Bergische Universität Wuppertal

Fakultät für Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics
(IMACM)

Preprint BUW-IMACM 19/38

Christian Budde, Retha Heymann

Extrapolation of operator-valued multiplication operators

December 16, 2019

http://www.math.uni-wuppertal.de

EXTRAPOLATION OF OPERATOR-VALUED MULTIPLICATION OPERATORS

CHRISTIAN BUDDE AND RETHA HEYMANN

Abstract. We discuss L^p fiber spaces which appear, e.g., as extrapolation spaces of unbounded multiplication operators which in turn are motivated, for instance, by non-autonomous evolution equations.

Introduction

Nagel-Nickel-Romanelli studied extrapolation spaces for unbounded operators in their paper [10] in Quaest. Math. in 1996. This paper extends this work by considering another scenario where extrapolation spaces occur.
It is motivated by non-autonomous Cauchy problems. Such a problem can be treated as an abstract Cauchy problem on a Banach space X. It takes the form

$$\text{(nACP)} \qquad \begin{cases} \dot{u}(t) = A(t)u(t), & t,s \in \mathbb{R},\ t \geq s, \\ u(s) = x, \end{cases}$$

where $(A(t), \mathrm{D}(A(t)))_{t\in\mathbb{R}}$ is a family of linear operators on X (see [3, Chapter VI, Section 9]). The solution of such a problem, if it exists, is given by a so-called evolution family $(U(t,s))_{t\geq s}$, see [4, Chapter VI, Def. 9.2]. One of the problems of evolution families is that, in contrast to C_0-semigroups, they do not yield a differentiable solution, e.g., it is possible that the map $t \mapsto U(t,s)x$ is differentiable only for $x = 0$. However, if we have a solution by means of an evolution family, we obtain a differentiable structure by means of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Bochner space $\mathrm{L}^p(\mathbb{R}, X)$ $(1 \leq p < \infty)$ by

$$(T(t)f)(s) := U(s, s-t)f(s-t), \quad t \geq 0,\ f \in \mathrm{L}^p(\mathbb{R}, X),\ s \in \mathbb{R}$$

(see [3, Chapter 9, Part b., especially Remark 9.12)]). One important challenge is to determine the exact domain of the corresponding generator $(G, \mathrm{D}(G))$, since it determines the space of differentiability for the semigroup, cf. [4, Chapter II, Lemma 1.3]. By [11, Thm. 3.4.7] operator-valued multiplication operators now arise naturally. Indeed, the evolution family provides a solution to (nACP) if and only if there exists an invariant core $D \subseteq \mathrm{W}^{1,p}(\mathbb{R}, X) \cap \mathrm{D}(\mathcal{A})$ with

$$Gf = \mathcal{A}f - f'$$

for $f \in D$. Here $\mathcal{A}$ is the multiplication operator on $\mathrm{L}^p(\mathbb{R}, X)$ defined by

$$(\mathcal{A}f)(\cdot) = A(\cdot)f(\cdot) \quad \text{for} \quad f \in \mathrm{L}^p(\mathbb{R}, X) \text{ with } A(\cdot)f(\cdot) \in \mathrm{L}^p(\mathbb{R}, X),$$

where $A(t), t \in \mathbb{R}$, is the family of linear operators in (nACP). One answer, in the special case where $A(t) \equiv A$ for some semigroup generator $(A, \mathrm{D}(A))$, is due to Nagel, Nickel and Romanelli [10, Sect. 4]. In [6] T. Graser studied bounded and unbounded operator-valued multiplication operators on the space of continuous functions $\mathrm{C}_0(\mathbb{R}, X)$ as well as their extrapolation spaces. We will see that extrapolation spaces of multiplication operators on $\mathrm{L}^p(\mathbb{R}, X)$ can be constructed similarly. Later on, S. Thomaschewski studied properties of such multiplication operators on Bochner L^p-spaces [11, Sect. 2.2 & 2.3] in connection with non-autonomous problems. In particular, she

2010 *Mathematics Subject Classification.* 47D06, 37B55, 47B38, 58D25.
Key words and phrases. extrapolation spaces, multiplication operators, L^p-fiber spaces, strongly continuous one-parameter operator semigroups, non-autonomous problems, evolution semigroups.
The first author was supported by the DAAD-TKA Project 308019 "*Coupled systems and innovative time integrators*".

connects multiplication semigroups with unbounded operator-valued multiplication operators. In order to construct our extrapolation spaces, the notion of fiber integrable functions is essential and leads to so-called L^p fibre spaces, see [8].
We start this paper with some preliminaries on fiber integrable functions and continue with unbounded multiplication operators in the second section. In Section 3 we discuss multiplication semigroups whose generators are multiplication operators. Furthermore, we determine the extrapolation spaces of such multiplication operators by means of L^p-fiber spaces. We remark that the results of this paper appear in similar fashion in [1, Chapter 3].

1. L^p-fiber spaces

Firstly, we recall the notion of a measurable Banach fiber bundle as it was introduced by R. Heymann, cf. [8, Def. VI.1.i]. To do so, let (Ω, Σ, μ) be a σ-finite measure space. Furthermore, let V be a complex vector space together with a family of seminorms $\{|||\cdot|||_s : s \in \Omega\}$ on V. Assume that there exists a countable set of elements $\mathcal{B} := \{b_k : k \in \mathbb{N}\} \subseteq V$ such that $\mathcal{B}$ is a vector space over $\mathbb{Q} + i\mathbb{Q}$ and such that for each $k \in \mathbb{N}$ the map $s \mapsto |||b_k|||_s$ is measurable as a map from Ω to $\mathbb{R}$. For every $s \in \Omega$ we define the set $N_s := \{b_k \in \mathcal{B} : |||b_k|||_s = 0\}$ and take the completion of the quotient space $\mathcal{B}/N_s$ with respect to the induced norms $\|\cdot\|_s$ on $\mathcal{B}/N_s$. This Banach space is denoted by X_s.

Definition 1.1. The family of Banach spaces $(X_s, \|\cdot\|_s)_{s\in\Omega}$ is called a *measurable Banach fiber bundle.*

Next, we follow [8, Def. VI.1.iii] in order to define what it means for a function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ to be measurable.

Definition 1.2. Let $(X_s, \|\cdot\|_s)_{s\in\Omega}$ be a measurable Banach fiber bundle. We define a function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ with $f(s) \in X_s$ for μ-almost every $s \in \Omega$ to be *fiber measurable* if it is almost everywhere a pointwise limit with respect to $\|\cdot\|_s$ of measurable simple functions with values in $\mathcal{B}$. More precisely, this means that there exists a sequence $(f_j)_{j\in\mathbb{N}}$ of functions $f_j : \Omega \to \bigcup_{s\in\Omega} X_s$ such that

1. $f_j = \sum_{i=1}^{n_j} \left(b_{\pi(i)} + N_s\right) \mathbf{1}_{\Omega_i}$, where $\pi : \mathbb{N} \to \mathbb{N}$, $n_j \in \mathbb{N}$, $\Omega_i \in \Sigma$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, and $b_i \in \mathcal{B}$ for each $1 \leq i \leq n_j$,
2. $f(s) = \lim_{j\to\infty} f_j(s)$ with respect to $\|\cdot\|_s$ for μ-almost every $s \in \Omega$.

The set of fiber measurable functions on Ω together with the pointwise addition and scalar multiplication is a $\mathbb{C}$-vector space, which we will call a *measurable Banach fiber bundle.*

Having the concept of measurability we continue with the notion of integrability for functions from Ω to measurable Banach fiber bundles, see [8, Def. VI.1.vi]. Especially, we define what it means to be p-integrable for $1 \leq p < \infty$. To do so, we remark that, for a fiber-measurable function f, the map $s \mapsto \|f(s)\|_s^p$ is measurable, since, by construction, the map $s \mapsto \|f(s)\|_s$ is a pointwise limit of measurable functions.

Definition 1.3. Let $1 \leq p < \infty$. We call a fiber measurable function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ *fiber p-integrable* if the integral

$$\int_\Omega \|f(s)\|_s^p \, \mathrm{d}\mu(s),$$

exists and is finite. In this case, we call

$$\|f\|_p := \left(\int_\Omega \|f(s)\|_s^p \, \mathrm{d}\mu(s)\right)^{\frac{1}{p}}$$

the L^p*-fiber norm* of f.

Remark 1.4. (i) Observe that the set of fiber p-integrable functions with pointwise addition and scalar multiplication is a vector space.

(ii) The relation defined by $f \sim g \Leftrightarrow f = g$ μ-almost everywhere is an equivalence relation on the set of fiber p-integrable functions.
(iii) The set of equivalence classes of fiber p-integrable functions with the canonical vector space structure is called a L^p-*fiber space* and is denoted by $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$.
(iv) By [8, Prop. VI.1.xi] the space $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$ is a Banach space with respect to the L^p-fiber norm.
(v) The notion of L^p-fiber spaces is closely related to the one of measurable Banach bundles, cf. [7] and [5].

2. Operator-valued multiplication operators

The main objects of this section are operator-valued multiplication operators, cf. [11, Def. 2.3.1].

Definition 2.1. Let X be a Banach space and let $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$ be a family of (possibly) unbounded linear operators on X, i.e., $M(s) : \mathrm{D}(M(s)) \subseteq X \to X$ for $s \in \Omega$. The operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ on $\mathrm{L}^p(\Omega, X)$ defined by

$$\mathrm{D}(\mathcal{M}) := \{f \in \mathrm{L}^p(\Omega, X) : \ f(s) \in \mathrm{D}(M(s)) \ \mu\text{-a.e.}, (s \mapsto M(s)f(s)) \in \mathrm{L}^p(\Omega, X)\},$$
$$(\mathcal{M}f)(s) := M(s)f(s), \quad f \in \mathrm{D}(\mathcal{M}), s \in \Omega, \ \mu\text{-almost everywhere},$$

is called the corresponding *operator-valued multiplication operator*. The operators $(M(s), \mathrm{D}(M(s)))$, $s \in \Omega$, are called *fiber operators*.

As already mentioned in the beginning, the concept of unbounded multiplication operators was studied by S. Thomaschewski [11] on Bochner L^p-spaces in connection with non-autonomous Cauchy problems. Here we summarize some important results.
Firstly, the closedness of the fiber operators implies the closedness of the multiplication operator, see [11, Lemma 2.3.4].

Lemma 2.2. *If $(M(s), \mathrm{D}(M(s)))$ is closed for μ-almost every $s \in \Omega$, then $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ is closed.*

In what follows we assume that $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ is a closed operator-valued multiplication operator with closed fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$. The following result [11, Lemma 2.3.5] shows that the resolvent operator of $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ also gives rise to a multiplication operator. For this we remind the reader of the following definition [11, Def. 2.2.3]:

$$\mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X)) := \{M : \Omega \to \mathscr{L}(X) : \ s \mapsto M(s)x \in \mathrm{L}^\infty(\Omega, X) \text{ for all } x \in X\}.$$

Lemma 2.3. *Let $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ be a multiplication operator and assume that $\lambda \in \rho(\mathcal{M})$ where $\rho(\mathcal{M})$ denotes the resolvent set of $\mathcal{M}$. Then $R(\lambda, \mathcal{M})$ is a bounded multiplication operator, i.e., there exists $M \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$ such that $(R(\lambda, \mathcal{M})f)(s) = M(s)f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$.*

Unfortunately, since the proof of the above lemma uses a characterization of bounded multiplication operators, the converse cannot be proved in a similar way and has not been proved in general, cf. [11, Thm. 2.2.17]. However, the following result [11, Thm. 2.3.6] holds.

Theorem 2.4. *Let $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ be a densely defined closed von $\mathrm{L}^p(\Omega, X)$. Assume that there exists an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(\mathcal{M})$ such that for all $f \in \mathrm{L}^p(\Omega, X)$ one has $\lim_{n\to\infty} \lambda_n R(\lambda_n, \mathcal{M})f = f$. If $R(\lambda_n, \mathcal{M})$ is a bounded multiplication operator for every $n \in \mathbb{N}$, then there exists a family $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$ of densely defined closed operators on X such that $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ is a multiplication operator with fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$. Furthere there exists a μ-null-set $\mathscr{N}$ such that for every $s \in \Omega \setminus \mathscr{N}$ and for each $n \in \mathbb{N}$ one has $\lambda_n \in \rho(M(s))$.*

Finally, if $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ is already supposed to be a multiplication operator on $\mathrm{L}^p(\Omega, X)$, then the resolvent of $\mathcal{M}$ and the resolvents of the fiber operators are related by the following result [11, Prop. 2.3.7].

Proposition 2.5. *Let $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ be a closed multiplication operator with closed fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$.*
(a) *If $\lambda \in \rho(M(s))$ for μ-almost every $s \in \Omega$ and $R(\lambda, M(\cdot)) \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$, then $\lambda \in \rho(\mathcal{M})$ and $(R(\lambda, \mathcal{M})f)(s) = R(\lambda, M(s))f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$ and μ-almost every $s \in \Omega$.*

(b) *If there exists an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(\mathcal{M})$ such that for all $f \in \mathrm{L}^p(\Omega, X)$ one has $\lambda_n R(\lambda_n, \mathcal{M})f \to f$ for $n \to \infty$, then for μ-almost all $s \in \Omega$ and all $n \in \mathbb{N}$ one has $\lambda_n \in \rho(M(s))$ and $(R(\lambda_n, \mathcal{M})f)(s) = R(\lambda_n, M(s))f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$ and μ-almost every $s \in \Omega$.*

In [11, Sect. 2.2.3] S. Thomaschewski studies multiplication semigroups on $\mathrm{L}^p(\Omega, X)$. The definition is as follows.

Definition 2.6. A C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathrm{L}^p(\Omega, X)$ is called a *multiplication semigroup* if for every $t \geq 0$ the operator $\mathcal{T}(t)$ is a bounded multiplication operator, i.e., for every $t \geq 0$ there exists $T_{(\cdot)}(t) \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$ such that $(\mathcal{T}(t)f)(s) = T_s(t)f(s)$ for μ-almost every $s \in \Omega$.

By [11, Thm. 2.3.15], stated below, these multiplication semigroups have unbounded multiplication operators as generators.

Theorem 2.7. *Let $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathrm{L}^p(\Omega, X)$ such that $\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega t}$ for some $M \geq 0$, $\omega \in \mathbb{R}$ and for all $t \geq 0$. The following are equivalent.*
(a) *$(\mathcal{T}(t))_{t\geq 0}$ is a multiplication semigroup.*
(b) *$(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ is an unbounded operator-valued multiplication operator with fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$. Moreover, for μ-almost every $s \in \Omega$, $\lambda \in \rho(M(s))$ whenever $\mathrm{Re}(\lambda) > \omega$, $(R(\lambda, \mathcal{M})f)(\cdot) = R(\lambda, M(\cdot))f(\cdot)$ and $(M(s), \mathrm{D}(M(s)))$ is the generator of a C_0-semigroup $(T_s(t))_{t\geq 0}$ such that $(\mathcal{T}(t)f)(s) = T_s(t)f(s)$ for all $t \geq 0$.*

3. Extrapolation of unbounded multiplication operators

We recall the construction of extrapolation spaces of unbounded operators, as described, for example, in [4, Chapter II, Sect. 5(a)], [9] or in a more general framework in [2]: Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator $(A, \mathrm{D}(A))$. Without loss of generality one may assume that $0 \in \rho(A)$. On X one defines a new norm $\|\cdot\|_{-1}$ by

$$\|x\|_{-1} := \left\|A^{-1}x\right\|, \quad x \in X.$$

The completion of X with respect to $\|\cdot\|_{-1}$ is called the *(first) extrapolation space* and will be denoted by X_{-1}. The original space X is densely embedded in X_{-1}. By continuity one can extend the original semigroup $(T(t))_{t\geq 0}$ to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1}. The corresponding generator is denoted by $(A_{-1}, \mathrm{D}(A_{-1}))$. As a matter of fact, one obtains $\mathrm{D}(A_{-1}) = X$ and $A_{-1} : X \to X_{-1}$ becomes an isometric isomorphism.
We now consider a multiplication operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ generating a multiplication semigroup $(\mathcal{T}(t))_{t\geq 0}$, cf. Theorem 2.7. By $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$ we denote the fiber operators corresponding to $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$, i.e., $(\mathcal{M}f)(s) = M(s)f(s)$, $f \in \mathrm{D}(\mathcal{M})$ and $s \in \Omega$. By Theorem 2.7 the operator $(M(s), \mathrm{D}(M(s)))$, $s \in \Omega$, generates a C_0-semigroup $(T_s(t))_{t\geq 0}$. We assume without loss of generality that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$. The extrapolated operators will be denoted by $(M_{-1}(s), \mathrm{D}(M_{-1}(s)))$, $s \in \Omega$. The extrapolation space of $\mathrm{L}^p(\Omega, X)$ corresponding to the operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ will be denoted by $\mathcal{X} := (\mathrm{L}^p(\Omega, X))_{-1}(\mathcal{M})$. By the extrapolation procedure described above, $\mathcal{X}$ is formally given by

$$\mathcal{X} = \left\{ f \in \prod_{s\in\Omega} X_{-1,s} : \ \exists g \in \mathrm{L}^p(\Omega, X) : \ f = M_{-1}(\cdot)g \right\}$$

The norm $\|f\|_{\mathcal{X}} := \left(\int_\Omega \|f(s)\|_{-1,s}^p \ \mathrm{d}\mu(s)\right)^{1/p}$ turns $\mathcal{X}$ into a Banach space.

Lemma 3.1. *Let $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ be a multiplication operator on $\mathrm{L}^p(\Omega, X)$ with fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$. Moreover, let $(\mathcal{T}(t))_{t\geq 0}$ the multiplication semigroup of type (M, ω) generated by $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$. Moreover, denote by $(T_s(t))_{t\geq 0}$ the C_0-semigroups of type (M, ω) generated by the fiber operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$. The associated extrapolated semigroups are denoted by $(T_{-1,s}(t))_{t\geq 0}$, $s \in \Omega$. Define*

$$(\mathcal{S}(t)f)(s) := T_{-1,s}(t)f(s), \quad t \geq 0, \ f \in \mathcal{X}, \ s \in \Omega.$$

This defines a C_0-semigroup on $\mathcal{X}$ which is generated by the operator $(\mathcal{M}_{-1}, \mathrm{D}(\mathcal{M}_{-1}))$ defined by

$$(\mathcal{M}_{-1}f)(s) = M_{-1}(s)f(s), \quad \mathrm{D}(\mathcal{M}_{-1}) = \mathrm{L}^p(\Omega, X). \tag{3.1}$$

Proof. First of all, to see that $(\mathcal{S}(t))_{t\geq 0}$ is indeed a semigroup is easy, since $(T_{-1,s}(t))_{t\geq 0}$ is a semigroup for each $s \in \Omega$. As a matter of fact, $(T_{-1,s}(t))_{t\geq}$ is an extension of $(T_s(t))_{t\geq 0}$ for each $s \in \Omega$ and hence $(\mathcal{S}(t))_{t\geq 0}$ extends $(\mathcal{T}(t))_{t\geq 0}$. Since, by construction, $\mathrm{L}^p(\Omega, X)$ is dense in $\mathcal{X}$, the semigroup $(\mathcal{S}(t))_{t\geq 0}$ is strongly continuous. In order to show that the generator of $(\mathcal{S}(t))_{t\geq 0}$ is of the form mentioned in the lemma, let us denote the generator of $(\mathcal{S}(t))_{t\geq 0}$ by $(\mathcal{A}, \mathrm{D}(\mathcal{A}))$. We will show that $(\mathcal{A}, \mathrm{D}(\mathcal{A})) = (\mathcal{M}_{-1}, \mathrm{D}(\mathcal{M}_{-1}))$. Firstly, assume that $f \in \mathrm{D}(\mathcal{A})$, then since $\lambda \in \rho(M_{-1}(s))$ for all $\lambda > \omega$ and almost every $s \in \Omega$ we conclude that $\lambda \in \rho(\mathcal{M}_{-1})$. By assumption $(\mathcal{A}, \mathrm{D}(\mathcal{A}))$ is the generator of $(\mathcal{S}(t))_{t\geq 0}$ with $\|\mathcal{S}(t)\| \leq M\mathrm{e}^{\omega t}$ meaning that $\lambda \in \rho(\mathcal{A})$. From surjectivity one obtain $f \in \mathrm{D}(\mathcal{A})$, $g \in \mathcal{X}$ and $h \in \mathrm{D}(\mathcal{M}_{-1})$ satisfying the following equality

$$f = R(\lambda, \mathcal{A})g = R(\lambda, \mathcal{A})(\lambda - \mathcal{M})h = R(\lambda, \mathcal{A})(\lambda - \mathcal{A})h = h \in \mathrm{D}(\mathcal{M}_{-1}),$$

showing that $\mathcal{A} \subseteq \mathcal{M}_{-1}$. For the converse, let $f \in \mathrm{D}(\mathcal{M}_{-1})$ and observe that from the fact $f(s) \in \mathrm{D}(M_{-1}(s))$ for almost every $s \in \Omega$ one conclude that

$$(\mathcal{S}(t)f)(s) - f(s) = T_{-1,s}(t)f(s) - f(s) = \int_0^t T_{-1,s}(t)M_{-1}(s)f(s)\,\mathrm{d}\mu(s) = \int_0^t (\mathcal{S}(t)\mathcal{M}_{-1}f)(s)\,\mathrm{d}\mu(s),$$

showing that $\mathcal{M}_{-1} \subseteq \mathcal{A}$, which concludes the proof. □

The previous result shows actually that the extrapolated multiplication operator is again a multiplication operator. In this case the fiber operators are the extrapolated fiber operators $(M_{-1}(s), \mathrm{D}(M_{-1}(s)))_{s\in\Omega}$, i.e., (3.1) holds. We now characterize the space $\mathcal{X} := (\mathrm{L}^p(\Omega, X))_{-1}(\mathcal{M})$. Assume that the Banach space X we are working with is separable, i.e., there exists a countable dense set in X. Denote the extrapolation spaces corresponding to the fiber operator $(M(s), \mathrm{D}(M(s))$ by $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)$, $s \in \Omega$. The following result prepares for the extrapolation procedure. .

Lemma 3.2. *Suppose X is a separable Banach space. If $0 \in \rho(M(s))$ for almost every $s \in \Omega$ and $s \mapsto M(s)^{-1}x$ is measurable for each $x \in X$, then the family $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber bundle.*

Proof. We make use of the separability of X and take a dense countable subset of X and make a $\mathbb{Q} + i\mathbb{Q}$ vector space $\mathcal{B}$ out of it. Observe that $\mathcal{B}$ is still countable, i.e, $\mathcal{B} := \{b_k : \ k \in \mathbb{N}\} \subseteq X$. We define a family of seminorms $\{|||\cdot|||_s : \ s \in \Omega\}$ on X by

$$|||x|||_s := \left\|M(s)^{-1}x\right\|, \quad x \in X, \ s \in \Omega.$$

Then $|||\cdot|||_s$ is actually a norm on X. By the assumption $s \mapsto M(s)^{-1}x$ is measurable for each $x \in X$ and hence so is the map $s \mapsto \|b_k\|_s$ for each $k \in \mathbb{N}$. Since $N_s = \{0\}$ for each $s \in \Omega$ we obtain $\mathcal{B}/N_s = \mathcal{B}$. Finally, the completion of $\mathcal{B}$ with respect to $|||\cdot|||_s$ is just the space $X_{-1,s}$, $s \in \Omega$. By Definition 1.1 we therefore obtain that $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber bundle. □

Since we know that $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber bundle, we can consider the space of fiber p-integrable functions over this set of Banach spaces. In what follows we relate this space to the extrapolation space of $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$ with respect to the operator-valued multiplication operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$.

Theorem 3.3. *Let $1 \leq p < \infty$ and consider the unbounded multiplication operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ on $\mathrm{L}^p(\Omega, X)$, induced by the family of unbounded operators $(M(s), \mathrm{D}(M(s)))_{s\in\Omega}$ on X. Let $(M(s), \mathrm{D}(M(s)))$ be a semigroup generator for μ-almost every $s \in \Omega$. Suppose that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$ and that $s \mapsto M(s)b$ and $s \mapsto M(s)^{-1}b$ are measurable for each $b \in \mathcal{B}$. Then*

$$[\mathrm{L}^p(\Omega, X)]_{-1}(\mathcal{M}) = \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega}).$$

Proof. Let $f \in [\mathrm{L}^p(\Omega, X)]_{-1}(\mathcal{M})$ and find $g \in \mathrm{L}^p(\Omega, X)$ such that $f = \mathcal{M}_{-1}g$, where $\mathcal{M}_{-1} : \mathrm{L}^p(\Omega, X) \to \mathrm{L}^p(\Omega, X)_{-1}(\mathcal{M})$. Since g is measurable, we can find a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions approximating g pointwise, i.e.,

$$g_n := \sum_{i=1}^{m_n} x_{n,i}\mathbf{1}_{\Omega_{n,i}} \text{ and } g_n \to g \ \mu\text{-almost everywhere.}$$

Without loss of generality, we assume that $x_{n,i} \in \mathcal{B}$ for every $i = 1, 2, \dots m_n$ and $n \in \mathbb{N}$. In order to show that $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$ define $f_n := \mathcal{M}_{-1}g_n$. Then

$$f_n(s) := (\mathcal{M}_{-1}g_n)(s) = \sum_{i=1}^{m_n} M_{-1}(s)x_{n,i}\mathbf{1}_{\Omega_{n,i}}(s),$$

where we use Lemma 3.1 as well as the fact that $M_{-1}(s)x_{n,i} \in X_{-1,s}$, $s \in \Omega$. Since the function $\Omega \to X; s \mapsto M(s)b$ is measurable for each $n \in \mathcal{B}$, it is easy to see that f_n is fiber-measurable for each $n \in \mathbb{N}$.
Furthemore,

$$\|f\|^p_{\mathrm{L}^p(\Omega,(X_{-1,s})_{s\in\Omega})} = \int_\Omega \|f(s)\|^p_{-1,s} \ \mathrm{d}s = \int_\Omega \|g(s)\|^p \ \mathrm{d}s = \|g\|^p_{\mathrm{L}^p(\Omega,X)} < \infty,$$

showing that f is a fiber p-integrable function, i.e., $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$.

For the converse inclusion, suppose that $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$. We have to show that there exists $g \in \mathrm{L}^p(\Omega, X)$ such that $f = \mathcal{M}_{-1}g$. Since f is fiber measurable, there exists a sequence $(f_j)_{j\in\mathbb{N}}$ of simple functions $f_j : \Omega \to \bigcup_{s\in\Omega} X_{-1,s}$ with $f(s) \in X_{-1,s}$ for μ-almost every $s \in \Omega$ and

$$f_j = \sum_{i=1}^{n_j} b_{k_i}\mathbf{1}_{\Omega_i},$$

where $b_{k_i} \in \mathcal{B}$, $\Omega_i \in \Sigma$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, for $1 \leq i \leq n_j$, and

$$\|f(s) - f_j(s)\|_{-1,s} \to 0, \tag{3.2}$$

for $j \to \infty$ and μ-almost every $s \in \Omega$. By the assumption that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$ we conclude by Proposition 2.5 that $0 \in \rho(\mathcal{M})$. So we define

$$g_j := (\mathcal{M}_{-1}^{-1}f_j)(\cdot) = \sum_{i=1}^{n_j} \left(M_{-1}^{-1}(\cdot)b_{k_i}\right)\mathbf{1}_{\Omega_i}(\cdot).$$

We observe that $M_{-1}^{-1}(s)b_{k_i} \in X$ for μ-almost every $s \in \Omega$ and hence g_j is a simple function. By (3.2) we conclude that $(g_j(s))_{j\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_s$ for μ-almost every $s \in \Omega$ and hence convergent. This yields a measurable function $g : \Omega \to X$ by taking the pointwise limit, i.e.,

$$g(s) := \lim_{j\to\infty} g_j(s), \quad s \in \Omega.$$

By the continuity of $M_{-1}(s)$, $s \in \Omega$, on X and the fact that $M_{-1}(s)M_{-1}^{-1}(s) = \mathrm{I}$ for μ-almost every $s \in \Omega$, we directly obtain that $\mathcal{M}_{-1}g = f$. Moreover,

$$\|g\|^p_{\mathrm{L}^p(\Omega,X)} = \int_\Omega \|g(s)\|^p \ \mathrm{d}s = \int_\Omega \|f(s)\|^p_{-1,s} \ \mathrm{d}s = \|f\|^p_{\mathrm{L}^p(\Omega,(X_{-1,s})_{s\in\Omega}} < \infty,$$

and therefore $g \in \mathrm{L}^p(\Omega, X)$. □

As a direct consequence we recover the following result, which has previously been used in in the continuous setting in [10, Sect. 4].

Corollary 3.4. *Let $(A, \mathrm{D}(A))$ be a generator of a C_0-semigroup on a Banach space X and denote ist first extrapolation space by X_{-1}^A. Define the operator $(\mathcal{M}, \mathrm{D}(\mathcal{M}))$ on $\mathrm{L}^p(\Omega, X)$ by*

$$(\mathcal{M}f)(s) := Af(s), \quad f \in \mathrm{D}(\mathcal{M}) := \mathrm{L}^p(\Omega, \mathrm{D}(A)), \ s \in \Omega.$$

Then $(\mathrm{L}^p(\Omega, X))_{-1}(\mathcal{M}) = \mathrm{L}^p(\Omega, X_{-1}^A)$.

Acknowledgement

We would like to thank Bálint Farkas for a discussion in which the idea for the collaboration which resulted in this paper was initiated. Furthermore, we are indepted to Rainer Nagel for helpful feedback.

References

[1] C. Budde. *General Extrapolation Spaces and Perturbations of Bi-Continuous Semigroups*. PhD thesis, Bergische Universität Wuppertal, 2019.

[2] C. Budde and B. Farkas. Intermediate and extrapolated spaces for bi-continuous operator semigroups. *Journal of Evolution Equations*, 19(2):321–359, Jun 2019.

[3] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[4] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[5] I. Ganiev and G. S. Mahmuod. The Bochner integral for measurable sections and its properties. *Ann. Funct. Anal.*, 4(1):1–10, 2013.

[6] T. Graser. Operator multipliers generating strongly continuous semigroups. *Semigroup Forum*, 55(1):68–79, Jan 1997.

[7] A. E. Gutman. Banach bundles in the theory of lattice-normed spaces. II. Measurable Banach bundles. *Siberian Adv. Math.*, 3(4):8–40, 1993. Siberian Advances in Mathematics.

[8] R. Heymann. *Multiplication operators on Bochner spaces and Banach fibre spaces*. PhD thesis, Eberhard-Karls-Universität Tübingen, 01 2015.

[9] R. Nagel. Extrapolation spaces for semigroups. *RIMS Kôkyûroku*, 1009:181–191, 1997.

[10] R. Nagel, G. Nickel, and S. Romanelli. Identification of extrapolation spaces for unbounded operators. *Quaest. Math.*, 19(1-2):83–100, 1996.

[11] S. Thomaschewski. *Form methods for autonomous and non-autonomous Cauchy problems*. PhD thesis, Universität Ulm, 2003.

North-West University, Department of Mathematics, Unit for BMI, Potchefstroom 2531, South Africa
E-mail address: `christian.budde@nwu.ac.za`

Stellenbosch University, Mathematics Division, Faculty of Science, Merriman Avenue, 7600 Stellenbosch, South Africa
E-mail address: `rheymann@sun.ac.za`