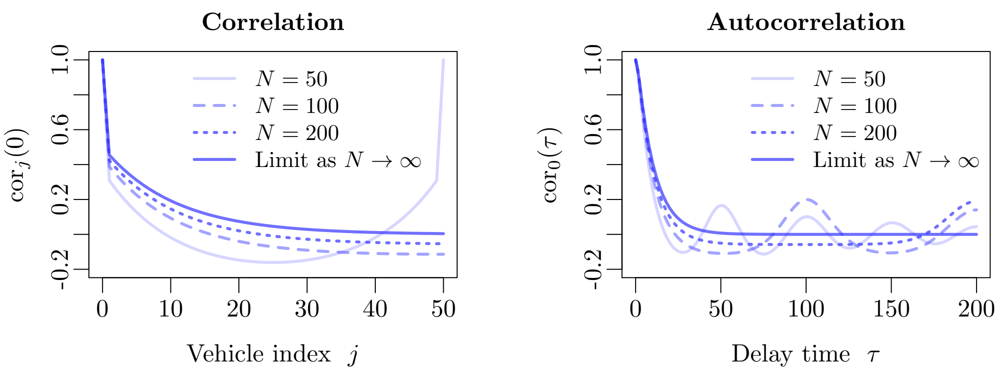

Figure 3. *Correlation and autocorrelation in stationary state for systems with $N = 50$, 100, 200 and at the limit $N, L \to \infty$ with L/N constant (see Eqs. (25) and (26)) . $\lambda = 1$ and $\beta = 0.1$.*

7. Estimation of the model parameters. The stochastic pedestrian model is based on four parameters: the time gap inverse λ, the pedestrian length ℓ, the noise relaxation rate β and the noise volatility σ. Let us suppose disposing of K observations (v_k, s_k), $k = 1, \ldots, K$, of the speed v_k and spacing s_k of pedestrians in a row. We suppose that the observations are realisations of time-dependent random variables (V, S) such that $V = \lambda^\star(S - \ell^\star) + Z(\beta^\star, \sigma^\star)$ for some parameter $\theta^\star = (\lambda^\star, \ell^\star, \beta^\star, \sigma^\star)$, $Z(a, b)$ being a Ornstein-Uhlenbeck process with relaxation a and volatility b.

Theorem 9. *We denote as $\tilde\theta_K = (\tilde\lambda_K, \tilde\ell_K, \tilde\beta_K, \tilde\sigma_K)$ the least squares estimates of the parameters of the model.*

Then

$$\tilde\theta_K \xrightarrow{a.s.} \theta^\star \qquad \textit{as} \quad K \to \infty. \tag{27}$$

Proof. The least squares estimations $(\tilde{\lambda}_K, \tilde{\ell}_K)$ of the time gap inverse and pedestrian length parameters are solutions of the quadratic optimisation problem

$$(\tilde{\lambda}_K, \tilde{\ell}_K) = \arg\min_{\lambda,\ell} H_K(\lambda,\ell), \qquad \text{with} \quad H_K(\lambda,\ell) = \frac{1}{K}\sum_{k=1}^{K}\left[\lambda(s_k-\ell)-v_k\right]^2.$$

Solving $\partial H_K/\partial\lambda = \partial H_K/\partial\ell = 0$, the optima are

$$\tilde{\lambda}_K = \frac{\frac{1}{K}\sum_k (s_k-\bar{s}_K)^2}{\frac{1}{K}\sum_k (s_k v_k - \bar{s}_K\bar{v}_K)} \qquad \text{and} \qquad \tilde{\ell}_K = \bar{s}_K - \bar{v}_K/\tilde{\lambda}_K,$$

with the mean values $\bar{s}_K = \frac{1}{K}\sum_{k=1}^{K} s_k$ and $\bar{v}_K = \frac{1}{K}\sum_{k=1}^{K} v_k$.

Thanks to the ergodic framework of the process,

$$H_K(\lambda,\ell) \to H(\lambda,\ell) = \mathbb{E}_\infty\left[(\lambda(S-\ell)-V)^2\right]$$

almost surely as $K\to\infty$, i.e. the least squares estimate corresponds asymptotically to minimising the variability of the model's residuum. Furthermore,

$$\tilde{\lambda}_K \to \lambda^\star = \frac{\mathrm{var}_\infty(S)}{\mathrm{cov}_\infty(S,V)} \qquad \text{while} \qquad \tilde{\ell}_K \to \ell^\star = \mathbb{E}_\infty(S) - \mathbb{E}_\infty(V)/\lambda^\star$$

almost surely as $K\to\infty$.

We denote in the following the model's residuals $r_k = \tilde{\lambda}_K(d_k - \tilde{\ell}_K) - v_k$. We suppose disposing of successive observations of the residual (r_k, r_k^τ) at time t_k and $t_k+\tau$ to estimate empirically the autocorrelation of the noise. The asymptotic autocorrelation of the Ornstein-Uhlenbeck process $\xi_n(t)$ is $e^{-\tau\beta}$ while the asymptotic variance is $\sigma^2\beta^{-1}/2$. The least-squares estimators for the noise relaxation rate β and the noise volatility σ are

$$\tilde{\beta}_K = -\frac{\log(\tilde{c}_K(\tau))}{\tau} \qquad \text{and} \qquad \tilde{\sigma}_K^2 = 2\tilde{\beta}_K H_K\big(\tilde{\lambda}_K, \tilde{\ell}_K\big),$$

with $\tilde{c}_K(\tau) = \frac{1}{K}\sum_{k=1}^{K} r_k r_k^\tau$ the empirical estimation of the autocorrelation of the residuals at time lag τ. Here again, the ergodic theorem allows to show that $\tilde{c}_K(\tau)\to c_\infty(\tau)$,

$$\tilde{\beta}_K \to \beta^\star = -\frac{\log(c_\infty(\tau))}{\tau}, \qquad \text{and} \qquad \tilde{\sigma}_K^2 \to (\sigma^\star)^2 = 2\beta^\star H(\lambda^\star,\ell^\star),$$

almost surely as $K\to\infty$. ■

Single-file experimental data are used to calibrate the parameters. The data come from experiments done on a quasi-circular geometry of length 27 m with soldiers in 2007 in Germany (see the schemes Fig. 4 and [32, 38] for details on the data). Extracting speed and spacing observations from the trajectories, the estimates of the parameters were $\tilde{\lambda} = 0.98$ s^{-1}, $\tilde{\ell} = 0.34$ m, $\tilde{\beta} = 0.23$ s^{-1} and $\tilde{\sigma} = 0.09$ ms$^{3/2}$ [38]. The trajectories for the experiments done with 28, 45 and 62 participants (corresponding to a density level of 1 ped/m, 1.7 ped/m and 2.3 ped/m) are plotted in Fig. 5, top panels, while the simulated trajectories obtained with the calibrated stochastic model are shown in the bottom panels. The simulation results are obtained using a Euler-Maruyama scheme with time step $\delta t = 0.01$ s.

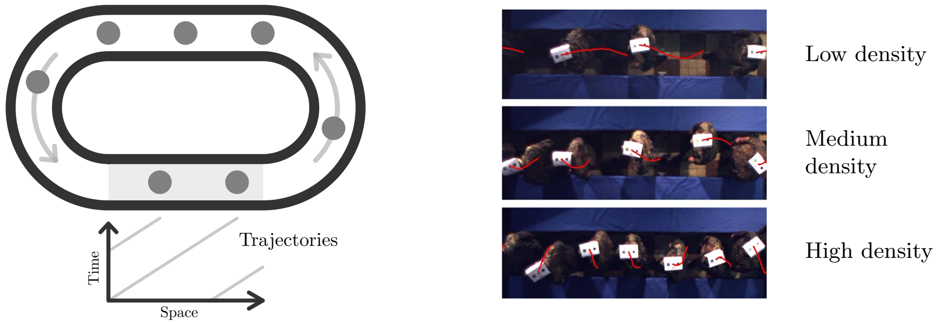

Figure 4. *Schemes for the single-motion experiment and the collection of the trajectory data.*

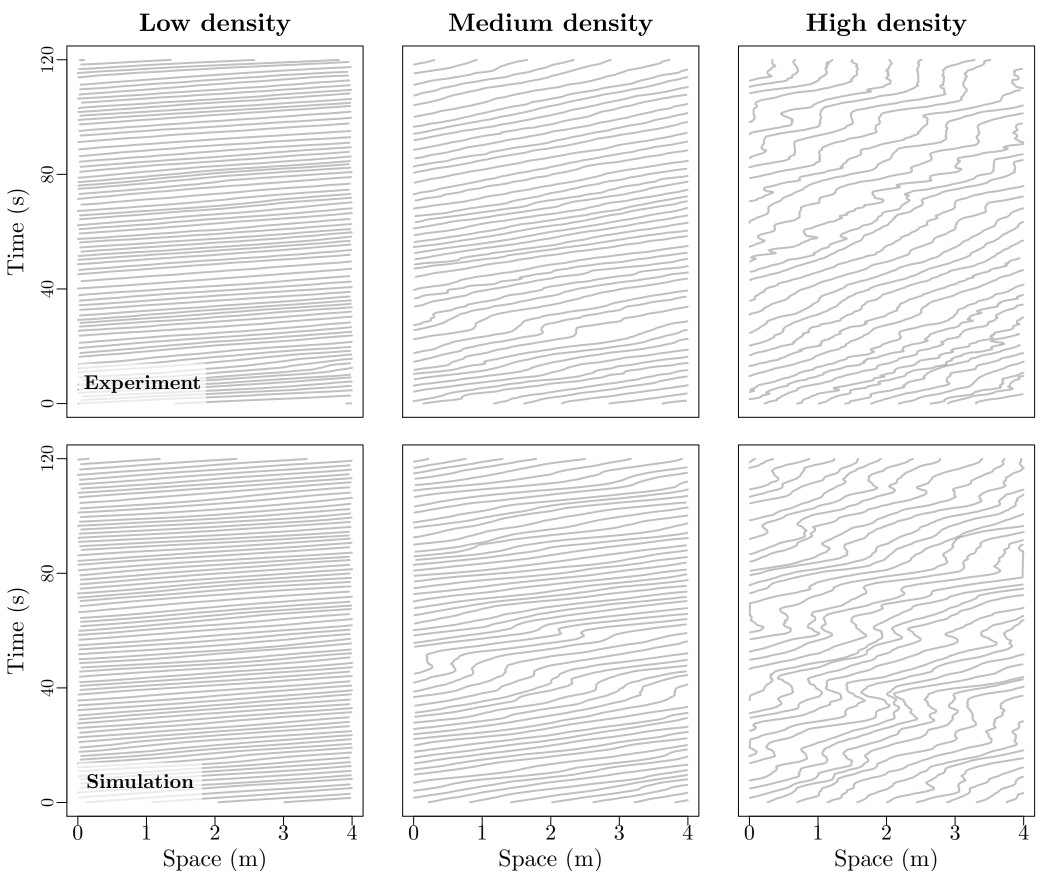

Figure 5. *Trajectories of single-file pedestrian motions with density levels 1 ped/m (left panels), 1.7 ped/m (central panels) and 2.3 ped/m (right panel). Top panels: Real experimental data. Bottom panels: Simulation of the calibrated stochastic pedestrian model. We observe stop-and-go waves for medium and high density levels in both real data and simulation.*

Appendix 1. The covariance of the spacing difference to the homogeneous solution is, by using Eq. (23),

$$
\begin{aligned}
&\frac{2\beta}{\sigma^2}\left[Ae^{\lambda At}\right]^{-1}\operatorname{cov}(\mathbf{y}(t),\mathbf{y}(s))\left[A^\top e^{\lambda A^\top s}\right]^{-1} \\
&= \int_0^t\int_0^s e^{-(\lambda A+\beta I_N)u}e^{-(\lambda A^\top+\beta I_N)v}\left(e^{2\beta\min\{u,v\}}-1\right)dv\,du \\
&= \int_0^t\int_0^u e^{-(\lambda A+\beta I_N)u}e^{-(\lambda A^\top+\beta I_N)v}\left(e^{2\beta v}-1\right)dv\,du \\
&\quad+\int_0^t\int_u^s e^{-(\lambda A+\beta I_N)u}e^{-(\lambda A^\top+\beta I_N)v}\left(e^{2\beta u}-1\right)dv\,du \\
&= \int_0^t e^{-(\lambda A+\beta I_N)u}\int_0^u\left[e^{-(\lambda A^\top-\beta I_N)v}-e^{-(\lambda A^\top+\beta I_N)v}\right]dv\,du \\
&\quad+\int_0^t\left[e^{-(\lambda A-\beta I_N)u}-e^{-(\lambda A+\beta I_N)u}\right]\int_u^s e^{-(\lambda A^\top+\beta I_N)v}\,dv\,du \\
&= \int_0^t e^{-(\lambda A+\beta I_N)u}\left[I_N-e^{-(\lambda A^\top-\beta I_N)u}\right]\left[\lambda A^\top-\beta I_N\right]^{-1} \\
&\qquad -e^{-(\lambda A+\beta I_N)u}\left[I_N-e^{-(\lambda A^\top+\beta I_N)u}\right]\left[\lambda A^\top+\beta I_N\right]^{-1}du \\
&+\int_0^t\left[e^{-(\lambda A-\beta I_N)u}-e^{-(\lambda A+\beta I_N)u}\right]\left[e^{-(\lambda A^\top+\beta I_N)u}-e^{-(\lambda A^\top+\beta I_N)s}\right] \\
&\qquad\left[\lambda A^\top+\beta I_N\right]^{-1}du^\top
\end{aligned}
\tag{A1}
$$

Finally,

$$
\begin{aligned}
\operatorname{cov}(\mathbf{y}(t),\mathbf{y}(s)) &= \frac{\sigma^2}{2\beta}Ae^{\lambda At}\int_0^t\left[e^{-(\lambda A+\beta I_N)u}-e^{-\lambda(A+A^\top)u}\right]du \\
&\qquad\left[\left[\lambda A^\top-\beta I_N\right]^{-1}-\left[\lambda A^\top+\beta I_N\right]^{-1}\right]e^{\lambda A^\top s}A^\top \\
&-\frac{\sigma^2}{2\beta}e^{-\beta s}Ae^{\lambda At}\int_0^t\left[e^{-(\lambda A-\beta I_N)u}-e^{-(\lambda A+\beta I_N)u}\right]du\left[\lambda A^\top+\beta I_N\right]^{-1}A^\top \\
&= \sigma^2\left[Ae^{\lambda At}-Ae^{-\beta t}\right][\lambda A+\beta I_N]^{-1}\left[\lambda^2(A^\top)^2-\beta^2 I_N\right]^{-1}e^{\lambda A^\top s}A^\top \\
&+\frac{\sigma^2}{\lambda}\left[e^{\lambda AA^\top t}-I_N\right]e^{\lambda A^\top(s-t)}\left[\lambda^2(A^\top)^2-\beta^2 I_N\right]^{-1} \\
&-\frac{\sigma^2}{2\beta}\left[\left[e^{-\beta s}Ae^{\lambda At}-e^{-\beta(s-t)}A\right][\lambda A-\beta]^{-1}\right. \\
&\left.-\left[e^{-\beta s}Ae^{\lambda At}-e^{-\beta(t+s)}A\right][\lambda A+\beta]^{-1}\right]\left[\lambda A^\top+\beta\right]^{-1}A^\top,
\end{aligned}
\tag{A2}
$$

since $A+A^\top=-AA^\top$ and $\left[\lambda A^\top-\beta I_N\right]^{-1}-\left[\lambda A^\top+\beta I_N\right]^{-1}=2\beta\left[\lambda^2(A^\top)^2-\beta^2 I_N\right]^{-1}$.

Appendix 2. The covariance and autocovariance of the spacing difference at the limit $N, L \to \infty$ with L/N constant are the Riemann integrals

$$\operatorname{cov}_j^\infty(\tau) = \frac{\sigma^2}{2\beta}\frac{1}{2\pi i}\int_{|z|=1} \frac{F(z)}{z}\, dz,$$

with

$$\begin{aligned}\frac{F(z)}{z} = \frac{z^j e^{-\beta\tau}}{\lambda}&\left(\frac{1}{(\lambda-\beta)z} - \frac{1}{(\lambda-\beta)\left(z-\frac{\lambda-\beta}{\lambda}\right)} + \frac{1}{(\lambda+\beta)\left(z-\frac{\lambda}{\lambda+\beta}\right)}\right)\\ &- \frac{z^j e^{\lambda(z-1)\tau}}{\lambda}\left(\frac{2\beta}{(\lambda^2-\beta^2)z} - \frac{1}{(\lambda-\beta)\left(z-\frac{\lambda-\beta}{\lambda}\right)} + \frac{1}{(\lambda+\beta)\left(z-\frac{\lambda+\beta}{\lambda}\right)}\right).\end{aligned}$$

In the following, the covariances and autocovariances are determined by using the Cauchy formula. We obtain the variance if $j = 0$ and $\tau = 0$

$$\operatorname{cov}_0^\infty(0) = \frac{\sigma^2}{2\lambda\beta}\left[\frac{1}{\lambda-\beta} + \frac{1}{\lambda+\beta} - \frac{2\beta}{\lambda^2-\beta^2}\right] = \frac{\sigma^2}{\lambda\beta(\lambda+\beta)}. \tag{A3}$$

For $j > 0$ and $\tau = 0$, the covariance is

$$\operatorname{cov}_j^\infty(0) = \frac{\sigma^2}{2\lambda\beta}\frac{\left(\frac{\lambda}{\lambda+\beta}\right)^j}{\lambda+\beta} = \frac{\sigma^2\lambda^{j-1}}{2\beta(\lambda+\beta)^{j+1}}, \tag{A4}$$

and $\operatorname{cor}_j^\infty(0) = \frac{1}{2}\left[\lambda/(\lambda+\beta)\right]^j$. For $j = 0$ and $\tau \geq 0$, the auto-covariance is if $\left|\frac{\lambda-\beta}{\lambda}\right| \leq 1$, i.e. if $\beta \leq 2\lambda$,

$$\begin{aligned}\operatorname{cov}_0^\infty(\tau) &= \frac{\sigma^2}{2\lambda\beta}\left[\frac{e^{-\beta\tau}}{\lambda+\beta} - \frac{2\beta e^{-\lambda\tau}}{\lambda^2-\beta^2} + \frac{e^{\lambda\left(\frac{\lambda-\beta}{\lambda}-1\right)\tau}}{\lambda-\beta}\right]\\ &= \frac{\sigma^2}{2\lambda\beta}\left[\frac{e^{-\beta\tau}}{\lambda+\beta} - \frac{2\beta e^{-\lambda\tau}}{\lambda^2-\beta^2} + \frac{e^{-\beta\tau\tau}}{\lambda-\beta}\right] = \frac{\sigma^2\left[\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}\right]}{\lambda\beta[\lambda^2-\beta^2]}.\end{aligned} \tag{A5}$$

Similarly, we get if $\beta > 2\lambda$

$$\operatorname{cov}_0^\infty(\tau) = \frac{\sigma^2}{2\lambda\beta}\left[\frac{e^{-\beta\tau}}{\lambda-\beta} + \frac{e^{-\beta\tau}}{\lambda+\beta} - \frac{2\beta e^{-\lambda\tau}}{\lambda^2-\beta^2}\right] = \frac{\sigma^2\left[\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}\right]}{\lambda\beta[\lambda^2-\beta^2]},$$

and $\operatorname{cor}_0^\infty(\tau) = [\lambda e^{-\beta\tau} - \beta e^{-\lambda\tau}]/[\lambda-\beta]$. Note that by taking $\lambda = \beta + \varepsilon$ and by calculating the auto-covariance at the limit $\varepsilon \to 0$ we obtain

$$\operatorname{cov}_0^\infty(\tau) = \frac{\sigma^2 e^{-\lambda\tau}}{2\lambda^3}[1 + \lambda\tau]$$

while $\operatorname{cor}_0^\infty(\tau) = e^{-\lambda\tau}[1 + \lambda\tau]$ if $\beta = \lambda$.

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