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OPERATOR SPLITTING FOR ABSTRACT CAUCHY PROBLEMS WITH DYNAMICAL BOUNDARY CONDITION

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ABSTRACT. In this work we study operator splitting methods for a certain class of coupled abstract Cauchy problems, where the coupling is such that one of the problems prescribes a "boundary type" extra condition for the other one. The theory of one-sided coupled operator matrices provides an excellent framework to study the well-posedness of such problems. We show that with this machinery even operator splitting methods can be treated conveniently and rather efficiently. We consider three specific examples: the Lie (sequential), the Strang, and the weighted splitting, and prove the convergence of these methods along with error bounds under fairly general assumptions.

1. INTRODUCTION

Operator splitting procedures provide an efficient way of solving differential equations which describe the combined effect of several processes. In this case the operator on the equation's right-hand side is the sum of certain sub-operators corresponding to the different processes. The main idea of operator splitting is that one solves the sub-problems corresponding to the sub-operators separately, and constructs the solution of the original problem from the sub-solutions.

Depending on how the sub-solutions define the solution itself, we distinguish several operator splitting procedures, such as sequential (proposed by Bagrinovskii and Godunov in [3]), Strang (proposed by Strang and Marchuk in [48] and [44]), or weighted ones (see e.g. in Csomós et al. [13]). An application of sequential splitting, for instance, results in the subsequent solution of the sub-problems using the previously obtained sub-solution as initial condition for the next sub-problem.

Although operator splitting procedures enable the numerical treatment of complicated differential equations, their application leads to an approximate solution which usually differs from the exact one. The accuracy can be increased by considering the sequence of the sub-problems on short time intervals in a cycle, which will in turn increase the computational effort. However, the analysis of the error, caused by the use of operator splitting, stands in the main focus of related research. For general overviews on splitting methods we refer the interested reader to the vast literature. For instance, Bjørhus analysed the consistency of Lie splitting in an abstract framework in [8], Sportisse also considered the stiff case in [47], Hansen and Ostermann also treated the abstract case in [24], while Bátkai et al. applied the splitting methods for non-autonomous evolution equations in [7].Error bounds in the abstract setting were proved by Jahnke and Lubich in [32] for the Strang

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splitting. While Hansen and Ostermann in [25] have treated higher order splitting methods. A survey can be found in Geiser [21].

Another challenging issue is what kinds of processes of the sub-operators describe. They can e.g. correspond to various physical, chemical, biological, financial, etc. phenomena. Hundsdorfer and Verwer analysed the splitting of advection–diffusion–reaction equations in [31, Chapter IV], Dimov et al. solved air pollution transport models in [14], Jacobsen et al. considered the Hamilton–Jacobi equations in [34], Holden et al. partial differential equations with Burgers nonlinearity in [30], while in [11] Csomós and Nickel and in [5, 6] Bátkai et al. applied splitting methods for delay equations.Splitting methods for Schrödinger equations are treated, e.g., in Hochbruck et al. [33], Caliari et al. [9].

The sub-operators can also refer to the change (derivative) with respect to various spatial coordinates or other variables such as Hansen and Ostermann has done in [24], [26]; or for the case of Maxwell equations, see, e.g., Jahnke et al [29] or Eilinghoff and Schnaubelt [15]. Furthermore, the sub-problems can also be originated from other properties of the problem itself, such as in the present case of dynamic boundary problems.

Let us emphasise that the analysis and the numerical treatment of dynamic boundary problems have attracted very recently a lot of interest among researchers, cf. the work of Hipp [27, 28] for wave-type equations or Knopf et al. [35, 36, 37] on the Cahn–Hilliard equation or Kovács et al. [38] and Kovács, Lubich [39] on parabolic equations. The literature is extensive, and we mention some very recent papers by Altmann [1], Epshteyn, Xia [19], Fukao et al. [20], Langa, Pierre [40], and refer to the references therein.

In the present work we focus on the *abstract setting* of coupled Cauchy problems, where one of the subproblems provides a extra condition, of boundary type, to the other. We consider equations of the form:

$$\begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \quad u(0) = u_0 \in E, \\ \dot{v}(t) = B v(t) & \text{for } t \geq 0, \quad v(0) = v_0 \in F, \\ Lu(t) = v(t) & \text{for } t \geq 0, \end{cases} \tag{1.1}$$

where E and F are Banach spaces over the complex field $\mathbb{C}$, A and B are unbounded linear operators on E and F, respectively. The coupling of the two problems involves the unbounded linear operator L acting between E and F. Moreover, this coupling is of "boundary type", i.e., as concrete examples we have in mind problems of the following form:

$$\dot{u}(t) = \Delta_\Omega u(t), \qquad u(0) = u_0 \in \mathrm{L}^2(\Omega), \tag{1.2}$$

$$\dot{v}(t) = \Delta_{\partial\Omega} v(t), \qquad v(0) = v_0 \in \mathrm{L}^2(\partial\Omega), \tag{1.3}$$

$$u(t)|_{\partial\Omega} = v(t),$$

where Ω is bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary and $A_m = \Delta_\Omega$, $B = \Delta_{\partial\Omega}$ are the (maximal) distributional Laplace and Laplace–Beltrami operators restricted to the respective L^2-space. In this example L denotes the trace operator (the precise ingredient will be discussed in Example 2.6 below.)

It is an obvious idea for the numerical treatment of (1.2)–(1.3) to apply operator splitting methods, i.e. to treat the first and second equations separately, see also in [39]. The purpose of this work is to investigate such possibilities, and as a splitting strategy we propose the following steps:

(1) Choose a time step $\tau > 0$.
(2) Solve the second equation (1.3) with the initial condition $v(0) = u_0|_{\partial\Omega} = v_0$, set $v_1 := v(\tau)$.
(3) Solve the first equation (1.2) with the inhomogeneous boundary condition $u(t)|_{\partial\Omega} = v_1$ on $[0, \tau]$ and the initial condition $u(0) = \widetilde{u}_0$. The methods determines how $\widetilde{u}_0$ is calculated from u_0 (and v_0), and in general $\widetilde{u}_0$ need not be equal to u_0. Set $u_1 = u(\tau)$.
(4) The new initial condition for equation (1.3) is then $u_1|_{\partial\Omega} = v_1$.
(5) Iterate this procedure for $n \in \mathbb{N}$ time steps.

The aim of this paper is to formulate this splitting method as *an operator splitting* in an abstract *operator semigroup theoretic* framework and investigate its convergence properties. The method then becomes applicable for a wider class of equations as in (1.1). Our choice for the auxiliary, modified initial value $\widetilde{u}_0$, in Step 3 above, is motivated by this approach. Indeed, the abstract theory will immediately yield the convergence of the method as an instance of the Lie–Trotter formula. However, we shall briefly touch upon other possible choices for $\widetilde{u}_0$ as well.

As a matter of fact our proposed methods, at a first sight, will be slightly different in that we decompose the system in not two but three sub-problems. This idea is nicely illustrated in the above example of diffusion: We separate the dynamics in the domain and assume homogeneous boundary conditions, the dynamics on the boundary, and as the third component the interaction between the two dynamics, i.e., how the boundary dynamics is fed into the domain. In fact, this decomposition is responsible for the modified form $\widetilde{u}_0$ of the initial condition. This approach will also have the advantage that the internal and boundary dynamics are completely separated. Hence well-established methods can be used for solving each of the subproblems. We also note that the splitting approach here enables a way to parallelization of the solution to the subproblems.

This work is organized as follows. In Section 2 we recall the necessary operator theoretic background for this program and in Section 3 we introduce the different splitting approaches for the dynamic boundary conditions: the Lie splitting, the Strang splitting and the weighted splitting. We also prove the convergence of these methods under fairly general assumptions. Finally, Section 4 contains error bounds for the above mentioned splitting methods.

2. Abstract dynamic boundary conditions

Before discussing splitting methods in more detail let us briefly recall a possible approach for treating such abstract dynamic boundary value problems. The abstract treatment of boundary perturbations, i.e., techniques for altering the domain of the generator of a C_0-semigroup goes back to the work of Greiner [22]. Many results have been building on his theory, and our main sources for describing the abstract setting will be the works by Casarino, Engel, Nagel, and Nickel [10] and Engel [16, 17]. In [10] the following set of conditions were posed for treating the well-posedness of the problem (1.1).

Hypothesis 2.1. The $\mathbb{C}$-vector spaces E and F are Banach spaces.

(i) The operators $A_m : \mathrm{dom}(A_m) \subseteq E \to E$ and $B : \mathrm{dom}(B) \subseteq F \to F$ are linear.
(ii) The linear operator $L : \mathrm{dom}(A_m) \to F$ is surjective and bounded with respect to the graph norm of A_m on $\mathrm{dom}(A_m)$.

(iii) The restriction A_0 of A_m to $\ker(L)$ generates a strongly continuous semigroup $\big(T_0(t)\big)_{t\geq 0}$ on E.

(iv) The operator B generates a strongly continuous semigroup $\big(S(t)\big)_{t\geq 0}$ on F.

(v) The operator (matrix) $\binom{A_m}{L} : \mathrm{dom}(A_m) \to E \times F$ is closed.

In this paper we make the following technical assumption to simplify the things a bit.

Hypothesis 2.2. The operators A_0 and B are invertible.

However, let us note that for the splitting procedures this makes *no theoretical difference*, since one always find sufficiently large $\lambda > 0$ such that $A_0 - \lambda$ and $B - \lambda$ become invertible. Then the numerical schemes can be applied in this rescaled situation.

Next, we recall the following definition from [10, Lemma 2.2], and note that under the previous assumption $0 \in \rho(A_0)$ (the resolvent set of A):

$$D_0 := L|_{\ker(A_m)}^{-1} : F \to \ker(A_m) \subseteq E. \tag{2.1}$$

The operator D_0 is called the *abstract Dirichlet operator*; the operator $L|_{\ker(A_m)}$ is indeed invertible, see the mentioned lemma in [10].

Remark 2.3. (a) The operator $D_0B : \mathrm{dom}(B) \to E$ is closed if $\mathrm{dom}(B)$ is supplied with the norm $\|\cdot\|_F$, and bounded if $\mathrm{dom}(B)$ is supplied with the graph-norm $\|\cdot\|_B$.

(b) We have $\mathrm{rg}(D_0) \cap \mathrm{dom}(A_0) = \{0\}$.

Following [10] we introduce the product space $E \times F$ and the operator $\mathcal{A}$ acting on it as

$$\mathcal{A} := \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix} \text{ with } \mathrm{dom}(\mathcal{A}) := \left\{ \tbinom{x}{y} \in \mathrm{dom}(A_m) \times \mathrm{dom}(B) : Lx = y \right\}. \tag{2.2}$$

Section 1.1 in [46] relates the well-posedness of (1.1) to the generation property of $\mathcal{A}$, see also [45].

The first thing to be settled is therefore, whether the abstract Cauchy problem

$$\dot{\boldsymbol{u}}(t) = \mathcal{A}u(t), \quad \text{for } t \geq 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_0, v_0)^\top,$$

is well-posed in the sense of C_0-semigroups, see [18, Section II.6]. In this case the solution satisfies $\boldsymbol{u}(t) = \mathcal{T}(t)\boldsymbol{u}_0$, where $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup generated by $\mathcal{A}$. The problem of well-posedness is solved in [10]. We briefly recall here the following results from [10, Theorem 2.7] and from its proof.

Theorem 2.4. *Let the operators $\mathcal{A}$, D_0 be as defined in* (2.2) *and* (2.1) *and assume Hypotheses* 2.1 *and* 2.2. *For $y \in \mathrm{dom}(B)$ define*

$$Q(t)y = D_0S(t)y - T_0(t)D_0y - \int_0^t T_0(t-s)D_0S(s)By \, \mathrm{d}s. \tag{2.3}$$

The operator $\mathcal{A}$ is a generator of a C_0-semigroup if and only if for each $t \geq 0$ the operator (extends to)

$$Q(t) \in \mathscr{L}(F, E) \quad \textit{and} \quad \limsup_{t \downarrow 0} \|Q(t)\| < \infty.$$

In this case the semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ generated by $\mathcal{A}$ is given by

$$\mathcal{T}(t) = \begin{pmatrix} T_0(t) & Q(t) \\ 0 & S(t) \end{pmatrix}. \tag{2.4}$$

Hypothesis 2.5. The operator A_0 generates a bounded analytic semigroup [43, 23].

If Hypothesis 2.1, 2.2 and 2.5 are fulfilled and also B is a generator of an analytic semigroup, then Theorem 2.4 applies and assures that the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is analytic too, see [10, Cor. 2.8].

The motivating example from the introduction is discussed in [10, Section 3] in detail. We recall here the ingredients, to illustrate that our proposed methods will be applicable also for this equation.

Example 2.6 (Laplace and Laplace–Beltrami operators)**.** Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^2.

- $E := \mathrm{L}^2(\Omega)$, $F := \mathrm{L}^2(\partial\Omega)$ are the L^2-spaces with respect to the Lebesgue and the surface measure, respectively.
- Δ_Ω and $B = \Delta_{\partial\Omega}$ are the (maximal) distributional Laplace and Laplace–Beltrami operators, respectively.
- $A_m = \Delta_\Omega$ with domain

$$\mathrm{dom}(A_m) = \{f : f \in \mathrm{H}^{1/2}(\Omega) \cap \mathrm{H}^2_{\mathrm{loc}}(\Omega) \text{ with } \Delta_\Omega f \in \mathrm{L}^2(\Omega)\}.$$

- $Lf = f|_{\partial\Omega}$ the trace of $f \in \mathrm{dom}(A_m)$ on $\partial\Omega$.
- $B = \Delta_{\partial\Omega}$ with domain

$$\mathrm{dom}(B) = \{g : g \in \mathrm{L}^2(\partial\Omega) \text{ with } \Delta_{\partial\Omega} g \in \mathrm{L}^2(\Omega)\}.$$

Hypotheses 2.1, 2.2, 2.5 are satisfied for these choices. In particular, $\mathcal{A}$ generates an analytic semigroup on $E \times F$, see [10, Section 3]. We also have the following:

- The Dirichlet operator $D_0 : \mathrm{L}^2(\partial\Omega) \to \mathrm{H}^{1/2}(\Omega)$ assigns to a prescribed boundary value g a function f with $f|_{\partial\Omega}$ and $\Delta_\Omega f = 0$.
- $A_0 = \Delta_{\mathrm{D}}$ is the Laplace operator with (homogeneous) Dirichlet boundary condition, generating the Dirichlet heat semigroup $(T_0(t))_{t\geq 0}$ on $\mathrm{L}^2(\Omega)$.
- The semigroup $(S(t))_{t\geq 0}$ is the heat semigroup on $\mathrm{L}^2(\partial\Omega)$.

The decisive tool, based on the theory of coupled operator matrices [16, 17], is to bring the *formally diagonal* operator $\mathcal{A}$ with a *non-diagonal domain* into an upper triangular form with the state space transformations

$$\mathcal{R}_0 = \begin{pmatrix} I & -D_0 \\ 0 & I \end{pmatrix}, \qquad \mathcal{R}_0^{-1} = \begin{pmatrix} I & D_0 \\ 0 & I \end{pmatrix}.$$

Accordingly, we obtain the following representation:

$$\mathcal{A} = \mathcal{R}_0^{-1} \mathcal{A}_0 \mathcal{R}_0, \tag{2.5}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & -D_0 B \\ 0 & B \end{pmatrix} \quad \text{with} \quad \mathrm{dom}(\mathcal{A}_0) = \mathrm{dom}(A_0) \times \mathrm{dom}(B),$$

see [10, Lemma 2.6 and the proof of Corollary 2.8].

3. Operator splitting methods for dynamic boundary conditions problems

Since the form of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ can be rarely determined in practice, our aim is to approximate it. To this end, we choose a time step $\tau > 0$, and denote at time $t = k\tau$ the approximation of $\boldsymbol{u}(k\tau)$ by $\boldsymbol{u}_k(\tau)$ for all $k \in \mathbb{N}$. It is a natural expectation that the approximate value should converge to the exact one when refining the temporal resolution. We recall the following definition from [41] due to Lax and Richtmyer.

Let us begin by stating what we mean by convergence:

Definition 3.1 (Convergence)**.** The approximation $\boldsymbol{u}_k$ is called convergent to the solution $\boldsymbol{u}$ of problem (1.1) if $\boldsymbol{u}(t) = \lim\limits_{n\to\infty} \boldsymbol{u}_n(\frac{t}{n})$ holds uniformly for all $t \in [0, t_{\max}]$ for some $t_{\max} \geq 0$.

Starting from the representation (2.5), we construct approximations of the form

$$\boldsymbol{u}_k(\tau) := \mathcal{R}_0^{-1}\mathbb{T}(\tau)^k\mathcal{R}_0\binom{u_0}{v_0}, \tag{3.1}$$

where the operator $\mathbb{T}(\tau)\colon E \times \operatorname{dom}(B) \to E \times \operatorname{dom}(B)$, $\tau \geq 0$ describes the actual numerical method, and $\boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_0, v_0)^\top$. In order to specify the operator $\mathbb{T}(\tau)$, we remark that the operator $\mathcal{A}_0$ can be written as the sum

$$\mathcal{A}_0 =: \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,$$

where

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

with

$$\operatorname{dom}(\mathcal{A}_1) = \operatorname{dom}(A_0) \times F,\ \operatorname{dom}(\mathcal{A}_2) = E \times \operatorname{dom}(B),\ \operatorname{dom}(\mathcal{A}_3) = E \times \operatorname{dom}(B).$$

We remark that $\mathcal{A}_1$ and $\mathcal{A}_3$ commute in the sense of resolvents. From Hypothesis 2.1 and Remark 2.3 we immediately obtain the following proposition.

Proposition 3.2. *The operator semigroups $(\mathcal{T}_i(t))_{t\geq 0}$, $i = 1, 2, 3$ given by*

$$\mathcal{T}_1(t) = \begin{pmatrix} T_0(t) & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{T}_2(t) = \begin{pmatrix} I & -tD_0B \\ 0 & I \end{pmatrix}, \quad \mathcal{T}_3(t) = \begin{pmatrix} I & 0 \\ 0 & S(t) \end{pmatrix}.$$

are strongly continuous on $E \times \operatorname{dom}(B)$ with generator

$$\mathcal{A}_1|_{E\times\operatorname{dom}(B)},\ \mathcal{A}_2 \text{ and } \mathcal{A}_3|_{E\times\operatorname{dom}(B)}, \text{ respectively.}$$

Here we consider the parts of the respective operators in the space $E \times \operatorname{dom}(B)$. The semigroups $(\mathcal{T}_1(t))_{t\geq 0}$ and $(\mathcal{T}_3(t))_{t\geq 0}$ are even strongly continuous on $E \times F$. Their generator is $\mathcal{A}_1$ and $\mathcal{A}_3$, respectively.

In this work we focus on methods (3.1) with the following choices for the operator $\mathbb{T}(\tau)$:

$$\mathbb{T}^{[\mathrm{Lie}]}(\tau) := \mathcal{T}_1(\tau)\mathcal{T}_2(\tau)\mathcal{T}_3(\tau) \tag{3.2}$$

for the *Lie (or sequential) splitting*;

$$\mathbb{T}^{[\mathrm{Str}]}(\tau) := \mathcal{T}_1(\tfrac{\tau}{2})\mathcal{T}_3(\tfrac{\tau}{2})\mathcal{T}_2(\tau)\mathcal{T}_3(\tfrac{\tau}{2})\mathcal{T}_1(\tfrac{\tau}{2}) \tag{3.3}$$

for the *Strang (or symmetrical) splitting*;

$$\mathbb{T}^{[\mathrm{wgh}]}(\tau) := \Theta\mathcal{T}_1(\tau)\mathcal{T}_2(\tau)\mathcal{T}_3(\tau) + (1-\Theta)\mathcal{T}_3(\tau)\mathcal{T}_2(\tau)\mathcal{T}_1(\tau) \tag{3.4}$$

for the *weighted splitting*, where the parameter $\Theta \in [0,1]$ is fixed. We note that the case $\Theta = 1$ corresponds to the Lie splitting, while $\Theta = 0$ gives the Lie splitting in the reverse order. Computing the composition of the operators leads to the common form

$$\mathbb{T}(\tau) = \begin{pmatrix} T_0(\tau) & V(\tau) \\ 0 & S(\tau) \end{pmatrix} \tag{3.5}$$

with the operators

$$\text{Lie splitting:} \quad V^{[\mathrm{Lie}]}(\tau) = -\tau T_0(\tau) D_0 B S(\tau), \tag{3.6}$$

$$\text{Strang splitting:} \quad V^{[\mathrm{Str}]}(\tau) = -\tau T_0(\tfrac{\tau}{2}) D_0 B S(\tfrac{\tau}{2}), \tag{3.7}$$

$$\text{weighted splitting:} \quad V^{[\mathrm{wgh}]}(\tau) = -\tau(\Theta T_0(\tau) D_0 B S(\tau) + (1-\Theta) D_0 B) \tag{3.8}$$

for all $\tau > 0$. The approximation (3.1) requires the powers of the operator $\mathbb{T}(\tau)$.

Proposition 3.3. *For the operator family* $\mathbb{T}(\tau)\colon E \times \mathrm{dom}(B) \to E \times \mathrm{dom}(B)$, $\tau > 0$, *from* (3.5) *we have the identity*

$$\mathbb{T}(\tau)^k = \begin{pmatrix} T_0(k\tau) & V_k(\tau) \\ 0 & S(k\tau) \end{pmatrix}$$

with

$$V_k(\tau) = \sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big) V(\tau) S(j\tau). \tag{3.9}$$

Proof. We show the assertion by induction. For $k = 1$ we have formula (3.5) with $V_1(\tau) = V(\tau)$. If the assertion is valid for some $k \geq 1$, then

$$\mathbb{T}(\tau)^{k+1} = \begin{pmatrix} T_0(k\tau) & V_k(\tau) \\ 0 & S(k\tau) \end{pmatrix} \begin{pmatrix} T_0(\tau) & V(\tau) \\ 0 & S(\tau) \end{pmatrix} = \begin{pmatrix} T_0\big((k+1)\tau\big) & V_{k+1}(\tau) \\ 0 & S\big((k+1)\tau\big) \end{pmatrix}$$

holds with

$$\begin{aligned} V_{k+1}(\tau) &= T_0(k\tau)V(\tau) + V_k(\tau)S(\tau) \\ &= T_0(k\tau)V(\tau) + \sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big)V(\tau)S(j\tau)S(\tau) \\ &= T_0(k\tau)V(\tau) + \sum_{j=1}^{k} T_0\big((k-j)\tau\big)V(\tau)S\big((j-1)\tau\big)S(\tau) \\ &= \sum_{j=0}^{k} T_0\big((k-j)\tau\big)V(\tau)S(j\tau). \end{aligned}$$

This proves the assertion for all $k \in \mathbb{N}$ by induction. □

The convergence of the approximation relies on the following result.

Proposition 3.4. *Under Hypotheses* 2.1, 2.2 *and with the notation* (3.5)*, the approximation* (3.1) *is convergent for* $y \in \mathrm{dom}(B)$ *if the condition*

$$\lim_{n\to\infty} V_n(\tfrac{t}{n})y = -\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \tag{3.10}$$

holds uniformly for t *in compact intervals.*

Proof. From Proposition 3.3, the approximation has the form

$$\begin{aligned}\boldsymbol{u}_k(\tau) &= \begin{pmatrix} I & D_0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_0(k\tau) & V_k(\tau) \\ 0 & S(k\tau)\end{pmatrix}\begin{pmatrix} I & -D_0 \\ 0 & I\end{pmatrix}\boldsymbol{u}_0 \\ &= \begin{pmatrix} T_0(k\tau) & V_k(\tau) - T_0(k\tau)D_0 + D_0S(k\tau) \\ 0 & S(k\tau)\end{pmatrix}\boldsymbol{u}_0.\end{aligned} \tag{3.11}$$

By comparing with formula (2.4) and using the relation (2.3), condition (3.10) implies the assertion. □

The convergence of the Riemann sums implies our next result concerning the approximation of the convolution in (3.10).

Lemma 3.5. *Let* $t_{\max} \ge 0$*, let* $F\colon [0,t_{\max}] \to \mathscr{L}(F)$ *be strongly continuous, and let* $G\colon [0,t_{\max}] \to F$ *be continuous. For each* $n \in \mathbb{N}$ *and* $t \in [0,t_{\max}]$ *define the following expressions*

$$\begin{aligned} C_n^{[1]}(t) &:= \tfrac{t}{n}\sum_{j=0}^{n-1} F\big((k-j)\tfrac{t}{n}\big)\,G(j\tfrac{t}{n}), \\ C_n^{[2]}(t) &:= \tfrac{t}{n}\sum_{j=0}^{n-1} F\big((k-j-\tfrac12)\tfrac{t}{n}\big)\,G\big((j+\tfrac12)\tfrac{t}{n}\big). \end{aligned}$$

Then for $j = 1,2$ *we have that*

$$\lim_{n\to\infty} C_n^{[j]}(t) = \int_0^t F(t-s)\,G(s)\,\mathrm{d}s$$

holds uniformly for $t \in [0,t_{\max}]$.

We can now state the main result of this section concerning convergent approximations of the solution to problem (1.1).

Proposition 3.6. *The approximations defined in* (3.2), (3.3)*, and* (3.4) *are convergent for all* $\boldsymbol{u}_0 \in E \times \mathrm{dom}(B)$.

Proof. It suffices to prove that condition (3.10) holds for the operators $V(\tau)$ defined in (3.6) and (3.7). By Proposition 3.3, we have the following identity for the Lie splitting:

$$\begin{aligned} V_k^{[\mathrm{Lie}]}(\tau)y &= -\tau\sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big)T_0(\tau)D_0BS(\tau)S(j\tau)y \\ &= -\tau\sum_{j=0}^{k-1} T_0\big((k-j)\tau\big)D_0BS\big((j+1)\tau\big)y, \end{aligned}$$

for the Strang splitting:

$$\begin{aligned} V_k^{[\mathrm{Str}]}(\tau)y &= -\tau\sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big)T_0(\tfrac{\tau}{2})D_0BS(\tfrac{\tau}{2})S(j\tau)y \\ &= -\tau\sum_{j=0}^{k-1} T_0\big((k-j-\tfrac{1}{2})\tau\big)D_0BS\big((j+\tfrac{1}{2})\tau\big)y, \end{aligned} \tag{3.12}$$

and for the weighted splitting:

$$\begin{aligned} V_k^{[\mathrm{wgh}]}(\tau)y &= -\tau\sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big)\big(\Theta T_0(\tau)D_0BS(\tau)+(1-\Theta)D_0B\big)S(j\tau)y \\ &= -\Theta\tau\sum_{j=0}^{k-1} T_0\big((k-j)\tau\big)D_0BS\big((j+1)\tau\big)y \\ &\quad -(1-\Theta)\tau\sum_{j=0}^{k-1} T_0\big((k-j-1)\tau\big)D_0BS(j\tau)y \end{aligned}$$

for all $y \in \mathrm{dom}(B)$, $\tau > 0$, and $\Theta \in [0,1]$. Since the operator B is the generator of the semigroup $(S(t))_{t\geq 0}$, they commute on $\mathrm{dom}(B)$, we have

$$\begin{aligned} V_n^{[\mathrm{Lie}]}(\tfrac{t}{n})y &= -\tau\sum_{j=0}^{n-1} T_0((n-j)\tfrac{t}{n})D_0S((j+1)\tfrac{t}{n})By, \\ V_n^{[\mathrm{Str}]}(\tfrac{t}{n})y &= -\tau\sum_{j=0}^{n-1} T_0((n-j-\tfrac{1}{2})\tfrac{t}{n})D_0S((j+\tfrac{1}{2})\tfrac{t}{n})By, \\ V_n^{[\mathrm{wgh}]}(\tfrac{t}{n})y &= -\Theta\tau\sum_{j=0}^{n-1} T_0((n-j)\tfrac{t}{n})D_0S((j+1)\tfrac{t}{n})By \\ &\qquad -(1-\Theta)\tau\sum_{j=0}^{n-1} T_0((n-j-1)\tfrac{t}{n})D_0S(j\tfrac{t}{n})By. \end{aligned}$$

By choosing $F(t) = T_0(t)$ and $G(t) = D_0S(t)By$ in Lemma 3.5 with the operators $C_n^{[1]}(t), C_n^{[2]}(t)$ we have

$$\begin{aligned} V_n^{[\mathrm{Lie}]}(\tfrac{t}{n})y &= -C_n^{[1]}(t)S(\tfrac{t}{n})By, \\ V_n^{[\mathrm{Str}]}(\tfrac{t}{n})y &= -C_n^{[2]}(t)By, \\ V_n^{[\mathrm{wgh}]}(\tfrac{t}{n})y &= -\Theta C_n^{[1]}(t)S(\tfrac{t}{n})By-(1-\Theta)T_0(\tfrac{t}{n})C_n^{[1]}(t)By. \end{aligned}$$

Now Lemma 3.5 yields the convergence to the convolution in (3.10). □

Remark 3.7. The stability of splitting methods for triangular operator matrices has been studied in [4]. If we write

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix} = \mathcal{B} + \mathcal{A}_2,$$

then $\mathcal{B}$ with $\mathrm{dom}(\mathcal{B}) = E \times \mathrm{dom}(B)$ generates the strongly continuous semigroup

$$\mathcal{S}(t) = \begin{pmatrix} T_0(t) & 0 \\ 0 & S(t) \end{pmatrix},$$

on $E \times \mathrm{dom}(B)$. Since $\mathcal{A}_2$ is bounded on this space, by [4, Prop. 2.4] we obtain that for some $M \geq 0$ and $\omega \in \mathbb{R}$

$$\big\|\big(\mathcal{S}(\tfrac{t}{n})\mathcal{T}_2(\tfrac{t}{n})\big)^n\big\|_{\mathscr{L}(E\times\mathrm{dom}(B))} \leq M\mathrm{e}^{\omega} \quad \text{for every } t \geq 0.$$

Thus we immediately obtain the convergence of the corresponding Lie splitting procedure on $E \times \mathrm{dom}(B)$ by the Lie–Trotter product formula, see [18, Section III.5] , or [49]. As a matter of fact, in this way we obtain also the generator property of $\mathcal{A}_0$ on $E \times \mathrm{dom}(B)$ without recurring to [10].

Remark 3.8. Let us comment on the relation between the previously proposed Lie splitting and the one from the introduction. Given $u_0 \in \mathrm{H}^{1/2}(\Omega)$ and $v_0 = u_0|_{\partial\Omega}$ belongs to $\mathrm{dom}(B) = \mathrm{H}^2(\partial\Omega)$, then we have that the Lie splitting corresponds to the choices $v_1 = S(\tau)v_0 \in \mathrm{dom}(B)$ and

$$\begin{aligned}\widetilde{u}_0 &= u_0 - D_0v_0 + D_0v_1 - \tau D_0Bv_1 = u_0 + D_0\Big(\int_0^\tau S(r)Bv_0\,\mathrm{d}r - \tau D_0Bv_1\Big)\\ &= u_0 + D_0\int_0^\tau \big(S(r) - S(\tau)\big)Bv_0\,\mathrm{d}r.\end{aligned}$$

If $v_0 \in \mathrm{dom}(B^2)$, we obtain $\tilde{u}_0 = u_0 + \mathrm{O}(\tau^2)$, where $\mathrm{O}(\tau^2)$ denotes a term with norm less than or equal to $C\|B^2v_0\|$.

It can be proven that if a method (more precisely the choice of $\tilde{u}_0$) satisfies $\tilde{u}_0 = u_0 + \mathrm{O}(\tau^2)$, then the corresponding splitting method (e.g. the one in the introduction with $\widetilde{u}_0 = u_0$) is convergent. In addition, its convergence order is the same as for the Lie splitting, cf. the next section.

4. Order of convergence

In this section we will investigate the order of convergence of the proposed splitting schemes. We begin with recalling the standard definition, see, e.g., [2].

Definition 4.1 (Order of Convergence)**.** The approximation $\boldsymbol{u}_n$ to $\boldsymbol{u}$ is called *convergent of order* $p > 1$ if for every $t_{\max} > 0$ there exists a constant $C \geq 0$ such that $\|\boldsymbol{u}(t) - \boldsymbol{u}_n(\frac{t}{n})\| \leq Cn^{-p}$ for every $t \in [0, t_{\max}]$, $n \to \infty$.

The remainder of this paper is devoted to the proof of such estimates for the approximations given in (3.1).

Remark 4.2. Jahnke and Lubich [32] studied the convergence order of the Strang splitting for generators of bounded analytic semigroups under certain commutator conditions (for the Lie splitting, see [12, Chapter 10]). If we split

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix} = \mathcal{B} + \mathcal{A}_2,$$

and assume that A_0, B are generators of bounded analytic semigroups, then in order to apply their result we need to calculate the commutator $[\mathcal{B}, \mathcal{A}_2]$. We have

$$\begin{aligned}[\mathcal{B}, \mathcal{A}_2] &= \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -D_0B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix}\\ &= \begin{pmatrix} 0 & -A_0D_0B \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_0B^2 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & D_0B^2 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

with the domain

$$\mathrm{dom}([\mathcal{B}, \mathcal{A}_2]) = \mathrm{dom}(A_0) \times \{0\},$$

by Remark 2.3. This renders the direct application of the Jahnke–Lubich result impossible. Moreover, in contrast to [32] we do not need to require that the operator B is also an analytic generator, only the well-posedness (1.1). The price to be paid for this simplification is the requirement of increased regularity conditions for the initial value.

Before proceeding to the error estimates we start with an important observation, whose proof is a small modification of the one of Lemma 3.4, cf. (3.11).

Proposition 4.3. *Let $V(\tau)$ be as in* (3.5) *and let $D \subseteq F$ be a subspace with a given norm $\|\cdot\|_D$. Let $r \geq 0$, let $t_{\max} > 0$ and $C \geq 0$ such that for every $y \in D$ and for every $t \in [0, t_{\max}]$*

$$\Big\|V_n(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \leq \frac{Ct^r\log(n)}{n^r}\|y\|_D. \tag{4.1}$$

Then

$$\Big\|\mathcal{R}_0^{-1}\mathbb{T}^n(\tfrac{t}{n})\mathcal{R}_0\tbinom{x}{y} - \mathcal{T}(t)\tbinom{x}{y}\Big\| \leq \frac{Ct^r\log(n)}{n^r}\|y\|_D,$$

for every $x \in E$, $y \in D$ and $t \in [0, t_{\max}]$. In particular, the numerical method $\boldsymbol{u}_k$ defined in (3.1) *is convergent of order p for any $p \in (0, r)$ and every initial value $\boldsymbol{u}_0 \in E \times D$.*

From now on we will focus on the error estimates concerning the approximation $V_n(\frac{t}{n})$, where the corresponding V is either given in (3.6), or (3.7) or (3.8) (but note that many other choices for V are possible, cf. Remark 3.8.)

Lemma 4.4 (Local error of splittings I)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively, and suppose Hypotheses* 2.1, 2.2, 2.5. *For every $t_{\max} > 0$ there is $C \geq 0$ such that for every $h \in [0, t_{\max}]$, for every $s_0, s_1 \in [0, h]$ and for very $y \in \mathrm{dom}(B^2)$ we have*

$$\Big\|\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)A_0^{-1}D_0S(s_1)By\Big\| \\ \leq Ch^2(\|By\| + \|B^2y\|).$$

Proof. For any $y \in \mathrm{dom}(B^2)$ we can write

$$\begin{aligned}
&\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)A_0^{-1}D_0S(s_1)By\\
&= \int_0^h \big(T_0(h-s)A_0^{-1}D_0S(s)By - T_0(h-s_0)A_0^{-1}D_0S(s_1)By\big)\,\mathrm{d}s\\
&= \int_0^h (T_0(h-s) - T_0(h-s_0))A_0^{-1}D_0S(s)By\,\mathrm{d}s\\
&\quad + \int_0^h T_0(h-s_0)A_0^{-1}D_0(S(s) - S(s_1))By\,\mathrm{d}s = I_1 + I_2,
\end{aligned}$$

where I_1, I_2 denote the occurring integrals on the right-hand side in the order of appearance. The first term I_1 can be estimated as

$$\|I_1\| = \Big\|\int_0^h (T_0(h-s) - T_0(h-s_0))A_0^{-1}D_0S(s)By\,\mathrm{d}s\Big\|$$

$$\leq \int_0^h \|(T_0(h-s) - T_0(h-s_0))A_0^{-1}\| \cdot \|D_0\| \cdot \|S(s)By\| \,\mathrm{d}s$$
$$\leq C_1 \int_0^h |s-s_0| \,\mathrm{d}s \cdot \|By\| = C_2 h^2 \|B^2 y\|.$$

For the second term I_2 we obtain the estimate:

$$\begin{aligned}\|I_2\| &= \Big\| \int_0^h T_0(h-s_0)A_0^{-1} D_0 (S(s) - S(s_1))By \,\mathrm{d}s \Big\| \\ &\leq C_3 \int_0^h \|(S(s)-S(s_1))By\| \,\mathrm{d}s = C_3 \int_0^h \Big\| \int_{s_1}^s S(r) B^2 y \,\mathrm{d}r \Big\| \,\mathrm{d}s \\ &\leq C_4 h^2 \|B^2 y\|.\end{aligned}$$

Putting these estimates together finishes the proof of the lemma. □

The validity of the following condition makes it possible to prove convergence rates of higher order splittings.

Hypothesis 4.5. (We suppose as in Hypothesis 2.5 that A_0 generates an bounded analytic semigroup.) The number $\gamma \geq 0$ is such that $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$.

We refer to [23, Chapter 3], [43, Chapter 4], [18, Chapter II.5] or [12, Chapter 9] for details concerning fractional powers of sectorial operators. In particular, at this point it is important to recall the following result.

Remark 4.6. If A_0 is the generator of a bounded analytic semigroup $(T(t))_{t\geq 0}$ then there exist a $M \geq 0$ such that for every $\alpha \geq 0$

$$\|t^\alpha (-A_0)^\alpha T(t)\| \leq M \quad \text{for each } t > 0.$$

Remark 4.7. (a) For $\gamma = 0$ the previous condition is always trivially satisfied, and this choice will suffice for the Lie splitting. The requirement $\gamma > 0$ is only needed for the cases of the Strang and the weighted splittings.

(b) Hypothesis 4.5 is fulfilled in the setting of Example 2.6 for the Laplace and the Laplace–Beltrami operators with $\gamma \in [0, 1/4)$. Indeed, we have $\mathrm{rg}(D_0) \subseteq \mathrm{H}^{1/2}(\Omega)$. For $\gamma \in [0, 1/4)$ we have by [42, Theorem 11.1] that

$$\mathrm{H}^{2\gamma}(\Omega) = \mathrm{H}_0^{2\gamma}(\Omega),$$

and then by by complex interpolation, [42, Theorem 11.6], we can write

$$\big[\mathrm{H}_0^2(\Omega), \mathrm{L}^2(\Omega)\big]_\gamma = \mathrm{H}^{2\gamma}(\Omega).$$

Moreover, since

$$\mathrm{H}_0^2(\Omega) \subseteq \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega) = \mathrm{dom}(\Delta_{\mathrm{D}})$$

with continuous inclusion, we obtain (see, e.g., [43, Chapter 4]) that

$$\mathrm{H}^{2\gamma}(\Omega) = \big[\mathrm{H}_0^2(\Omega), \mathrm{L}^2(\Omega)\big]_\gamma \subseteq \big[\mathrm{dom}(A_0), \mathrm{L}^2(\Omega)\big]_\gamma \subseteq \mathrm{dom}\big((-\Delta_{\mathrm{D}})^\gamma\big).$$

Finally, this yields

$$\mathrm{rg}(D_0) \subseteq \mathrm{H}^{1/2}(\Omega) \subseteq \mathrm{H}^{2\gamma}(\Omega) \subseteq \mathrm{dom}\big((-\Delta_{\mathrm{D}})^\gamma\big).$$

(c) It is important to note that if for some $\gamma \geq 0$ the Hypothesis 4.5 is satisfied, then $(-A_0)^\gamma D_0 : F \to E$ is a closed, and hence bounded, linear operator.

Lemma 4.8 (Local error of splittings II)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Suppose Hypotheses 2.1, 2.2, 2.5 and also 4.5, i.e., that $\operatorname{rg}(D_0) \subseteq \operatorname{dom}((-A_0)^\gamma)$ for some $\gamma \in [0,1]$. For every $t_{\max} > 0$ there is $C \geq 0$ such that for every $h \in [0, t_{\max}]$, for every $s_0, s_1 \in [0,h]$ and for very $y \in \operatorname{dom}(B^2)$ we have*

$$\Big\| \int_0^h T_0(h-s)D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)D_0S(s_1)By \Big\| \leq Ch^{1+\gamma}(\|By\| + \|B^2y\|).$$

Proof. For any $y \in \operatorname{dom}(B^2)$ we can write

$$\begin{aligned}
&\Big\| \int_0^h T_0(h-s)D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)D_0S(s_1)By \Big\| \\
&\quad = \Big\| \int_0^h \big(T_0(h-s)D_0S(s)By - T_0(h-s_0)D_0S(s_1)By\big)\,\mathrm{d}s \Big\| \\
&\quad \leq \Big\| \int_0^h \big(T_0(h-s) - T_0(h-s_0)\big)D_0S(s)By\,\mathrm{d}s \Big\| \\
&\qquad + \Big\| \int_0^h T_0(h-s_0)D_0\big(S(s) - S(s_1)\big)By\,\mathrm{d}s \Big\|.
\end{aligned}$$

The second term can be further estimated as

$$\begin{aligned}
&\Big\| \int_0^h T_0(h-s_0)D_0\big(S(s) - S(s_1)\big)By\,\mathrm{d}s \Big\| \\
&\quad \leq C_1 \int_0^h \|\big(S(s) - S(s_1)\big)By\|\,\mathrm{d}s = C_1 \int_0^h \Big\| \int_{s_1}^s S(r)B^2y\,\mathrm{d}r \Big\|\,\mathrm{d}s \\
&\quad \leq C_2h^2\|B^2y\|.
\end{aligned} \tag{4.2}$$

It remains to estimate the first term. Since $(-A_0)^\gamma D_0$ is closed and everywhere defined, it is bounded (see Remark 4.7) and hence we can write

$$\begin{aligned}
&\Big\| \int_0^h \big(T_0(h-s) - T_0(h-s_0)\big)D_0S(s)By\,\mathrm{d}s \Big\| \\
&\quad \leq \int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s) - T_0(h-s_0)\big)(-A_0)^\gamma D_0S(s)By\|\,\mathrm{d}s \\
&\quad \leq \int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s) - T_0(h-s_0)\big)\| \cdot \|(-A_0)^\gamma D_0S(s)By\|\,\mathrm{d}s \\
&\quad \leq C_3\|By\| \int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s) - T_0(h-s_0)\big)\|\,\mathrm{d}s.
\end{aligned}$$

Now, by Remark 4.6 we have

$$\|(-A_0)^{-\gamma}\big(T_0(h-s) - T_0(h-s_0)\big)\| \leq C_4(s-s_0)^\gamma.$$

Inserting this back into the previous inequality and integrating with respect to s we finally obtain the statement. □

The next result yields that the order of Lie splitting is (at most 1 but) as near to 1 as we wish, provided the initial data is smooth enough.

Theorem 4.9 (Convergence of the Lie splitting)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Suppose*

Hypotheses 2.1, 2.2, 2.5 and also 4.5, i.e., that $\operatorname{rg}(D_0) \subseteq \operatorname{dom}((-A_0)^\gamma)$ *for some* $\gamma \in [0,1]$. *For each* $t_{\max} > 0$ *there is* $C \geq 0$ *such that for every* $n \in \mathbb{N}$, $y \in \operatorname{dom}(B^2)$ *and* $t \in [0, t_{\max}]$ *we have*

$$\Big\| V_n^{[\mathrm{Lie}]}(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \Big\| \leq C\frac{t\log(n)}{n}(\|By\| + \|B^2y\|).$$

Proof. With $\tau = \frac{t}{n}$ we have

$$\begin{aligned} V_n^{[\mathrm{Lie}]}(\tau)y &+ \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \\ &= -\tau\sum_{j=0}^{n-1} T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\ &\quad + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \\ &= -\sum_{j=0}^{n-1}\Big(\tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\ &\quad - \int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big). \end{aligned}$$

Notice that for $j \in \{0, \ldots, n-1\}$ we have

$$\begin{aligned} &\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\ &= T_0(t-(j+1)\tau)\int_{j\tau}^{(j+1)\tau} T_0((j+1)\tau - s)D_0S(s)By\,\mathrm{d}s \\ &\quad - \tau T_0(t-(j+1)\tau)T_0(\tau)D_0S((j+1)\tau)By. \end{aligned}$$

If $j \in \{0, \ldots, n-2\}$ then by Lemma 4.4 we conclude that

$$\begin{aligned} &\Big\|\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y\Big\| \\ &\leq \|A_0T_0(t-(j+1)\tau)\| \cdot \Big\|\int_{j\tau}^{(j+1)\tau} T_0((j+1)\tau - s)A_0^{-1}D_0S(s)By\,\mathrm{d}s \\ &\qquad - \tau T_0(\tau)D_0S((j+1)\tau)By\Big\| \\ &\leq C_1\frac{1}{t-(j+1)\tau}\Big\|\int_0^\tau T_0(\tau - s)A_0^{-1}D_0S(s+j\tau)By\,\mathrm{d}s \\ &\qquad - \tau T_0(\tau)A_0^{-1}D_0S(\tau)S(j\tau)By\Big\| \\ &\leq C_2\frac{1}{t-(j+1)\tau}\tau^2(\|BS(j\tau)y\| + \|B^2S(j\tau)y\|) \\ &\leq C_3\frac{t}{n(n-(j+1))}(\|By\| + \|B^2y\|). \end{aligned}$$

Whereas for $j = n-1$ we have by Lemma 4.8 (with $\gamma = 0$, $h = \tau$, $s_0 = s_1 = \tau$) that

$$\begin{aligned}
&\Big\|\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y\Big\|\\
&= \Big\|\int_0^{\tau} T_0(\tau-s)D_0S((n-1)\tau+s)By\,\mathrm{d}s - \tau T_0(\tau)D_0S(\tau)S((n-1)\tau)By\Big\|\\
&\le C_4\frac{t}{n}(\|By\| + \|B^2y\|).
\end{aligned}$$

Summing these terms for $j = 0, \ldots, n-1$ we obtain that

$$\begin{aligned}
&\Big\|V_n^{[\mathrm{Lie}]}(\tau)y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\|\\
&\le \sum_{j=0}^{n-2} C_3\frac{t}{n(n-(j+1))}(\|By\| + \|B^2y\|) + C_4\frac{t}{n}(\|By\| + \|B^2y\|)\\
&\le C\frac{t\log(n)}{n}(\|By\| + \|B^2y\|),
\end{aligned}$$

as asserted. □

Lemma 4.10 (Local error of the Strang splitting)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\ge0}$ and $(S(t))_{t\ge0}$, respectively. Suppose Hypotheses 2.1, 2.2, 2.5 and also 4.5, i.e., that $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$ for some $\gamma \in [0,1]$. For every $t_{\max} > 0$ there is $C \ge 0$ such that for every $h \in [0, t_{\max}]$ and for very $y \in \mathrm{dom}(B^3)$ we have*

$$\Big\|\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\Big\| \le Ch^{2+\gamma}(\|By\| + \|B^2y\| + \|B^3y\|).$$

Proof. We have

$$\begin{aligned}
&\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\\
&= \int_0^h T_0(h-s)A_0^{-1}D_0S(s)By - T_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s\\
&= \int_0^h T_0(h-s)A_0^{-1}D_0\big(S(s) - S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad + \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s\\
&= \int_0^h \Big(T_0(h-s) - T_0(\tfrac{h}{2})\Big)A_0^{-1}D_0\big(S(s) - S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad + \int_0^h T_0(\tfrac{h}{2})A_0^{-1}D_0\big(S(s) - S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad + \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s = I_1 + I_2 + I_3,
\end{aligned}$$

where I_1, I_2, I_3 denote the integrals on the right-hand side in the respective order of appearance.

We start with the estimation of I_1. Inserting the Taylor remainder

$$(S(s) - S(\tfrac{h}{2}))By = \int_{\frac{h}{2}}^{s} S(r)B^2 y \,\mathrm{d}r,$$

and the analogous formula for $T_0(h-s) - T_0(\frac{h}{2})$, in the definition of I_1 yields that

$$\begin{aligned} I_1 &= \int_0^h \Big(T_0(h-s) - T_0(\tfrac{h}{2})\Big) A_0^{-1} D_0 \big(S(s) - S(\tfrac{h}{2})\big) By \,\mathrm{d}s \\ &= \int_0^h \int_{\frac{h}{2}}^{h-s} T(t) A_0 A_0^{-1} D_0 \int_{\frac{h}{2}}^{s} S(r) B^2 y \,\mathrm{d}r \,\mathrm{d}t \,\mathrm{d}s \\ &= \int_0^h \int_{\frac{h}{2}}^{h-s} T(t) D_0 \int_{\frac{h}{2}}^{s} S(r) B^2 y \,\mathrm{d}r \,\mathrm{d}t \,\mathrm{d}s. \end{aligned}$$

Whence we conclude

$$\begin{aligned} \|I_1\| &\le \int_0^h \Big| \int_{\frac{h}{2}}^{h-s} \|T(t) D_0\| \Big| \int_{\frac{h}{2}}^{s} \|S(r)\| \|B^2 y\| \,\mathrm{d}r \Big| \,\mathrm{d}t \Big| \,\mathrm{d}s \\ &\le C_1 \|D_0\| \int_0^h |h - s - \tfrac{h}{2}| \cdot |\tfrac{h}{2} - s| \,\mathrm{d}s \cdot \|B^2 y\| \le C_2 h^3 \|B^2 y\|, \end{aligned} \tag{4.3}$$

where C_1 and C_2 depend only on the growth bounds of $(T_0(t))_{t\ge}$ and $(S(t))_{t\ge 0}$ and on $t_{\max}$ and $\|D_0\|$.

The next is the estimation of the integral I_2. Now instead of inserting a first order Taylor approximation for $S(s)$ in the definition of I_2 we make use of the special structure of Strang splitting and recall the following Taylor formula

$$S(s)By = S(\tfrac{h}{2})By + (s - \tfrac{h}{2})S(\tfrac{h}{2})B^2 y + \frac{1}{2}\int_0^{s-\frac{h}{2}} (s - \tfrac{h}{2} - r) S(r) S(\tfrac{h}{2}) B^3 y \,\mathrm{d}r.$$

If we substitute this into the definition of I_2, we arrive at

$$\begin{aligned} I_2 &= \int_0^h T_0(\tfrac{h}{2}) A_0^{-1} D_0 \big(S(s) - S(\tfrac{h}{2})\big) By \,\mathrm{d}s \\ &= T_0(\tfrac{h}{2}) A_0^{-1} D_0 \int_0^h \big(S(s) - S(\tfrac{h}{2})\big) By \,\mathrm{d}s \\ &= T_0(\tfrac{h}{2}) A_0^{-1} D_0 \int_0^h \Big(S(\tfrac{h}{2})By + (s - \tfrac{h}{2}) S(\tfrac{h}{2}) B^2 y \\ &\quad + \frac{1}{2}\int_0^{s-\frac{h}{2}} (s - \tfrac{h}{2} - r) S(r) S(\tfrac{h}{2}) B^3 y \,\mathrm{d}r - S(\tfrac{h}{2})By\Big) \,\mathrm{d}s \\ &= T_0(\tfrac{h}{2}) A_0^{-1} D_0 \Big(\int_0^h (s - \tfrac{h}{2}) S(\tfrac{h}{2}) B^2 y \,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^h \int_0^{s-\frac{h}{2}} (s - \tfrac{h}{2} - r) S(r) S(\tfrac{h}{2}) B^3 y \,\mathrm{d}r \,\mathrm{d}s\Big) \\ &= \frac{1}{2} T_0(\tfrac{h}{2}) A_0^{-1} D_0 \int_0^h \int_0^{s-\frac{h}{2}} (s - \tfrac{h}{2} - r) S(r) S(\tfrac{h}{2}) B^3 y \,\mathrm{d}r \,\mathrm{d}s, \end{aligned}$$

the last equality being true since the first integral on the right-hand side on the line before is 0. This immediately implies the desired norm-estimate for I_2:

$$\begin{aligned}\|I_2\| &\le \frac{1}{2}\|T_0(\tfrac{h}{2})A_0^{-1}D_0\| \cdot \Big\|\int_0^h \int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(r)S(\tfrac{h}{2})B^3y \,\mathrm{d}r\,\mathrm{d}s\Big\| \\ &\le C_3h^3\|B^3y\|,\end{aligned}$$

where C_3 is an appropriate constant independent of y and $h \in [0, t_{\max}]$.

We finally turn to the estimation of the term I_3, and this is only where the order reduction by $1-\gamma$ occurs. If we abbreviate $z = D_0S(\frac{h}{2})By$, then

$$I_3 = \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}z.$$

By analyticity we have $T_0(\frac{h}{2})z \in \operatorname{dom}(A_0^2)$ so, similarly to the case of the term I_2, we can use the Taylor expansion

$$\begin{aligned}T_0(h-s)A_0^{-1}z = T(\tfrac{h}{2})A_0^{-1}z &+ (s-\tfrac{h}{2})A_0T_0(\tfrac{h}{2})A_0^{-1}z \\ &+ \frac{1}{2}\int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)T_0(r)A_0^2T_0(\tfrac{h}{2})A_0^{-1}z\,\mathrm{d}r.\end{aligned}$$

Whence we conclude

$$\begin{aligned}I_3 &= \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}z\,\mathrm{d}s \\ &= \int_0^h \Big((s-\tfrac{h}{2})AT_0(\tfrac{h}{2})A_0^{-1}z \\ &\quad + \frac{1}{2}\int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)T_0(r)A_0^2T_0(\tfrac{h}{2})A_0^{-1}z\,\mathrm{d}r\Big)\,\mathrm{d}s,\end{aligned}$$

since the first integral here is 0, we arrive at

$$I_3 = \frac{1}{2}\int_0^h \int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)T_0(r)A_0^2T_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}r\,\mathrm{d}s.$$

We take the norm here and estimate trivially:

(4.4)

$$\begin{aligned}\|I_3\| &= \frac{1}{2}\Big\|\int_0^h \int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)T_0(r)(-A_0)^{1-\gamma}T_0(\tfrac{h}{2})(-A_0)^{\gamma}D_0S(\tfrac{h}{2})By\,\mathrm{d}r\,\mathrm{d}s\Big\| \\ &\le \frac{C_4}{2}\|(-A_0)^{1-\gamma}T_0(\tfrac{h}{2})\| \cdot \|(-A_0)^{\gamma}D_0\| \cdot \|By\| \int_0^h \int_0^{s-\frac{h}{2}} s-\tfrac{h}{2}-r\,\mathrm{d}r\,\mathrm{d}s \\ &= C_5\frac{h^3}{h^{1-\gamma}}\|By\| = C_5h^{2+\gamma}\|By\|.\end{aligned}$$

The proof of the lemma is now finished by putting together the estimates for I_1, I_2 and I_3. □

Theorem 4.11 (Convergence of the Strang splitting)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\ge0}$ and $(S(t))_{t\ge0}$, respectively. Suppose Hypotheses* 2.1, 2.2, 2.5 *and also* 4.5, *i.e., that* $\operatorname{rg}(D_0) \subseteq \operatorname{dom}((-A_0)^{\gamma})$

for some $\gamma \in [0,1]$. *For each* $t_{\max} > 0$ *there is* $C \geq 0$ *such that for every* $n \in \mathbb{N}$, $y \in \operatorname{dom}(B^3)$ *and* $t \in [0, t_{\max}]$ *we have*

$$\Big\| V_n^{[\mathrm{Str}]}(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \Big\| \leq C\frac{t^{1+\gamma}\log(n)}{n^{1+\gamma}}(\|By\| + \|B^2y\| + \|B^3y\|).$$

Proof. Set $\tau := \frac{t}{n}$. Recall from (3.12) that for $y \in \operatorname{dom}(B)$ we have

$$V_n^{[\mathrm{Str}]}(\tau)y = -\tau \sum_{j=0}^{n-1} T_0((n-j-\tfrac{1}{2})\tau)D_0S((j+\tfrac{1}{2})\tau)By, \tag{4.5}$$

so that

$$\begin{aligned}
&\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s + V_n^{[\mathrm{Str}]}(\tau)y \\
&\quad = \sum_{j=0}^{n-1} \int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By - T_0((n-j-\tfrac{1}{2})\tau)D_0S((j+\tfrac{1}{2})\tau)By\,\mathrm{d}s \\
&\quad = \sum_{j=0}^{n-1} T_0((n-j-1)\tau) \int_0^\tau T_0(\tau-s)D_0S(s)S(j\tau)By \\
&\qquad\qquad\qquad\qquad - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s.
\end{aligned}$$

We first consider the term for $j = n-1$. By Lemma 4.8, with $h = \tau$ and $s_0 = s_1 = \frac{\tau}{2}$ we have that

$$\begin{aligned}
&\Big\| \int_0^\tau T_0((\tau-s)D_0S(s)S((n-1)\tau)By - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S((n-1)\tau)By\,\mathrm{d}s \Big\| \\
&\qquad \leq C_1\tau^{1+\gamma}(\|BS((n-1)\tau)y\| + \|B^2S((n-1)\tau)y\| \\
&\qquad \leq C_2\tau^{1+\gamma}(\|By\| + \|B^2y\|)
\end{aligned}$$

for $t \in [0, t_{\max}]$. Next we consider the summands in (4.5) for $j \in \{0, \ldots, n-2\}$. In these cases we can write

$$\begin{aligned}
&\Big\| T_0\big((n-j-1)\tau\big) \int_0^\tau T_0(\tau-s)D_0S(s)S(j\tau)By - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s \Big\| \\
&\quad = \|A_0T_0\big((n-j-1)\tau\big) \int_0^\tau T_0(\tau-s)A_0^{-1}D_0S(s)S(j\tau)By \\
&\qquad - T_0(\tfrac{\tau}{2})A_0^{-1}D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s \Big\| \\
&\quad \leq \|A_0T_0\big((n-j-1)\tau\big)\| \cdot \Big\| \int_0^\tau T_0(\tau-s)A_0^{-1}D_0S(s)S(j\tau)By \\
&\qquad - T_0(\tfrac{\tau}{2})A_0^{-1}D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s \Big\|,
\end{aligned}$$

and by Lemma 4.10 and Remark 4.6 we can continue as follows:

$$\begin{aligned}
&\leq \frac{C_3}{(n-j-1)\tau}\tau^{2+\gamma}\big(\|BS(j\tau)y\| + \|B^2S(j\tau)y\| + \|B^3S(j\tau)y\|\big) \\
&\leq \frac{C_4}{(n-j-1)\tau}\tau^{2+\gamma}\big(\|By\| + \|B^2y\| + \|B^3y\|\big)
\end{aligned}$$

for constants C_3, C_4 independent of y, n and $t \in [0, t_{\max}]$. Summing up these estimates we arrive at

$$\begin{aligned}
&\Big\| \int_0^t T_0(t-s) D_0 S(s) B y \, \mathrm{d}s + V_n^{[\mathrm{Str}]}(\tau) y \Big\| \\
&\quad \le C_2 \tau^{1+\gamma} \big(\|By\| + \|B^2 y\| \big) + \sum_{j=0}^{n-2} \frac{C_4}{(n-j-1)\tau} \tau^{2+\gamma} \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big) \\
&\quad \le \Big(C_2 \frac{t^{1+\gamma}}{n^{1+\gamma}} + \frac{C_4 t^{2+\gamma}}{n^{2+\gamma}} \sum_{k=1}^{n-1} \frac{n}{tk} \Big) \cdot \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big) \\
&\quad \le C \frac{t^{1+\gamma} \log(n)}{n^{1+\gamma}} \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big),
\end{aligned}$$

with an appropriate constant $C \ge 0$. The proof is complete. □

Finally, let us turn to the weighted splittings. For any $\Theta \in [0,1]$ the weighted splitting possess at least the convergence properties as the Lie splitting. For the case $\Theta = 1/2$ one can prove even more.

Lemma 4.12 (Local error of the symmetrically weighted Splitting)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$, respectively. Suppose Hypotheses* 2.1, 2.2, 2.5 *and also* 4.5, *i.e., that* $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$ *for some $\gamma \in [0,1]$. For every $t_{\max} > 0$ there is $C \ge 0$ such that for every $h \in [0, t_{\max}]$ and for very $y \in \mathrm{dom}(B^3)$ we have*

$$\begin{aligned}
\Big\| \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) B y \, \mathrm{d}s - \frac{1}{2} \Big(h T_0(h) A_0^{-1} D_0 B S(h) y + h A_0^{-1} D_0 B y \Big) \Big\| \\
\le C h^{2+\gamma} (\|By\| + \|B^2 y\| + \|B^3 y\|).
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
&2 \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) B y \, \mathrm{d}s - \Big(h T_0(h A_0^{-1}) D_0 B S(h) y + h A_0^{-1} D_0 B y \Big) = \\
&\quad = \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) B y - h T_0(h) A_0^{-1} D_0 B S(h) y \, \mathrm{d}s \\
&\qquad + \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) B y - A_0^{-1} D_0 B y \, \mathrm{d}s \\
&\quad = \int_0^h (T_0(h-s) - T_0(h)) A_0^{-1} D_0 S(s) B y \, \mathrm{d}s \\
&\qquad + \int_0^h T_0(h) A_0^{-1} D_0 (S(s) B y - S(h) B y) \, \mathrm{d}s \\
&\qquad + \int_0^h (T_0(h-s) - \mathrm{I}) A_0^{-1} D_0 S(s) B y \, \mathrm{d}s \\
&\qquad + \int_0^h (\mathrm{I} - T_0(h)) A_0^{-1} D_0 (S(s) B y - B y) \, \mathrm{d}s \\
&\qquad + \int_0^h T_0(h) A_0^{-1} D_0 (S(s) B y - B y) \, \mathrm{d}s = I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}$$

where I_j denotes the respective term on the right-hand side in order of occurrence.

The terms I_1 and I_3 can be estimated as

$$\|I_1\| \leq C_1 h^{2+\gamma} \|By\|,$$
$$\|I_3\| \leq C_1 h^{2+\gamma} \|By\|,$$

cf. (4.4) in the estimation of the term I_3 in Lemma 4.10.

The term I_4 can be estimated as

$$\|I_4\| \leq C_2 h^3 \|B^2 y\|,$$

cf. (4.3) in the estimation of the term I_1 in Lemma 4.10.

For the sum $I_2 + I_5$ we can write

$$\begin{aligned} I_2 + I_5 &= \int_0^h T_0(h) A_0^{-1} D_0 (S(s)By - S(h)By)\,\mathrm{d}s \\ &\quad + \int_0^h T_0(h) A_0^{-1} D_0 (S(s)By - By)\,\mathrm{d}s \\ &= T_0(h) A_0^{-1} D_0 \int_0^h (2S(s)By - S(h)By - By)\,\mathrm{d}s. \end{aligned}$$

Since $y \in \mathrm{dom}(B^3)$ for any $t > 0$ we have the Taylor expansion

$$S(t)By = By + tB^2 y + \frac{1}{2} \int_0^t r S(r) B^3 y\,\mathrm{d}r.$$

Substituting this into the above formula for $I_2 + I_5$ we obtain that

$$\begin{aligned} I_2 + I_5 = T_0(h) A_0^{-1} D_0 &\int_0^h \Big(2By + 2sB^2 y + \int_0^s r S(r) B^3 y\,\mathrm{d}r - By - hB^2 y \\ &- \frac{1}{2} \int_0^h r S(r) B^3 y\,\mathrm{d}r - By \Big)\,\mathrm{d}s \\ = T_0(h) A_0^{-1} D_0 &\Big[\int_0^h \Big((2s - h) B^2 y + \int_0^s r S(r) B^3 y\,\mathrm{d}r \Big)\,\mathrm{d}s \\ &- \frac{h}{2} \int_0^h r S(r) B^3 y\,\mathrm{d}r \Big] \\ = T_0(h) A_0^{-1} D_0 &\int_0^h \Big(\int_0^s r S(r) B^3 y\,\mathrm{d}r - \frac{h}{2} s S(s) B^3 y \Big)\,\mathrm{d}s, \end{aligned}$$

since the integral of the first term is 0. Whence we conclude

$$\|I_2 + I_5\| \leq C_3 h^3 \|B^3 y\|.$$

Putting the estimates for the terms I_1, I_3, I_4, $I_2 + I_5$ together finishes the proof of the lemma. □

Based on Lemma 4.12 we immediately obtain the following error estimate for the symmetrically weighted splitting, the proof is analogous to the one of Theorem 4.11.

Theorem 4.13 (Convergence of the symmetrically weighted splitting)**.** *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Suppose Hypotheses* 2.1, 2.2, 2.5 *and also* 4.5, *i.e., that* $\mathrm{rg}(D_0) \subseteq$

$\mathrm{dom}((-A_0)^\gamma)$ *for some* $\gamma \in [0,1]$. *For each* $t_{\max} > 0$ *there is* $C \geq 0$ *such that for every* $n \in \mathbb{N}$, $y \in \mathrm{dom}(B^3)$ *and* $t \in [0, t_{\max}]$ *we have*

$$\left\| V_n^{[\mathrm{wgh}]}(\tfrac{t}{n})y + \int_0^t T_0(t-s) D_0 S(s) By \, \mathrm{d}s \right\| \leq C \frac{t^{1+\gamma}\log(n)}{n^{1+\gamma}} (\|By\| + \|B^2 y\| + \|B^3 y\|).$$

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