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Remarks on input-to-state stability of collocated systems with saturated feedback

Birgit Jacob, Felix L. Schwenninger and Lukas A. Vorberg *

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We investigate input-to-state stability of infinite-dimensional collocated control systems subject to saturated feedback where the unsaturated closed loop system is dissipative and uniformly globally asymptotically stable. We review recent results from the literature and explore limitations thereof.

1 Introduction

In this note we continue the study of the stability of systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) - B\sigma\big(B^*x(t) + d(t)\big), \\ x(0) = x_0, \end{cases} \tag{Σ_{SLD}}$$

derived from the linear collocated open-loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= B^*x(t). \end{aligned}$$

by the nonlinear feedback law $u(t) = -\sigma(y(t) + d(t))$. Here X and U are Hilbert spaces, $A : D(A) \subset X \to X$ is the generator of a strongly continuous contraction semigroup and B is a bounded linear operator from U to X, i.e. $B \in \mathcal{L}(U, X)$. The function $\sigma : U \to U$ is Lipschitz continuous and maximal monotone with $\sigma(0) = 0$. Of particular interest is the case in which σ is even locally linear. In the following we are interested in stability with respect to both the initial value x_0, that is *internal stability*, and the

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disturbance d; *external stability*. This is combined in the notion of *input-to-state stability* (ISS), which has recently been studied for infinite-dimensional systems e.g. in [6, 8, 18, 19] and particularly for semilinear systems in [4, 5, 22], see also [17] for a survey. The effect of feedback laws acting (approximately) linearly only locally is known in the literature as *saturation*, and first appeared in [25, 23] in the context of stabilization of infinite-dimensional linear systems, see also [9]. There, internal stability of the closed-loop system was studied using nonlinear semigroup theory, a natural tool to establish existence and uniqueness of solutions for equations of the above type, see also the more recent works [10, 14, 15]. The simultaneous study of internal stability and the robustness with respect to additive disturbances in the saturation seems to be rather recent. This notion clearly includes uniform global (internal) stability, which by far is not guaranteed for such nonlinear systems. In [21] this was studied for a wave equation and in [13] Korteweg-de Vries type equation was rigorously discussed, building on preliminary works in [11, 12], see also [10].
The combination of saturation and ISS was initiated in [15] and, as for internal stability, complemented in [14]. For the rich finite-dimensional theory on ISS for related semilinear systems, we refer e.g. to [4, 5] and the references therein. For (infinite-dimensional) nonlinear systems, ISS is typically assessed by Lyapunov functions, see e.g. [3, 7, 16, 19, 22]. These are often constructed by energy-based L^2 norms, but also Banach space methods exist [19], which are much easier to handle in the sense of L^∞-estimates as present in ISS. We will use some of these constructions here.

In this note we investigate the question whether internal stability of the corresponding linear undisturbed system, that is, (Σ_{SLD}) with $\sigma(u) = u$ and $d \equiv 0$, implies input-to-state stability of (Σ_{SLD}). In doing so we try to shed light on limitations of existing results. Because the linear system has a bounded input operator, the above question is equivalent to asking whether ISS of the linear system yields that $(\Sigma_{SLD})$is ISS, see e.g. [8]. For nonlinear systems, uniform global (internal) stability is only a necessary condition for ISS, which, however, may fail in presence of saturation. The following system can be seen as a show-case example,

$$\begin{cases} \dot{x}(t,\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}x(t,\xi) - \mathrm{sat}_{\mathbb{R}}\big(x(t,\xi)\big), & (t,\xi) \in (0,\infty) \times [0,1], \\ x(t,0) = x(t,1), \\ x(0,\xi) = f(\xi), \end{cases} \tag{$\Sigma_{\mathfrak{sat}}$}$$

where

$$\mathrm{sat}_{\mathbb{R}}(z) := \begin{cases} 1, & z \geq 1 \\ z, & z \in (-1,1) \\ -1, & z \leq -1. \end{cases} \tag{1}$$

2 ISS for saturated systems

Definition 1. *We call $\sigma : U \to U$ an* admissible feedback function *if*

i) $\sigma(0) = 0$,

ii) σ *is* Lipschitz continuous*, i.e. there exists a $k > 0$ such that*

$$\|\sigma(u) - \sigma(v)\|_U \le k\|u - v\|_U \quad \forall\, u, v \in U,$$

iii) σ *is* maximal monotone*, i.e.* $\Re\langle \sigma(u) - \sigma(v), u - v\rangle_U \ge 0 \quad \forall\, u, v \in U$.

If additionally a Banach space S is continuously, densely embedded in U with

iv) $\|\sigma(u) - u\|_{S'} \le \Re\langle \sigma(u), u\rangle_U \quad \forall\, u \in U$,

v) *there exists a positive value C_0 such that*

$$\Re\langle u, \sigma(u + v) - \sigma(u)\rangle_U \le C_0\|v\|_U \quad \forall\, u, v \in U,$$

we call σ a saturation function*. Here $U \subset S'$ in the sense of Gelfand triples.*

Example 1. Let $\mathrm{sat}_{\mathbb{R}}$ be the function from (1). It is easy to see that the function

$$\mathfrak{sat} : L^2(0,1) \to L^2(0,1), \quad u \mapsto \mathrm{sat}_{\mathbb{R}}(u(\cdot))$$

is an admissible feedback function. Moreover, for $S = L^\infty(0,1)$ we have

$$\begin{aligned}\|\mathfrak{sat}(u) - u\|_{L^1(0,1)} &= \int_0^1 \mathfrak{sat}(u)(\xi) - u(\xi)\,\mathrm{d}\xi \\ &\le \int_{\{u\ge 1\}} u(\xi)\,\mathrm{d}\xi + \int_{\{-1\le u\le 1\}} u^2(\xi)\,\mathrm{d}\xi + \int_{\{u\le 1\}} -u(\xi)\,\mathrm{d}\xi \\ &= \langle \mathfrak{sat}(u), u\rangle_U \qquad \forall u \in U.\end{aligned}$$

As Property (v) from Definition 1 follows similarly, $\mathfrak{sat}$ is a saturation function.

Let σ be an admissible feedback function. In the rest of the paper we will be interested in the following two types of systems: The *unsaturated system*,

$$\begin{cases} \dot{x}(t) = Ax(t) - BB^*x(t), \\ x(0) = x_0, \end{cases} \tag{Σ_L}$$

and the *disturbed saturated system*

$$\begin{cases} \dot{x}(t) = Ax(t) - B\sigma\big(B^*x(t) + d(t)\big), \\ x(0) = x_0. \end{cases} \tag{Σ_{SLD}}$$

with $d \in L^\infty(0,\infty;U)$. We abbreviate

$$\widetilde{A} : D(\widetilde{A}) \subset X \to X, \quad \widetilde{A}x := Ax - BB^*x.$$

By Lumer–Phillips theorem, $\widetilde{A}$ generates a strongly continuous semigroup of contractions $(\widetilde{T}(t))_{t\ge 0}$ as $-BB^* \in L(X)$ is dissipative. Clearly, (Σ_L) is a special case of (Σ_{SLD}) with $d = 0$, as $\sigma(u) = u$ is an admissible feedback function.

Definition 2. *Let $x_0 \in X$ and $d \in L^\infty(0,\infty;U)$. A continuous function $x : [0,\infty) \to X$ satisfying*

$$x(t) = T(t)x_0 - \int_0^t T(t-s)B\sigma\big(B^*x(s) + d(s)\big)\,\mathrm{d}s, \quad t \geq 0.$$

is called a mild solution *of (Σ_{SLD}). If x is differentiable almost everywhere such that $x' \in L^1(0,\infty;X)$, $x(0) = x_0$ and (Σ_{SLD}) holds for almost every $t \geq 0$, we say that x is a* strong solution.

By our assumptions, (Σ_{SLD}) has a unique mild solution for any $x_0 \in X$ and $u \in L^\infty(0,\infty;U)$, [2, Prop. 4.3.3]. In order to introduce the external stability notions, the following well-known comparison functions are needed,

$$\begin{aligned}
\mathcal{K} &:= \{\alpha \in C(\mathbb{R}_+,\mathbb{R}_+) \mid \alpha \text{ is strictly increasing}, \alpha(0) = 0\},\\
\mathcal{K}_\infty &:= \{\alpha \in \mathcal{K} \mid \alpha \text{ is unbounded}\},\\
\mathcal{L} &:= \{\alpha \in C(\mathbb{R}_+,\mathbb{R}_+) \mid \alpha \text{ is strictly decreasing with } \lim_{t\to\infty} \alpha(t) = 0\},\\
\mathcal{KL} &:= \{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R}_+) \mid \beta(\cdot,t) \in \mathcal{K} \;\; \forall t > 0,\; \beta(r,\cdot) \in \mathcal{L} \;\; \forall r > 0\},
\end{aligned}$$

where $C(\mathbb{R}_+,\mathbb{R}_+)$ refers to the continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Definition 3. i) (Σ_{SLD}) *is called* globally asymptotically stable *if for every mild solution x for $d = 0$ we have* $\lim_{t\to\infty} \|x(t)\|_X = 0$,

ii) (Σ_{SLD}) *is called* semi-globally exponentially stable *if for $d = 0$ and any $r > 0$ there exist two $\mu(r) > 0$ and $K(r) > 0$ such that for any mild solution x with initial value $x_0 \in D(A)$ satisfying $\|x_0\|_{D(A)} := \|x_0\|_X + \|Ax_0\|_X \leq r$,*

$$\|x(t)\|_X \leq K(r)\mathrm{e}^{-\mu(r)t}\|x_0\|_X \qquad \forall t \geq 0,$$

iii) (Σ_{SLD}) *is called* locally input-to-state stable (LISS) *if there exist $r > 0$, $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}_\infty$ such that for every mild solution x with initial value satisfying $\|x_0\|_X \leq r$ and all $t \geq 0$, we have*

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t) + \rho(\|d\|_{L^\infty(0,t;U)}). \tag{2}$$

(Σ_{SLD}) *is called* input-to-state stable (ISS) *if $r = \infty$.*
If (2) holds for (Σ_{SLD}) with $d \equiv 0$ and $r = \infty$, the system is called uniformly globally asymptotically stable (UGAS).

Compared to the other notions, semi-global exponential stability seems to be less common in the literature, but appeared already in the context of saturated systems in [14]. Note that for the linear System (Σ_L) UGAS is equivalent to the existence of constants $M,\omega > 0$ such that $\|\widetilde{T}(t)\|_X \leq M\mathrm{e}^{-\omega t}$ for all $t \geq 0$. Clearly, if (Σ_{SLD}) is UGAS, then it is globally asymptotically stable. Moreover, (Σ_L) is UGAS if and only if it is semi-globally exponentially stable, and semi-global exponential stability implies global asymptotical stability.

Next we investigate the question whether (semi)-global exponential stability or UGAS of system (Σ_L) implies (semi)-global exponential stability or UGAS of System (Σ_{SLD}).

Under the assumption that σ is an admissible feedback function with the additional properties that for all $u \in U$, $\Re\langle u, \sigma(u)\rangle = 0$ implies $u = 0$ and $D(A)$ equipped with the norm $\|\cdot\|_{D(A)} = \|\cdot\|_X + \|A\cdot\|_X$ is a Banach space compactly embedded in X, in [10, Theorem 2] it is shown that global asymptotic stability of (Σ_L) implies global asymptotic stability of (Σ_{SLD}). Note, that by [24, Lemma 2.1, pp. 165] and an argument in the proof of [1, Theorem 1.2, pp. 102] the other assumptions of [10, Theorem 2] are automatically satisfied in our case. In [18, Section V] it is shown, that under these conditions and in finite dimensions, i.e. $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$, (Σ_{SLD}) is UGAS.

Here we are interested in results for general admissible feedback functions and saturation functions. The following result was proved in [14] and [15].

Proposition 1 ([15, Theorem 1], [14, Theorem 2])**.** *Let (Σ_L) be UGAS and $\sigma : U \to U$ be a saturation function.*

i) *If $S = U$, then (Σ_{SLD}) is ISS and there exists an ISS Lyapunov function.*

ii) *If $S \neq U$ and*

$$\exists c > 0\, \forall x \in D(A): \quad \|B^* x\|_S \le c\|x\|_{D(A)}, \tag{3}$$

then (Σ_{SLD}) is semi-globally exponentially stable.

We will show next that Proposition 1 does not hold without assuming (3) and moreover, that (3) does neither imply UGAS nor ISS for (Σ_{SLD}).

Proposition 2. *Let $X = U = L^2(0,1)$, $A = 0$, $B = I$ and $\sigma = \mathfrak{sat}$. Then System (Σ_L) is UGAS and System (Σ_{SLD}) is neither semi-globally exponentially stable, nor UGAS nor ISS.*

Proof. As System (Σ_L) is given by $\dot{x}(t) = -x(t)$, it is UGAS. System (Σ_{SLD}) is given by

$$\begin{cases} \dot{x}(t,\xi) = -\mathrm{sat}_{\mathbb{R}}\big(x(t,\xi)\big), & t \ge 0, \xi \in (0,1), \\ x(0,\xi) = f(\xi), \end{cases} \tag{4}$$

with the unique mild solution $x \in C([0,\infty); L^2(0,1))$

$$x(t,\xi) = \begin{cases} f(\xi) - t, & \text{if } f(\xi) \ge 1 + t, \\ \mathrm{e}^{-t} f(\xi), & \text{if } f(\xi) \in (-1, 1), \\ f(\xi) + t, & \text{if } f(\xi) \le -1 - t, \\ \mathrm{e}^{f(\xi)-1-t}, & \text{if } f(\xi) \in [1, 1+t), \\ -\mathrm{e}^{1-t-f(\xi)}, & \text{if } f(\xi) \in (-1-t, -1]. \end{cases} \tag{5}$$

We will show that there exists a sequence $(f_n)_n \in L^2(0,1)$ with $\|f_n\|_{D(A)} = \|f_n\|_{L^2(0,1)} = 1$ such that for all $t > 0$ there exists an $n \in \mathbb{N}$ such that $\|x_n(t)\|_{L^2(0,1)} > \frac{1}{2}$ where x_n denotes the corresponding solution of (4) with initial function f_n. For this purpose we will only consider the restriction of x_n to $\{\xi \in [0,1] \mid f(\xi) \geq 1+t\}$ and define

$$f_n(\xi) := \frac{1}{\sqrt{n}}\xi^{-\alpha_n}$$

with $\alpha_n := \frac{1}{2}\left(1 - \frac{1}{n}\right)$. Clearly, $f_n \in L^2(0,1)$, $\|f_n\|_{L^2} = 1$ and f_n is decreasing. Because of $f_n(\xi_{t,n}) = 1+t$ if and only if

$$\xi_{t,n} = \frac{1}{(\sqrt{n}(1+t))^{\frac{1}{\alpha_n}}}$$

we have $\{\xi \in [0,1] \mid f_n(\xi) \geq 1+t\} = \{\xi \in [0,1] \mid \xi \leq \xi_{t,n}\}$. Hence,

$$\begin{aligned}
\|x_n(t)\|^2_{L^2(0,1)} &\geq \int_0^{\xi_{t,n}} x_n(t,\xi)^2 \mathrm{d}\xi \\
&= \int_0^{\xi_{t,n}} (f_n(\xi) - t)^2 \,\mathrm{d}\xi \\
&= \int_0^{\xi_{t,n}} \left(\frac{1}{\sqrt{n}}\xi^{-\alpha_n} - t\right)^2 \mathrm{d}\xi \\
&= \frac{1}{n}\int_0^{\xi_{t,n}} \xi^{-2\alpha_n}\mathrm{d}\xi - \frac{2t}{\sqrt{n}}\int_0^{\xi_{t,n}} \xi^{-\alpha_n}\mathrm{d}\xi + \int_0^{\xi_{t,n}} t^2 \mathrm{d}\xi \\
&= \frac{1}{n}\frac{1}{1-2\alpha_n}\xi_{t,n}^{1-2\alpha_n} - \frac{2t}{\sqrt{n}}\frac{1}{1-\alpha_n}\xi_{t,n}^{1-\alpha_n} + t^2\xi_{t,n} \\
&= n^{\frac{1}{1-n}}(1+t)^{\frac{2}{1-n}} - \frac{1}{n+1}2n^{\frac{1}{1-n}}2t(1+t)^{\frac{1+n}{1-n}} + n^{\frac{n}{1-n}}t^2(1+t)^{\frac{2n}{1-n}}.
\end{aligned}$$

Taking the limit $n \to \infty$ we conclude

$$\lim_{n\to\infty} \|x_n(t)\|^2_{L^2(0,1)} \geq 1.$$

Thus System (4) is neither semi-globally exponentially stable nor UGAS. □

The following theorem shows that UGAS of System (Σ_L) together with Condition (3) is not sufficient to guarantee UGAS of System (Σ_{SLD}).

Theorem 1. *Let $X = U = L^2(0,1)$, $A = \frac{d}{d\xi}$ with $D(A) = \{y \in H^1(0,1) | y(0) = y(1)\}$, $B = I$, $S = L^\infty(0,1)$ and $\sigma = \mathfrak{sat}$. Then System (Σ_L) is UGAS, Condition (3) is satisfied. Furthermore, System (Σ_{SLD}) is semi-globally exponentially stable, but neither UGAS nor ISS.*

We note, that System (Σ_{SLD}) of Theorem 1 equals $(\Sigma_{\mathfrak{sat}})$. Further, in [15, Theorem 1] it has been wrongly stated that the saturated system is UGAS.

Proof. It is easy to see that system (Σ_L) is UGAS. System (Σ_{SLD}) is given by $(\Sigma_{\mathfrak{sat}})$ in the introduction. Condition (3) is fulfilled, because $H^1(0,1)$ is continuously embedded in $L^\infty(0,1)$. Hence, (Σ_{SLD}) is semi-globally exponentially stable by Proposition 1. Note that A generates the periodic shift semigroup on $L^2(0,1)$. By extending the initial function f periodically to $\mathbb{R}_+$, the unique mild solution $y \in C([0,\infty); L^2(0,1))$ of $(\Sigma_{\mathfrak{sat}})$ is given by

$$y(t,\xi) = x(t, \xi + t),$$

where x is defined in (5). By the particular form of (5), this implies that

$$\|x(t)\|_{L^2(0,1)} = \|y(t)\|_{L^2(0,1)}$$

holds for all $t \geq 0$. We can therefore choose the same sequence $(f_n)_n \in L^2(0,1)$ with $\|f_n\|_{L^2(0,1)} = 1$ as in the proof of Proposition 2 in order to conclude

$$\lim_{n\to\infty} \|y_n(t)\|^2_{L^2(0,1)} \geq 1.$$

This shows that System $(\Sigma_{\mathfrak{sat}})$ is not UGAS and thus not ISS.

□

An important tool for the verification of ISS of System (Σ_{SLD}) are ISS Lyapunov functions.

Definition 4. *Let $U_r = \{x \in X \colon \|x\| \leq r\}$ and $r \in (0,\infty]$. A continuous function $V : U_r \to \mathbb{R}_{\geq 0}$ is called an* LISS Lyapunov function *for (Σ_{SLD}), if there exists $\psi_1, \psi_2, \alpha, \rho \in \mathcal{K}_\infty$, such that for all $x_0 \in U_r$, $\|d\|_{L^\infty(0,\infty;U)} \leq r$,*

$$\psi_1(\|x_0\|_X) \leq V(x_0) \leq \psi_2(\|x_0\|_X)$$

and

$$\dot{V}_d(x_0) := \limsup_{t\searrow 0} \frac{1}{t}\big(V(x(t)) - V(x_0)\big) \leq -\alpha(\|x_0\|_X) + \rho(\|d\|_{L^\infty(0,\infty;U)}). \tag{6}$$

If $r = \infty$, then V is called an ISS Lyapunov function*. For System (Σ_{SLD}) with $d \equiv 0$ we will call an ISS Lyapunov function a* UGAS Lyapunov function.

In [16, Theorem 4] it is shown that system (Σ_{SLD}) with an admissible feedback function is LISS if and only if there exists an LISS Lyapunov function for (Σ_{SLD}) which is Lipschitz continuous. Moreover, system (Σ_{SLD}) with an admissible feedback function is ISS if and only if there exists an ISS Lyapunov function for (Σ_{SLD}) which is locally Lipschitz continuous [19, Theorem 5].

In this paper, we are mainly interested in the contruction of ISS Lyapunov functions. In the setting of Theorem 1 the operator A generates a semigroup which is not exponentially stable. The following result shows that this is not accidental if the saturated system is not UGAS.

Proposition 3. *Suppose that there exists $\alpha > 0$ such that $\|T(t)\| \leq \mathrm{e}^{-\alpha t}$ for all $t > 0$ and let σ be an admissible feedback function. Then the function*

$$V(x) = \|x\|_X^2, \quad x \in X,$$

is an UGAS Lyapunov function for (Σ_{SLD}) and thus System (Σ_{SLD}) is UGAS.

Proof. Assume first that $x_0 \in D(A)$. Then by [20, Theorem 1.6, p. 189] there exists a strong solution x for (Σ_{SLD}). We obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_X^2 &= 2\Re(\langle Ax(t), x(t)\rangle_X - \langle B\sigma(B^*x(t)), x(t)\rangle_X) \\ &\leq -2\alpha\|x(t)\|^2\end{aligned}$$

for a.e. $t \geq 0$ by using Definition 1.iii). For $x_0 \in X$ there exists a sequence $(x_{0,n})$ in $D(A)$ with $x_{0,n} \to x_0$. Denoting by x the mild solution of (Σ_{SLD}) with initial value x_0 and by x_n the strong solution of (Σ_{SLD}) with initial value $x_{0,n}$ we have

$$\begin{aligned}\|x(t) - x_n(t)\|_X = \Big\|&T(t)x_0 - T(t)x_{0,n} \\ &+ \int_0^t T(t-s)B\big(\sigma(B^*x(s)) - \sigma(B^*x_n(s))\big)\mathrm{d}s\Big\|_X \\ \leq \|&x_0 - x_{0,n}\|_X + \int_0^t \|B\|_{\mathcal{L}(X,U)}^2 k\|x(s) - x_n(s)\|_X \mathrm{d}s.\end{aligned}$$

Application of Gronwall's Lemma yields

$$\|x(t) - x_n(t)\|_X \leq (\|x_0 - x_{0,n}\|_X)\, \mathrm{e}^{t\|B\|_{\mathcal{L}(X,U)}^2 k} \to 0$$

for $n \to \infty$. Thus

$$|V(x(t)) - V(x_n(t))| \to 0$$

for $n \to \infty$ uniformly on bounded intervalls. We can therefore conclude

$$|\dot{V}_0(x_0) - \dot{V}_0(x_{0,n})| \to 0$$

for $n \to \infty$. UGAS of (Σ_{SLD}) now follows from [19, Theorem 5] with $d \equiv 0$. $\square$

Note that the assumption on the semigroup made in Theorem 3 is strictly stronger than the condition that $(T(t))_{t\geq 0}$ is an exponentially stable contraction semigroup as can be seen e.g. for a nilpotent shift-semigroup on $X = L^2(0,1)$. However, note that by the Lumer–Phillips theorem the following assertions are equivalent for a semigroup $(T(t))_{t\geq 0}$ generated by A.

i) $\Re\langle Ax, x\rangle \leq -\omega\|x\|^2$ for some $\omega > 0$ and all $x \in D(A)$.

ii) $\sup_{t>0} \|e^{\omega t}T(t)\| \leq 1$ for some $\omega > 0$.

Next we study the question whether UGAS of (Σ_L), implies that System (Σ_{SLD}) has an ISS Lyapunov function. In [19] the following ISS Lyapunov function was shown to be an ISS Lyapunov function for System (Σ_L). By adapting the proof we obtain the following.

Theorem 2. *Let (Σ_L) be UGAS with constants $M, \omega > 0$ such that $\|\widetilde{T}(t)\| \leq M\mathrm{e}^{-\omega t}$ for all $t \geq 0$ and let σ be admissible with Lipschitz constant k. If $M^2\|B\|^2(k+1) < \omega$ then (Σ_{SLD}) is ISS with Lipschitz continuous ISS Lyapunov function $V(x) = \max_{s\geq 0} \|\mathrm{e}^{\delta s}\widetilde{T}(s)x\|$ for all $\delta \in (M^2\|B\|^2(k+1), \omega)$.*

Proof. We can rewrite (Σ_{SLD}) in the form

$$\begin{cases} \dot{x}(t) = \widetilde{A}x(t) + B\big(B^*x(t) - \sigma(B^*x(t) + d(t))\big), \\ x(0) = x_0. \end{cases} \tag{7}$$

Hence, the mild solution satisfies

$$x(h) = \widetilde{T}(h)x_0 + \int_0^h \widetilde{T}(h-s)B\big(B^*x(s) - \sigma(B^*x(s) + d(s)\big)\mathrm{d}s.$$

Denoting the integral by I_h, we have, using that σ is admissible,

$$\begin{aligned} \|I_h\|_X &\leq \int_0^h M\mathrm{e}^{-\omega(h-s)}\|B\|^2(k+1)\|x(s)\|_X + M\mathrm{e}^{-\omega(h-s)}\|B\|k\|d(s)\|_U\mathrm{d}s \\ &\leq \int_0^h M\|B\|^2(k+1)\|x(s)\|_X\mathrm{d}s + M\frac{1}{\omega}\left(1 - \mathrm{e}^{-\omega h}\right)\|B\|k\|d\|_{L^\infty(0,h;U)}. \end{aligned}$$

With $\|x\| \leq V(x) \leq M\|x\|$ and $V\big(\widetilde{T}(t)x\big) \leq \mathrm{e}^{-\delta t}V(x)$ for all $x \in X$ we obtain

$$\begin{aligned} \dot{V}_d(x_0) &= \limsup_{h\searrow 0} \frac{1}{h}\big(V(\widetilde{T}(h)x_0 + I_h) - V(x_0)\big) \\ &\leq \limsup_{h\searrow 0} \frac{1}{h}\left(\mathrm{e}^{-\delta h} - 1\right)V(x_0) + M\limsup_{h\searrow 0}\frac{1}{h}\|I_h\|_X \\ &\leq -\delta\|x_0\|_X + M^2\|B\|^2(k+1)\|x_0\|_X + M^2\|B\|k\|d\|_{L^\infty(0,\varepsilon;U)} \end{aligned}$$

for every $\varepsilon > 0$ due to the continuity of the mild solution.
The Lipschitz continuity of V follows from

$$\begin{aligned} |V(x) - V(y)| &\leq |\max_{s\geq 0}\|\mathrm{e}^{\delta s}\widetilde{T}(s)x\| - \max_{s\geq 0}\|\mathrm{e}^{\delta s}\widetilde{T}(s)y\|| \\ &\leq \max_{s\geq 0}\|\mathrm{e}^{\delta s}\widetilde{T}(s)(x-y)\| \\ &\leq M\|x-y\|. \end{aligned}$$

Applying [19, Thm. 5] yields input-to-state stability of (Σ_{SLD}). ∎

Example 2. Let $X = U = \mathbb{R}$ and $A = 0$. Then for every $B \in \mathbb{R}\backslash\{0\}$ the operator $A - BB^* = -B^2$ generates the semigroup $(\mathrm{e}^{-B^2 t})_{t\geq 0}$. By choosing $\sigma = 0$ and $x_0 \neq 0$, the constant function $x(t) = x_0$ solves System (Σ_{SLD}). In this case $\omega = B^2 = M^2\|B\|^2(k+1)$ holds, as $k = 0$ and $M = 1$, and the system is not ISS. Thus, the lower bound for ω required in Theorem 2 is optimal.

Locally linear admissible feedback functions yield LISS Lyapunov functions.

Theorem 3. *Let* (Σ_L) *be UGAS with constants* $M, \omega > 0$ *such that* $\|\widetilde{T}(t)\| \leq M\mathrm{e}^{-\omega t}$ *for all* $t \geq 0$ *and* σ *an admissible feedback function with* $\sigma(u) = u$ *for all* $\|u\|_U \leq \delta$ *and some* $\delta > 0$. *Then* (Σ_{SLD}) *is LISS with Lipschitz continuous LISS Lyapunov function* $V(x) := \max_{s\geq 0} \|\mathrm{e}^{\frac{\omega}{2}s}\widetilde{T}(s)x\|_X$.

Proof. Let $\|x_0\|_X \leq \|B\|^{-1}\delta$. Just like in the proof of Theorem 2 we rewrite (Σ_{SLD}) in the form (7) and use the abbreviation I_h. We have

$$\begin{aligned}
\limsup_{h\searrow 0} \frac{1}{h}\|I_h\|_X \leq \limsup_{h\searrow 0} \frac{1}{h}\Bigg(&\int_0^h M\|B\|\|B^*x(s) - \sigma(B^*x(s))\|_U \mathrm{d}s \\
&+ \int_0^h M\|B\|\|\sigma(B^*x(s)) - \sigma(B^*x(s) + d(s))\|_U \mathrm{d}s\Bigg) \\
\leq\ & M\|B\|\|B^*x_0 - \sigma(B^*x_0)\|_U + M\|B\|k\|d\|_{L^\infty(0,\varepsilon;U)} \\
=\ & M\|B\|k\|d\|_{L^\infty(0,\varepsilon;U)},
\end{aligned}$$

where the continuity of x, the Lipschitz continuity of σ as well as the condition $\sigma(u) = u$ if $\|u\| \leq \delta$ have been used.
With $\|x\| \leq V(x) \leq M\|x\|$ and $V\big(\widetilde{T}(t)x\big) \leq \mathrm{e}^{-\frac{\omega}{2}t}V(x)$ for all $x \in X$ we obtain

$$\begin{aligned}
\dot{V}_d(x_0) &= \limsup_{h\searrow 0} \frac{1}{h}\big(V(\widetilde{T}(h)x_0 + I_h) - V(x_0)\big) \\
&\leq \limsup_{h\searrow 0} \frac{1}{h}\left(\mathrm{e}^{-\frac{\omega}{2}h} - 1\right)V(x_0) + M\limsup_{h\searrow 0}\frac{1}{h}\|I_h\|_X \\
&\leq -\frac{\omega}{2}\|x_0\|_X + M^2\|B\|k\|d\|_{L^\infty(0,\varepsilon;U)}
\end{aligned}$$

for every $\varepsilon > 0$. The Lipschitz continuity of V can be shown by using the same argumentation as in the proof of Theorem 2. Application of [16, Theorem 4] yields local input-to-state stability of (Σ_{SLD}). □

Note that property iii) of Definition 1 has not been used in the proofs of Theorems 2 and 3.

3 Conclusion

A general assumption of this article is the contractivity of the underlying C_0-semigroup generated by A. It is an open question whether the results of the paper also hold in the case of bounded semigroups. The following example shows that for general strongly continuous semigroup it may happen that the nonlinear system (Σ_{SLD}) is not uniformly globally asymptotically stable, but the underlying linear system (Σ_L) is UGAS.

Example 3. Let $X = U = \mathbb{R}$ and choose $A = 1$ and $B = 2$. Then $A - BB^* = -3$ generates the uniformly globally asymptotically stable semigroup $(\mathrm{e}^{-3t})_{t\geq 0}$. Consider the system

$$\begin{cases} \dot{x}(t) = x(t) - 2\mathrm{sat}_{\mathbb{R}}\big(2x(t)\big) \\ x(0) = 2 \end{cases}$$

with saturation function $\mathrm{sat}_{\mathbb{R}}$ as in (1). Then $x(t) = 2$ is a solution of this system, but $x(t) \nrightarrow 0$ for $t \to \infty$.

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