Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 20/07

Anna Clevenhaus, Matthias Ehrhardt, Michael Günther and Daniel Ševčovič

Pricing American Options with a Non-constant Penalty Parameter

April 2020

http://www.math.uni-wuppertal.de

Pricing American Options with a Non-constant Penalty Parameter

Anna Clevenhaus[a,*], Matthias Ehrhardt[a], Michael Günther[a], Daniel Ševčovič[b]

[a] *Chair of Applied Mathematics and Numerical Analysis, University of Wuppertal, Wuppertal, Germany*
[b] *Department of Applied Mathematics and Statistics, Division of Applied Mathematics; Comenius University, Bratislava, Slovakia*

Abstract

As the American early exercise results in a free boundary problem, we add a penalty term to obtain a partial differential equation. In this article we focus on an improved definition of the penalty term for American options. We replace the constant penalty parameter by a time dependent function. To gain insight into the accuracy of our proposed extension, we compare the solution of the extension to standard reference solutions from the literature. This illustrates the improvement of using a penalty function instead of a penalising constant.

Keywords: American Options; PDE option pricing; Penalty term; projected SOR, penalization strategy

1. Introduction

As American options give the holder the right to exercise the option before and at the maturity, this leads to free boundary value problems which have to be solved numerically. Several schemes for solving American option problems have been proposed, e.g. the projected SOR scheme [1], the binomial method, front-fixing schemes [2], the power penalty method [3] and Monte Carlo simulation techniques. These schemes compute the free boundary value implicitly. Other researchers focused on an explicit representation of the free boundary value [4, 5, 6].

Another approach to solve American option problems is to add a penalty term to the problem [2, 7, 3]. A penalty term forces the problem to fulfill the free boundary constraint asymptotically. If the free boundary constraint is fulfilled, the penalty term is zero, otherwise it penalizes the problem with a factor. Until now the penalty term included a penalisation constant being

*Corresponding author

Email addresses: `clevenhaus@uni-wuppertal.de` (Anna Clevenhaus), `ehrhardt@uni-wuppertal.de` (Matthias Ehrhardt), `guenther@uni-wuppertal.de` (Michael Günther), `sevcovic@fmph.uniba.sk` (Daniel Ševčovič)

roughly estimated by an optimization. We improve the performance of the penalization by replacing the penalty constant by a time dependent penalty function. An explicit formulation for the derivation of the parameters of the penalty function is part of further research.

Once we add a penalty term to the American option problem, the problem reduces to a partial differential equation (PDE) on a fixed domain and we can apply standard numerical methods, e.g. finite differences methods, finite element methods, alternating direction implicit schemes. The usual approach to penalisation of the American put option free boundary problem involves a small parameter making the numerical analysis harder. The novelty and advantage of our approach consist in introducing a bounded penalty function enabling us to construct an efficient and stable numerical approximation scheme.

The outline of this article is as follows. Section 2 reviews the mathematical modeling for an American option with and without a penalty term. We choose the classical Black-Scholes equation and use a variable transformation to simplify our computations and add the penalty function. In Section 3 the model is discretized and the numerical results of the different test cases are presented in Section 4. In Section 5 we conclude this work with a brief outlook.

2. Mathematical Modeling

American options are more expensive than European options, as American options give the holder the right to exercise the option also before the maturity T. In the following we will focus on American put options for clearness of the idea, but as it can be seen in the numerical results all assumptions also hold analogously for the American call options. For pricing an American put option P with the Black-Scholes model we are seeking for a pair of functions $\big(P(S,t), S_f(t)\big)$ such that

$$\begin{aligned} \mathcal{L}_{BSM}[P] \equiv \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r-q) S \frac{\partial P}{\partial S} - rP \geq 0, \quad 0 \leq t \leq T, \\ (K-S)^+ = P(S,t) \quad \text{for} \quad S \leq S_f(t), \\ (K-S)^+ < P(S,t) \quad \text{for} \quad S > S_f(t), \end{aligned} \tag{1}$$

where K denotes the predefined strike price, r is the risk free interest rate, q the dividend rate, σ is the volatility, S is the price of an asset and $S_f(t)$ is the free boundary value at time t with $0 \leq t \leq T$. In (1) we used the standard notation $(f)^+ := \max(f, 0)$. The differential operator appearing in the Black-Scholes PDE (1) is abbreviated by $\mathcal{L}_{BSM}$.

The terminal condition at the maturity $t = T$ reads

$$P(S,T) = (K-S)^+ \tag{2}$$

and the 'spatial' boundary conditions at $S = S_f(t)$, $S \to \infty$, are given by

$$P\big(S_f(t), t\big) = \big(K - S_f(t)\big)^+, \ \frac{\partial P}{\partial S}\big(S_f(t), t\big) = -1, \ \lim_{S \to \infty} P(S,t) = 0, \ 0 \leq t \leq T.$$

If the free boundary value position $S_f(t)$ of the American put option problem is known, we can write

$$\mathcal{L}_{BSM}[P](S,t) = \begin{cases} -rK, & 0 < S \leq S_f(t), \\ 0, & S > S_f(t), \end{cases} \tag{3}$$

We transform the option price for the American put option $P(S,t)$ by the change of variables with τ being the scaled reversed time to maturity $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T-t)$, $k = 2r/\sigma^2$, and obtain for the new unknown function

$$u(x,\tau) = \frac{1}{K}\exp\Big(\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau\Big)P(S,t)$$

the transformed constraint

$$f(x,\tau) = \exp\Big(\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau\Big)\left(1-\exp(x)\right)^+, \tag{4}$$

and the transformed right hand side of (3)

$$g(x,\tau) = \begin{cases} -k\exp\Big(\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau\Big), & x < x_f(\tau), \\ 0, & x \geq x_f(\tau). \end{cases} \tag{5}$$

The transformed system is given as follows:

$$u_\tau - u_{xx} = g(x,\tau). \tag{6}$$

It is supplied with the initial and boundary conditions

$$u(x,0) = f(x,0),\ x \in \mathbb{R}, \quad \lim_{x\to\pm\infty}\big(u(x,\tau) - f(x,\tau)\big) = 0, \quad 0 \leq \tau \leq T. \tag{7}$$

Recall that we are seeking for a pair of functions $\big(u(x,\tau), x_f(\tau)\big)$ satisfying the free boundary problem (5)-(7).

2.1. The Penalty Term

Here we outline an approach to explore a penalty term $p(x,\tau)$ in the transformed problem in equation (6). For the penalty term we use the right hand side of equation (5). We the penalty function by multiplying it with $\delta(\tau)$, i.e.:

$$p(x,\tau) = \delta(\tau)\cdot g(x,\tau). \tag{8}$$

where the modification factor $\delta(\tau)$ is assumed to be an affine-linear function

$$\delta(\tau) = a\tau + b, \tag{9}$$

where $a, b \in \mathbb{R}$ are constants. As the inversion of the transformed right hand side of equation (6)

$$K\exp\Big(-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau\Big)\cdot g(x,\tau) = \begin{cases} -kK, & x < x_f(\tau), \\ 0, & x \geq x_f(\tau), \end{cases} \tag{10}$$

should coincide with the right hand side of (3), we deduce the equation $r = b\frac{2r}{\sigma^2}$ from which we conclude $b = \frac{\sigma^2}{2}$. The multiplier $\delta(\tau)$ is therefore given by an affine-linear function

$$\delta(\tau) = a\tau + \frac{\sigma^2}{2}, \tag{11}$$

where $a \in \mathbb{R}$ is constant.

As the transformed right hand side (5) is also space dependent, the penalty term $p(x, \tau)$ is space dependent, too. By including the penalty term (8) to the heat equation, we obtain the following penalised PDE formulation

$$u_\tau - u_{xx} = p(x, \tau). \tag{12}$$

A solution is subject to the initial and boundary conditions (7).

Since our penalty term requires an initial guess for the free boundary, we compute the exact solution for the Black-Scholes equation for the European put option and compute the point of intersection between the payoff and the solution as an initial guess $\bar{x}_f(\tau)$ for the free boundary $x_f(\tau)$. Then we compute the penalty term, solve the penalised heat equation (12) and obtain finally the solution of the American option problem.

3. Discretization

Let us introduce a temporal discretization $\tau_j = T - j\Delta\tau$, $\Delta\tau = T/M$, $j = 0, \ldots, M$, and a spatial grid between the points x_{min} and $x_{\max}$

$$x_i = x_{\min} + i\Delta x, \quad \Delta x = \frac{x_{\max} - x_{\min}}{N}, \quad i = 0, \ldots, N.$$

We use the finite difference θ-scheme for discretization and simplify notation by

$$\alpha_1 = \theta\frac{\Delta\tau}{(\Delta x)^2} \quad \text{and} \quad \alpha_2 = (1-\theta)\frac{\Delta\tau}{(\Delta x)^2} \quad \text{with} \quad 0 \le \theta \le 1.$$

We obtain $w^j = (w_1^j, \ldots, w_{N-1}^j)^\top$ with w_i^j as the approximation for $u(x_i, \tau_j)$, $f^j = (f_1^j, \ldots, f_{N-1}^j)^\top$ with $f_i^j \sim f(x_i, \tau_j)$ and the diagonal matrices A and B

$$A = \operatorname{diag}(-\alpha_1, 2\alpha_1 + 1, -\alpha_1), \qquad B = \operatorname{diag}(\alpha_2, -2\alpha_2 + 1, \alpha_2)$$

as well as the vector d^j containing the boundary values

$$(d^j)^\top = \left(\alpha_1 w_0^{j+1} + \alpha_2 w_0^j, 0, \ldots, 0, \alpha_1 w_{N+1}^{j+1} + \alpha_2 w_{N+1}^j\right).$$

The discretized penalty term (8) is given by

$$p^j = \delta^j \cdot g^j \quad \text{with} \quad \delta^j = \left(a\tau_j + \frac{\sigma^2}{2}\right) \quad \text{and with}$$

$$g_i^j = \begin{cases} -k\exp\Big(\frac{1}{2}(k-1)x_i + \frac{1}{4}(k+1)^2\tau_j\Big), & x_i < \bar{x}_f^j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\bar{x}_f^j$ is the unique solution to the European put option problem

$$\bar{u}\big(\bar{x}_f(\tau), \tau\big) = f\big(\bar{x}_f(\tau), \tau\big) \tag{13}$$

where $\bar{u}$ is a solution to the Cauchy problem: $\bar{u}_\tau - \bar{u}_{xx} = 0,\ \bar{u}(x,0) = f(x,0)$, which is given in a closed explicit form. Including all the components we obtain the θ-scheme discretized formulation for the penalised heat equation (12)

$$Aw^{j+1} - Bw^j - d^j = \Delta\tau \cdot p^j,$$

where the multiplication of $\Delta\tau$ results from the discretization.

4. Numerical Results

In this section we consider the example for pricing American put options from Nielson et al. [2]. All results are computed on a Intel® Core™ i7-5557U CPU with 3.10 GHz. We choose $x_{\min} = -4$, $x_{\max} = 4$, $M = 5000$, and use the parameter sets from Table 1. The penalty term parameters are obtained by an optimization, as a deterministic expression is a goal of our future research. To facilitate the optimization, we summarised $\Delta\tau p^j$ to

$$\Delta\tau p^j = \Big(\tilde{a}\tau_j + \Delta\tau \cdot r\Big) \cdot \begin{cases} -\exp\Big(\frac{1}{2}(k-1)x_i + \frac{1}{4}(k+1)^2\tau_j\Big), & x_i < \bar{x}_f^j, \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

where $\tilde{a} = \Delta\tau \cdot k \cdot a$. The mean square error (MSE) is given by

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(P^{\text{PSOR}}(S_i, 0) - P^{\text{Pen}}(S_i, 0))^2, \quad S_i = K\exp(x_i),$$

where P^{PSOR} is the solution obtained by the projected SOR algorithm and P^{Pen} the maximum of the solution of the penalised system and the payoff-function. The maximum is used to gain comparable results to the PSOR algorithm.

Our numerical results illustrate the accuracy of the method. The best results are obtained by the sample sets with a small volatility and short time maturity. The observation of the short time maturity is based on the fact, that the number of points is different. The dependence to the volatility is caused by the simplification of the term p since we cancel out $\sigma^2/2$ and include $2/\sigma^2$ into $\tilde{a}$. We observed that the differences are in the range between the estimated free boundary value and the final free boundary value. They are caused by the time-dependent movement of the free boundary position.

5. Conclusion

The numerical results give a clear evidence that using a non-constant penalty parameter δ is both feasible and beneficial. Future work will focus on the inclusion of the free boundary movement, a deterministic penalty function and on the extension to multi-asset American options.

Table 1: Numerical results of the corresponding parameter sets.

Example	T	K	r	σ	N	$\tilde{a}(\times 10^{-4})$	S_f	MSE
1	3	100	0.08	0.2	1000	7.5	81.87	9.6×10^{-3}
					2500	7.4	82.00	6.2×10^{-3}
2	1	1	0.1	0.2	1000	10.2	0.862	5.2×10^{-5}
					2500	10.0	0.863	3.3×10^{-5}
3	0.05	10	0.1	0.25	1000	8.0	9.158	3.5×10^{-5}
					2500	8.5	9.142	3.5×10^{-5}
4	0.1	100	0.1	0.3	1000	5.5	86.59	1.2×10^{-3}
					2500	5.4	86.87	7.6×10^{-4}
5	1	100	0.1	0.4	1000	2.6	66.49	1.4×10^{-2}
					2500	2.63	66.60	9.2×10^{-3}
6	0.05	50	0.1	0.4	1000	3.0	42.61	3.8×10^{-4}
					2500	3.1	42.61	2.5×10^{-4}

Acknowledgements

The authors were supported by the German-Slovakian Project *ENANEFA – Efficient Numerical Approximation of Nonlinear Equations in Financial Applications* financed by the DAAD and the Slovakian Ministry of Education.

References

[1] C. W. Cryer, The solution of a quadratic programming problem using systematic overrelaxation, SIAM J. Control 9 (3) (1971) 385–392.

[2] B. F. Nielsen, O. Skavhaug, A. Tveito, Penalty and front-fixing methods for the numerical solution of American option problems, J. Comput. Fin. 5 (4) (2002) 69–97.

[3] S. Wang, X. Q. Yang, K. L. Teo, Power penalty method for a linear complementarity problem arising from American option valuation, J. Optim. Theor. Appl. 129 (2) (2006) 227–254.

[4] R. Stamicar, D. Ševčovič, J. Chadam, The early exercise boundary for the American put near expiry: Numerical approximation, Canad. Appl. Math. Quaterly 7 (4) (1999) 427–444.

[5] S.-P. Zhu, A new analytical approximation formula for the optimal exercise boundary of American put options, Int. J. Theor. Appl. Fin. 09 (07) (2006) 1141–1177.

[6] M. Lauko, D. Ševčovič, Comparison of numerical and analytical approximation of the early boundary of American put options, The ANZIAM Journal 51 (4) (2010) 430–448.

[7] M. Günther, A. Jüngel, Finanzderivate mit MATLAB: Mathematische Modellierung und numerische Simulation, 2nd Edition, Vieweg, 2010.