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Birgit Jacob, Sebastian Möller and Christian Wyss

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Stability radius for infinite-dimensional interconnected systems

Birgit Jacob[a,*], Sebastian Möller[a], Christian Wyss[a]

[a] *School of Mathematics and Natural Sciences, University of Wuppertal, Wuppertal, Germany*

Abstract

The stability radius for finitely many interconnected linear exponentially stable well-posed systems with respect to static perturbations is studied. If the output space of each system is finite-dimensional, then a lower bound for the stability radius in terms of the norm of the corresponding transfer functions is given. Moreover, for regular linear systems with zero feedthrough operator and finite-dimensional output spaces a formula for the stability radius is developed.

Keywords: Stability radius, well-posed linear systems, interconnected systems, static perturbations.

1. Introduction

This paper is concerned with a finite number of exponentially stable well-posed linear systems Σ_i, $i = 1, \ldots, N$, which are interconnected by a given structure. For the notion of well-posed linear system we refer the reader to Section 4. Let $\Sigma := \operatorname{diag}(\Sigma_1, \ldots, \Sigma_N)$, which is again an exponentially stable well-posed linear system. Clearly, Σ represents the uncoupled system. We assume that the magnitudes of the couplings between the systems Σ_i are uncertain. The matrix $\mathcal{E} = (e_{ij}) \in \mathbb{R}^{N\times N}$ describes the structure and the strength of the interconnection of the systems, that is, the entry e_{ij} of $\mathcal{E}$ can be interpreted as the strength of the connection of the output of system Σ_j to the input of system Σ_i. If $e_{ij} = 0$, then the output of Σ_j does not influence Σ_i. We denote the input and output of the system Σ_i by u_i and y_i, respectively, and consider interconnections of the form

$$u_i(t) = \sum_{j=1}^{N} \Delta_{ij} e_{ij} y_j(t).$$

*corresponding author

Email addresses: bjacob@uni-wuppertal.de (Birgit Jacob), smoeller@uni-wuppertal.de (Sebastian Möller), wyss@math.uni-wuppertal.de (Christian Wyss)

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where Δ_{ij} are unknown linear, bounded operators, describing the magnitude of the coupling. For short we write $\Delta = (\Delta_{ij})$ and we denote the interconnected system by $\Sigma^{\Delta\circ\mathcal{E}}$.

It is easy to see, that for operators Δ which are small in some norm, the interconnected system $\Sigma^{\Delta\circ\mathcal{E}}$ is again an exponentially stable well-posed linear system. In applications it is natural to ask for the largest bound $r > 0$ such that exponential stability is preserved for all magnitudes of the coupling Δ of norm strictly less than r in a given normed perturbation set. This largest bound $r > 0$ is called the *stability radius*.

The stability radius was introduced in 1986 by Hinrichsen and Pritchard [2, 3] for finite-dimensional time-invariant systems. We note that for a fixed system the stability radius may depend on the normed set of perturbations and on the notion of stability. For finite-dimensional time-invariant systems there are formulas available for the stability radius with respect to different classes of perturbations, see [4] for a comprehensive survey. Pritchard and Townley introduced the stability radii for infinite-dimensional systems [9, 10]. The concept of the stability radius for interconnected systems was introduced by Hinrichsen and Pritchard in [5] for the finite-dimensional case. In this paper we consider interconnected infinite-dimensional systems and characterize the stability radius $r(\Sigma, \mathcal{E})$, which is given by

$$\begin{aligned} r(\Sigma,\mathcal{E}) = \sup\{r > 0 \mid \Delta\circ\mathcal{E} &\text{ is an admissible feed-}\\ &\text{back and } \Sigma^{\Delta\circ\mathcal{E}} \text{ is exponentially stable}\\ &\text{for all } \Delta \text{ with } \|\Delta\| < r\}. \end{aligned}$$

Here the notion of an admissible feedback guarantees that $\Sigma^{\Delta\circ\mathcal{E}}$ is again a well-posed linear system, see Section 4. Moreover, the perturbation class under consideration (in particular the choice for the norm of Δ) will be introduced in Section 5.

The main result of this paper is as follows. Let

$$\Theta = \left[\sup_{\omega\in\mathbb{R}} \rho\left(\begin{bmatrix} \|\mathbf{G}_1(i\omega)\|^2 & & 0 \\ & \ddots & \\ 0 & & \|\mathbf{G}_N(i\omega)\|^2 \end{bmatrix} \mathcal{E}^{\circ 2}\right)\right]^{\frac{1}{2}},$$

where $\rho(\cdot)$ denotes the spectral radius and $\mathbf{G}_i$ is the transfer function of system Σ_i.

Theorem 1.1. *If Σ is an exponentially stable well-posed linear system and each system Σ_i has a finite-dimensional output space, then*

$$r\left(\Sigma,\mathcal{E}\right) \geq \frac{1}{\Theta},$$

If additionally Σ is a regular linear system with feedthrough operator 0, then the

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stability radius satisfies

$$r(\Sigma, \mathcal{E}) = \frac{1}{\Theta}.$$

This theorem comprises the results of Theorems 5.3 and 6.1. We note that Theorem 1.1 is even new in the finite-dimensional setting as we study the stability radius also for a norm that is different to the one in [5]. Further, Theorem 1.1 generalizes the main result of [9] slightly as in [9] by assumption system Σ has feedthrough operator 0.

We proceed as follows. In Section 2 we summarize some known facts on positive matrices and Hardy spaces. The spectrum of multiplication operators are studied in Section 3. A short introduction to well-posed linear and regular systems is given in 4. The main results of this paper are formulated and proved in Sections 5 and 6.

2. Preliminaries

In this section we give a short summary of results concerning the spectrum and the spectral radius for non-negative matrices as well as operator matrices which are useful for the proof of the main results of this paper. Moreover, we summarize some results on Hardy spaces.

A matrix $A = (a_{ij})_{i,j\in\{1,\dots,N\}} \in \mathbb{R}^{N\times N}$, $N \in \mathbb{N}$, is called *non-negative* (we briefly write $A \in \mathbb{R}^{N\times N}_{\geq 0}$), if $a_{ij} \geq 0$ for every entry a_{ij} of A. Further, A is called *positive* ($A \in \mathbb{R}^{N\times N}_{>0}$) if $a_{ij} > 0$ for every entry a_{ij} of A. Similarly, we define non-negative (resp. positive) vectors and denote the set of such vectors by $\mathbb{R}^N_{\geq 0}$(resp. $\mathbb{R}^N_{>0}$). For a matrix $A \in \mathbb{R}^{N\times N}$, the *spectrum* $\sigma(A)$ is defined by $\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda$ is an eigenvalue of $A\}$ and $\varrho(A) := \mathbb{C} \setminus \sigma(A)$ denotes the *resolvent set* of A. Further, we denote the *spectral radius* by $\rho(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$.

Lemma 2.1 (Perron-Frobenius [7, Chapter 8])**.** *Suppose that $A, B \in \mathbb{R}^{N\times N}_{\geq 0}$. Then the following results hold:*

(i) $\rho(A) \in \sigma(A)$ and there exists a non-negative eigenvector z of A corresponding to the eigenvalue $\rho(A)$. The vector z is called Perron vector*.*

(ii) If there exists $\alpha \geq 0$ and $z \in \mathbb{R}^N_{\geq 0}$, $z \neq 0$, such that $Az \geq \alpha z$, then $\rho(A) \geq \alpha$.

(iii) If there exists $\beta \geq 0$ and $z \in \mathbb{R}^N_{>0}$ such that $Az \leq \beta z$, then $\rho(A) \leq \beta$.

(iv) If $A \leq B$, i.e. $a_{ij} \leq b_{ij}$ for $i, j \in \{1, \dots, N\}$ with $A = (a_{ij})$ and $B = (b_{ij})$, then $\rho(A) \leq \rho(B)$.

For $A, B \in \mathbb{R}^{N\times N}$ the *Hadamard product* $\circ$ of A and B is defined by

$$A \circ B = \begin{bmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN} \end{bmatrix} \circ \begin{bmatrix} b_{11} & \dots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \dots & b_{NN} \end{bmatrix}$$

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$$:= \begin{bmatrix} a_{11} \cdot b_{11} & \dots & a_{1N} \cdot b_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} \cdot b_{N1} & \dots & a_{NN} \cdot b_{NN} \end{bmatrix}$$

as the componentwise multiplication of the entries of both matrices. For $A \circ A$ we briefly write $A^{\circ 2}$.

Let X and Y be complex Banach spaces. We denote the set of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. The operator norm of an operator $A \in \mathcal{L}(X, Y)$ is denoted by $\|\cdot\|$. In the case of $X = Y$ we briefly write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. For an operator $A \in \mathcal{L}(X)$ we denote its spectrum by $\sigma(A)$, its approximate point spectrum by $\sigma_{\text{app}}(A)$, its resolvent by $\varrho(A)$ and its spectral radius by $\rho(A)$.

Suppose that $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$ are complex Banach spaces and $A = (A_{ij})_{i,j \in \{1,\dots,N\}} \in \mathcal{L}(X, Y)$ with $X = \bigoplus_{i=1}^N X_i$, $Y = \bigoplus_{i=1}^N Y_i$ and $A_{ij} \in \mathcal{L}(X_j, Y_i)$. A matrix of this form is called an *operator matrix*.

Let $\mathcal{E} = (e_{ij}) \in \mathbb{R}_{\geq 0}^{N \times N}$ and A an operator matrix. The Hadamard product of A with $\mathcal{E}$, denoted by $\circ$, is defined via

$$\circ : \mathcal{L}(X, Y) \times \mathbb{R}_{\geq 0}^{N \times N} \to \mathcal{L}(X, Y)$$

$$A \circ \mathcal{E} = \begin{bmatrix} e_{11} A_{11} & \dots & e_{1N} A_{1N} \\ \vdots & \ddots & \vdots \\ e_{N1} A_{N1} & \dots & e_{NN} A_{NN} \end{bmatrix}.$$

Let H be a complex separable Hilbert space and Z a Banach space. For $\omega \in \mathbb{R}$ let $\mathbb{C}_\omega = \{s \in \mathbb{C} \mid \text{Re}(s) > \omega\}$. We define the *Hardy spaces* $H^2(H)$ and $H^\infty(Z)$ by

$$\mathcal{H}^2(H) := \{f : \mathbb{C}_0 \to H \mid f \text{ is holomorphic and } \|f\|_2^2 = \sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(x + i\omega)\|^2 d\omega < \infty\}$$

and

$$H^\infty(\mathbb{C}_\omega; Z) := \{G : \mathbb{C}_\omega \to Z \mid G \text{ is holomorphic and } \sup_{Re(s)>\omega} \|G(s)\| < \infty\}$$

Lemma 2.2 ([8])**.** *$\mathcal{H}^2(H)$ has the following properties:*

- *For each $f \in \mathcal{H}^2(H)$ there exists a unique function $\tilde{f} \in L^2((-i\infty, i\infty); H)$ such that*
$$\lim_{x \searrow 0} f(x + i\omega) = \tilde{f}(i\omega)$$
for almost all $\omega \in \mathbb{R}$,
- $\lim_{x \searrow 0} \|f(x + \cdot) - \tilde{f}(\cdot))\|_{L^2((-i\infty, i\infty); H)} = 0$

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- *The mapping $f \to \tilde{f}$ is linear, injective and satisfies*

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\tilde{f}(i\omega)\|^2 d\omega.$$

- *$\mathcal{H}^2(H)$ is a Hilbert space with the inner product*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \tilde{f}(i\omega), \tilde{g}(i\omega)\rangle d\omega.$$

- *Let $f \in \mathcal{H}^2(H)$ be a function different from the zero function. Then $\tilde{f}$ is non-zero almost everywhere on the imaginary axis.*

For a function $f \in L^2([0,\infty); H)$ the *Laplace transform* is a function $\hat{f}$ defined by

$$\hat{f}(s) = \int_0^{\infty} f(t)e^{-st}dt, \quad s \in \mathbb{C}_0.$$

Theorem 2.3 (Paley-Wiener theorem [12])**.** *The Laplace transform is an isometric isomorphism from $L^2([0,\infty); H)$ to $\mathcal{H}^2(H)$.*

3. The spectrum of a multiplication operator

Let (X, Σ, μ) be a σ-finite measure space and $q : X \to \mathbb{C}^{N\times N}$ a measurable matrix-valued function. The operator A_q on $L^2(X; \mathbb{C}^N)$, defined by $A_q : f \mapsto qf$, i.e.

$$(A_q f)(s) = q(s)f(s), \quad s \in X,$$

for $f \in D(A_q) = \{f \in L^2(X; \mathbb{C}^N) \mid qf \in L^2(X; \mathbb{C}^N)\}$ is called a matrix multiplication operator. We have the following relation between the spectrum of the multiplication operator A_q and the pointwise computed spectra of the function q.

Proposition 3.1 ([6, Prop. 1])**.** *For $X = i\mathbb{R}$, μ the Lebesgue-measure and q a continuous and bounded matrix-valued function on X, we have*

$$\sigma(A_q) = \overline{\bigcup_{\omega\in\mathbb{R}} \sigma(q(i\omega))}.$$

Let now $q \in \mathcal{H}^\infty(\mathbb{C}_{-\delta}; \mathbb{C}^{N\times N})$, where $\delta > 0$, and let $Y = \mathbb{C}^N$. Let A_q be the multiplication operator on $L^2(i\mathbb{R}; Y)$, defined as above. Note that $A_q \in \mathcal{L}(L^2(i\mathbb{R}; Y))$ since q is bounded. Let $\tilde{A}_q \in \mathcal{L}(\mathcal{H}^2(Y))$ be the operator of multiplication by q on $\mathcal{H}^2(Y)$, i.e.

$$(\tilde{A}_q f)(s) = q(s)f(s), \quad s \in \mathbb{C}_0,$$

for $f \in \mathcal{H}^2(Y)$. Then the following result holds.

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Lemma 3.2. $\sigma_{\mathrm{app}}(\tilde{A}_q) \subseteq \sigma(A_q)$.

Proof. Consider the mapping $j : \mathcal{H}^2(Y) \to L^2(i\mathbb{R}; Y)$, $jf = \tilde{f}$ introduced in Lemma 2.2. Note that $\|jf\|_{L^2(i\mathbb{R};Y)} = \sqrt{2\pi}\|f\|_2$. The diagram

$$\begin{array}{ccc} \mathcal{H}^2(Y) & \xrightarrow{\tilde{A}_q} & \mathcal{H}^2(Y) \\ \big\downarrow j & & \big\downarrow j \\ L^2(i\mathbb{R};Y) & \xrightarrow{A_q} & L^2(i\mathbb{R};Y) \end{array}$$

is commutative since for $f \in \mathcal{H}^2(Y)$

$$\begin{aligned} (A_q\tilde{f})(i\omega) &= q(i\omega) \lim_{r\searrow 0} f(r+i\omega) \\ &= \lim_{r\searrow 0} q(r+i\omega) f(r+i\omega) \\ &= (j\tilde{A}_q f)(i\omega). \end{aligned}$$

Let $\lambda \in \sigma_{\mathrm{app}}(\tilde{A}_q)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^2(Y)$ such that $\|f_n\|_2 = 1$ for $n \in \mathbb{N}$ and $\|(\lambda - \tilde{A}_q)f_n\|_2 \to 0$ for $n \to \infty$. Then $g_n = \frac{1}{\sqrt{2\pi}} jf_n$ satisfies $\|g_n\|_{L^2(i\mathbb{R};Y)} = 1$ and

$$\begin{aligned} \|(\lambda - A_q)g_n\|_{L^2(i\mathbb{R};Y)} &= \frac{1}{\sqrt{2\pi}}\|(\lambda - A_q)jf_n\|_{L^2(i\mathbb{R};Y)} \\ &= \frac{1}{\sqrt{2\pi}}\|j\lambda f_n - j\tilde{A}_q f_n\|_{L^2(i\mathbb{R};Y)} \\ &= \|\lambda f_n - \tilde{A}_q f_n\|_2 \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence $\lambda \in \sigma_{\mathrm{app}}(A_q) \subseteq \sigma(A_q)$. □

4. On well-posed and regular systems

In this section we provide a brief review of well-posed linear as well as regular linear systems. For more results we refer the reader to [11, 13]. Let Z be a Banach space. For two functions $u, v \in L^2([0,\infty); Z)$ we define the τ-concatenation $\diamond$ by

$$u \underset{\tau}{\diamond} v = \mathbf{P}_{[0,\tau]}u + \mathbf{S}_\tau v,$$

where $\mathbf{P}_{[0,\tau]}$ denotes the truncation of the function $u \in L^2([0,\infty); Z)$ to the interval $[0,\tau]$ while $\mathbf{S}_\tau$ denotes the operator of right shift by τ.

Definition 4.1 (Well-posed linear system)**.** *Let X, U and Y be Banach spaces. A* well-posed linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{L})$ *on (X, U, Y) is a family of operators, where $\mathbb{T} = (T(t))_{t\geq 0}$ is a C_0-semigroup on X, $\Phi = (\Phi_t)_{t\geq 0}$ with*

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$\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X)$ *such that*

$$\Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = T(t)\Phi_\tau u + \Phi_t v$$

for $u, v \in L^2([0,\infty);U), t, \tau \geq 0$, $\Psi = (\Psi_t)_{t\geq 0}$ *with* $\Psi_t \in \mathcal{L}(X, L^2([0,\infty);Y))$ *such that*

$$\Psi_{\tau+t}x = \Psi_\tau x \underset{\tau}{\diamond} \Psi_t T(\tau)x$$

for $x \in X, t, \tau \geq 0, \Psi_0 = 0$, *and*
$\mathbb{L} = (\mathbb{L}_t)_{t\geq 0}$ *with* $\mathbb{L}_t \in \mathcal{L}(L^2([0,\infty);U), L^2([0,\infty);Y))$ *satisfying*

$$\mathbb{L}_{t+\tau}(u \underset{\tau}{\diamond} v) = \mathbb{L}_\tau u \underset{\tau}{\diamond} (\Psi_t \Phi_\tau u + \mathbb{L}_t v)$$

for $u, v \in L^2([0,\infty);U), t, \tau \geq 0, \mathbb{L}_0 = 0$.

We call X the *state space*, U the *input space* and Y the *output space* of Σ. The operators Φ_t are called *input operators*, the operators Ψ_t are called *output operators* whereas the operators $\mathbb{L}_t$ are called *input-output operators*. Given an initial state $x_0 \in X$ and an input $u \in L^2_{\text{loc}}([0,\infty);U)$ the state and output trajectories $x : [0,\infty) \to X$ and $y : [0,\infty) \to Y$ of Σ are defined by

$$\begin{bmatrix} x(t) \\ \mathbf{P}_{[0,t]}y \end{bmatrix} = \Sigma_t \begin{bmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{bmatrix} \tag{1}$$

with

$$\Sigma_t = \begin{bmatrix} T(t) & \Phi_t \\ \Psi_t & \mathbb{L}_t \end{bmatrix}.$$

Let X_1 be the space $D(A)$ equipped with norm $\|x\|_1 = \|(\beta I - A)x\|$, where A is the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ and $\beta \in \varrho(A)$ fixed. The space X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$. Then $X_1 \subseteq X \subseteq X_{-1}$ are continuous dense embedded. Note that different β yield equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_{-1}$. Each operator $T(t)$, $t \geq 0$, can be uniquely extended to a linear bounded operator on X_{-1}. We denote this extension by $T_{-1}(t)$ and we note that $\mathbb{T}_{-1} = (T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}. The generator A_{-1} of $\mathbb{T}_{-1}$ has domain $D(A_{-1}) = X$ and is an extension of A to X_{-1}.
For a well-posed linear system Σ there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$, the *control operator* of Σ, such that for every $t \geq 0$ the operator Φ_t can be represented via

$$\Phi_t u = \int_0^t T_{-1}(t-s)Bu(s)ds. \tag{2}$$

We remark that the integration in (2) is in X_{-1}, but the value of the integral is an element of X. Moreover, we obtain the existence of a unique operator $\Psi_\infty : X \to L^2_{\text{loc}}([0,\infty);Y)$, the *extended output operator*, with $\mathbf{P}_{[0,t]}\Psi_\infty = \Psi_t$ for every $t \geq 0$. It can be shown that there exists a unique operator $C \in \mathcal{L}(X_1, Y)$

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such that

$$(\Psi_\infty x_0)(t) = CT(t)x_0, \qquad x_0 \in X_1.$$

Similarly, there exists a uniquely determined operator $\mathbb{L}_\infty : L^2_{\text{loc}}([0,\infty);U) \to L^2_{\text{loc}}([0,\infty);Y)$, the *extended input-output operator*, such that $\mathbf{P}_{[0,t]}\mathbb{L}_\infty = \mathbb{L}_t\mathbf{P}_{[0,t]}$ for $t \geq 0$. Using the Laplace transform we are able to represent $\mathbb{L}_\infty$ by the *transfer function* $\mathbf{G}$ of Σ. Let $\omega > \omega_0(\mathbb{T})$, where $\omega_0(\mathbb{T})$ denotes the growth bound of the C_0-semigroup $\mathbb{T}$ and $L^2_\omega([0,\infty);U) = e_\omega L^2([0,\infty);U)$ with $(e_\omega)v(t) = e^{\omega t}v(t)$ and $\|e_\omega v\|_{L^2_\omega} = \|v\|_{L^2}$. $\mathbf{G}$ is a bounded analytic $\mathcal{L}(U,Y)$-valued function on $\mathbb{C}_\omega$. For $s \in \mathbb{C}_{\omega_0(\mathbb{T})}$ and $u \in L^2_\omega([0,\infty);U)$ the Laplace-integral of $\mathbb{L}_\infty u$ at s converges absolutely and

$$(\widehat{\mathbb{L}_\infty u})(s) = \mathbf{G}(s)\hat{u}(s)$$

for $Re(s) > \omega$. Furthermore, for $\alpha, \beta \in \mathbb{C}_{\omega_0(T)}$ the transfer function $\mathbf{G}$ satisfies

$$\begin{aligned}
&\mathbf{G}(\alpha) - \mathbf{G}(\beta) \\
&= (\beta - \alpha)C(\beta I - A_{-1})^{-1}(\alpha I - A_{-1})^{-1}B \\
&= C\left((\alpha I - A_{-1})^{-1} - (\beta I - A_{-1})^{-1}\right)B.
\end{aligned}$$

If the C_0-semigroup $(T(t))_{t\geq 0}$ is exponentially stable, i.e. $\omega_0(\mathbb{T}) < 0$, then $\Psi_\infty \in \mathcal{L}(X, L^2([0,\infty);Y))$ and $\mathbb{L}_\infty \in \mathcal{L}(L^2([0,\infty);U), L^2([0,\infty);Y))$ with

$$\|\mathbb{L}_\infty\| = \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(i\omega)\| \tag{3}$$

using the Paley-Wiener theorem and the maximum modulus principle.

Remark 4.2. *Under the assumption that the C_0-semigroup of the well-posed linear system is exponentially stable, the operator*
$\Phi_\infty : L^2_{\text{loc}}([0,\infty);U) \to L^2_{\text{loc}}([0,\infty);X)$*, defined by*

$$(\Phi_\infty u)(t) := \Phi_t u,$$

satisfies

$$\Phi_\infty \in \mathcal{L}(L^2([0,\infty);U), L^2([0,\infty);X)).$$

Indeed, considering $\Sigma(\mathbb{T},\Phi,\Psi,\mathbb{L})$ with an exponentially stable C_0-semigroup it is, as mentioned above, true that $\mathbb{L}_\infty \in \mathcal{L}(L^2([0,\infty),U), L^2([0,\infty),Y))$.
Defining the new well-posed system $\widetilde{\Sigma} = \left(\mathbb{T},\Phi,\widetilde{\Psi},\widetilde{\mathbb{L}}\right)$ with $\mathbb{T}$ and Φ as in the original system, Ψ replaced by

$$(\widetilde{\Psi}_t x)(\tau) := \mathbb{1}_{[0,t]}(\tau)T(\tau)x,$$

and $\mathbb{L}$ replaced by

$$(\widetilde{\mathbb{L}}_t u)(\tau) := \mathbb{1}_{[0,t]}(\tau)\Phi_\tau u,$$

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we get $\Phi_\infty = \widetilde{\mathbb{L}}_\infty \in \mathcal{L}(L^2([0,\infty),U), L^2([0,\infty),X))$.

A well-posed linear system together with a feedback law $u = Ky + v$, where $K \in \mathcal{L}(Y,U)$, defines a closed loop system Σ^K, see Figure 1.

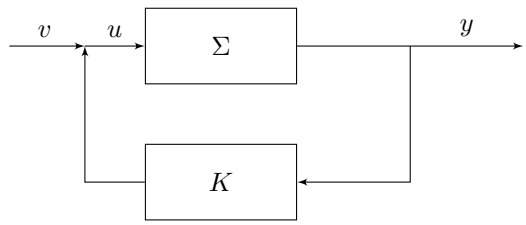

Figure 1: Closed-loop system Σ^K

K is called *admissible* if the closed-loop system Σ^K is also well-posed. If K is admissible, the closed-loop system

$$\Sigma_t^K = \begin{bmatrix} T^K(t) & \Phi_t^K \\ \Psi_t^K & \mathbb{L}_t^K \end{bmatrix}$$

fulfills (see [14])

$$\Sigma_t^K - \Sigma_t = \Sigma_t \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_t^K, \qquad t \geq 0. \tag{4}$$

For explicit formulas for $\mathbb{T}^K, \Phi^K, \Psi^K$ and $\mathbb{L}^K$ we refer to [11, Thm. 7.1.2.].
We note that if the semigroup of Σ is exponentially stable and the operator $I - \mathbb{L}_\infty K$ is bijective on $L^2([0,\infty);Y)$ then K is admissible ([11, Thm.7.1.8]). Here, the operator K in $I - \mathbb{L}_\infty K$ acts as the multiplication operator induced by K.
A special case of well-posed linear systems are *regular linear systems*. We call a well-posed linear system *regular*, if for every $v \in U$ the limit

$$Dv := \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \left(\mathbb{L}_\infty(\mathbb{1}_{[0,\infty)} v)\right)(s) ds$$

exists. Then $D \in \mathcal{L}(U,Y)$ and this operator is called the *feedthrough operator* of the system Σ.
For a well-posed linear system, we define the *Lebesgue extension* C_L of $C \in \mathcal{L}(X_1, Y)$ by

$$C_L x = \lim_{\tau \searrow 0} C \frac{1}{\tau} \int_0^\tau T(s) x ds$$

with domain $D(C_L)$ defined as the set of all $x \in X$ for which the limit exists in Y. We note that $D(C) \subseteq D(C_L)$.

Theorem 4.3 ([11, Thm. 5.6.5])**.** *If Σ is regular, then the output of Σ is given by*

$$y(t) = C_L x(t) + Du(t). \tag{5}$$

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Thus a regular linear system with generating operators A, B, C and D is completely determined by

$$\begin{aligned}\dot{x}(t) &= A_{-1}x(t) + Bu(t)\\ y(t) &= C_L x(t) + Du(t)\end{aligned}$$

Accordingly, we also denote the system by $\Sigma = (A, B, C, D)$. For $\alpha \in \mathbb{C}_{\omega_0(T)}$ the transfer function $\mathbf{G}$ satisfies

$$\mathbf{G}(\alpha) = C_L(\alpha I - A_{-1})^{-1}B + D,$$

in particular $(\alpha I - A)^{-1}Bu \in D(C_L)$ for $u \in U$.
As in the well-posed case we are interested in the closed loop system generated by an admissible feedback operator $K \in \mathcal{L}(Y, U)$. If K is admissible, then the semigroup corresponding to the closed-loop system is generated by $A^K : D(A^K) \subseteq X \to X$ with

$$A^K x = \left(A_{-1} + BK(I - DK)^{-1}C_L\right)x,\ x \in D(A^K),$$

$$\begin{aligned}D(A^K) =&\{x \in D(C_L) \mid\\ &(A_{-1} + BK(I - DK)^{-1}C_L)x \in X\}.\end{aligned}$$

5. Stability radius for well-posed systems

Let $X_1, \ldots, X_N$, $U_1, \ldots, U_N$, $Y_1, \ldots, Y_N$ be Hilbert spaces. Furthermore, we assume that $\Sigma_i = (\mathbb{T}_i, \Phi_i, \Psi_i, \mathbb{L}_i)$, $i = 1, \ldots, N$, are well-posed, exponentially stable linear systems, i.e. for $i = 1, \ldots, N$, Σ_i is a well-posed linear system and $\mathbb{T}_i$ is exponentially stable. We will now investigate the stability of interconnections of these systems. Let

$$X = \bigoplus_{i=1}^{N} X_i, \quad Y = \bigoplus_{i=1}^{N} Y_i \quad \text{and} \quad U = \bigoplus_{i=1}^{N} U_i \tag{6}$$

equipped with the norms

$$\|x\|^2 = \sum_{i=1}^{N} \|x_i\|^2,\ \|y\|^2 = \sum_{i=1}^{N} \|y_i\|^2,\ \|u\| = \max_{i=1}^{N} \|u_i\|.$$

It is easy to see that $\Sigma := (\mathbb{T}, \Phi, \Psi, \mathbb{L})$, defined by
$(\mathrm{diag}(\mathbb{T}_i), \mathrm{diag}(\Phi_i), \mathrm{diag}(\Psi_i), \mathrm{diag}(\mathbb{L}_i))$ is an exponentially stable well-posed linear system as well.
We consider interconnections of the form

$$u_i(t) = \sum_{j=1}^{N} \Delta_{ij} e_{ij} y_j(t).$$

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with $\Delta_{ij} \in \mathcal{L}(Y_j, U_i)$ and $e_{ij} \geq 0$. Setting $\mathcal{E} = \left(e_{ij}\right)_{i,j=1,\ldots,N}$, $\Delta = (\Delta_{ij})_{ij} \in \mathcal{L}(Y, U)$, $u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$ this leads to $u = (\Delta \circ \mathcal{E})\, y$ and hence to the closed-loop system $\Sigma^{\Delta \circ \mathcal{E}}$ as described in Figure 1 with $K = \Delta \circ \mathcal{E}$.
The matrix $\mathcal{E}$ describes the structure and the strength of the interconnection of the systems. More precisely, the entry e_{ij} of $\mathcal{E}$ can be interpreted as the strength of the connection of the output of system Σ_j to the input of system Σ_i. If $e_{ij} = 0$, then the output of Σ_j does not influence Σ_i.

Example 5.1 (construction of $\mathcal{E}$)**.** *Consider four subsystems with an interconnection structure given by Figure 2.*

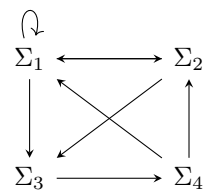

Figure 2: Example of an interconnection structure

Then the matrix $\mathcal{E}$ is given by

$$\mathcal{E} = \begin{bmatrix} e_{11} & e_{12} & 0 & e_{14} \\ e_{21} & 0 & 0 & e_{24} \\ e_{31} & e_{32} & 0 & 0 \\ 0 & 0 & e_{43} & 0 \end{bmatrix}.$$

When determining the stability of the interconnected systems we regard $\mathcal{E}$ and $\Sigma_i,\ i = 1, \ldots, N$ as fixed, while Δ is unknown. Clearly, for $\Delta = 0$ the system $\Sigma^{\Delta \circ \mathcal{E}}$ is exponentially stable and one may expect that this remains true if Δ is small.
We are interested in the largest r such that $\Sigma^{\Delta \circ \mathcal{E}}$ is exponentially stable for all Δ of norm smaller than r. This leads to the concept of the stability radius; for finite-dimensional interconnected systems it was introduced by Hinrichsen and Pritchard in [5].
We consider the stability radius for two different norms of Δ: The first is the operator norm $\|\Delta\|$, which is induced by the chosen norms on U and Y and satisfies

$$\|\Delta\| = \max_{i=1}^{N} \|\Delta_i\|,$$

where $\|\Delta_i\|$ is the operator norm of the ith row $\Delta_i = [\Delta_{i1}, \ldots, \Delta_{iN}] \in \mathcal{L}(Y, U_i)$

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of Δ. The second norm is given by

$$\|\Delta\|_{2,\infty} = \max_{i=1}^{N} \left(\sum_{j=1}^{N} \|\Delta_{ij}\|^2 \right)^{\frac{1}{2}}. \tag{7}$$

Note that both norms are related by $\|\Delta\| \leq \|\Delta\|_{2,\infty}$.

Definition 5.2. *Let*

$$\Sigma = (\mathrm{diag}(\mathbb{T}_i), \mathrm{diag}(\Phi_i), \mathrm{diag}(\Psi_i), \mathrm{diag}(\mathbb{L}_i))$$

be a well-posed, exponentially stable linear system and $\mathcal{E} = (e_{ij}) \in \mathbb{R}_{\geq 0}^{N\times N}$. *Then the* stability radius $r(\Sigma, \mathcal{E})$ *with respect to the operator norm* $\|\Delta\|$ *is defined by*

$$\begin{aligned} r(\Sigma, \mathcal{E}) = \sup\{ r > 0 \mid \Delta \circ \mathcal{E} \text{ is admissible and} \\ \Sigma^{\Delta\circ\mathcal{E}} \text{ is exponentially stable} \\ \text{for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| < r \}. \end{aligned}$$

The stability radius $r_{2,\infty}(\Sigma, \mathcal{E})$ *is defined in the same way with respect to the norm* $\|\Delta\|_{2,\infty}$.

It is easy to see that

$$r(\Sigma, \mathcal{E}) \leq r_{2,\infty}(\Sigma, \mathcal{E}).$$

Theorem 5.3. *If* $\Sigma = (\mathrm{diag}(\mathbb{T}_i), \mathrm{diag}(\Phi_i), \mathrm{diag}(\Psi_i), \mathrm{diag}(\mathbb{L}_i))$ *is a well-posed exponentially stable linear system and* Y *is finite-dimensional, then the stability radii satisfy*

$$r_{2,\infty}(\Sigma, \mathcal{E}) \geq r(\Sigma, \mathcal{E}) \geq \frac{1}{\Theta},$$

where

$$\Theta = \left[\sup_{\omega\in\mathbb{R}} \rho \left(\begin{bmatrix} \|\mathbf{G}_1(i\omega)\|^2 & & 0 \\ & \ddots & \\ 0 & & \|\mathbf{G}_N(i\omega)\|^2 \end{bmatrix} \mathcal{E}^{\circ 2} \right) \right]^{\frac{1}{2}}$$

and $\mathbf{G}_k$ *denotes the transfer function of* Σ_k.

Remark 5.4. *We use the convention* $1/0 = \infty$ *here. In particular, if* $\Theta = 0$ *then we obtain* $r(\Sigma, \mathcal{E}) = \infty$, *i.e., the system remains stable for every* $\Delta \in \mathcal{L}(Y, U)$. *We note that* $\Theta = 0$ *holds if the graph corresponding to* $\mathcal{E}$ *does not contain any cycles [5, Remark 4.6].*

Proof of Theorem 5.3. As a first step we show that for $\|\Delta\| < \frac{1}{\Theta}$ the operator $I - \mathbb{L}_\infty(\Delta\circ\mathcal{E})$ is invertible in $\mathcal{L}\left(L^2([0,\infty); Y)\right)$. By the Paley-Wiener theorem 2.3 this is equivalent to the fact that $I - \mathbf{G}(\cdot)(\Delta\circ\mathcal{E})$, with $\mathbf{G} = \mathrm{diag}(\mathbf{G}_1, \ldots, \mathbf{G}_N)$ is invertible in $\mathcal{L}\left(\mathcal{H}^2(Y)\right)$. Consider $\tilde{A}_q = \mathbf{G}(\cdot)(\Delta\circ\mathcal{E})$ as a multiplication operator on $\mathcal{H}^2(Y)$ and $A_q = \mathbf{G}(\cdot)(\Delta \circ \mathcal{E})$ as a multiplication operator on $L^2(i\mathbb{R}; Y)$.

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By Lemma 3.2 $\sigma_{\text{app}}(\tilde{A}_q) \subseteq \sigma(A_q)$. Using the assumption that Y is finite-dimensional, we can then apply Proposition 3.1 to obtain

$$\sigma_{\text{app}}(\tilde{A}_q) \subseteq \overline{\bigcup_{\omega\in\mathbb{R}} \sigma(\mathbf{G}(i\omega)(\Delta\circ\mathcal{E}))}.$$

To prove the invertibility of $I - \mathbf{G}(\cdot)(\Delta\circ\mathcal{E})$, it is now sufficient to show

$$\overline{\bigcup_{\omega\in\mathbb{R}} \sigma\left(\mathbf{G}(i\omega)(\Delta\circ\mathcal{E})\right)} \subseteq \overline{B_r(0)} \tag{8}$$

for some $r < 1$, where $B_r(0)$ denotes the ball around zero with radius r. Indeed, since $\tilde{A}_q$ is bounded and the boundary of the spectrum is contained in the approximate point spectrum, this implies $\sigma(\tilde{A}_q) \subseteq \overline{B_r(0)}$ and hence $1 \in \varrho(\tilde{A}_q)$. So let $\lambda \in \bigcup_{\omega\in\mathbb{R}} \sigma\left(\mathbf{G}(i\omega)(\Delta\circ\mathcal{E})\right)$. Then there exist $\omega \in \mathbb{R}$ and $y \neq 0$ such that

$$\lambda y = \mathbf{G}(i\omega)(\Delta\circ\mathcal{E})y.$$

Componentwise this implies for $k = 1, \dots, N$

$$\begin{aligned}
|\lambda|^2\|y_k\|^2 &= \|\mathbf{G}_k(i\omega)\sum_{j=1}^N e_{kj}\Delta_{kj}y_j\|^2 \\
&\leq \|\mathbf{G}_k(i\omega)\|^2\left\|\Delta_k\begin{bmatrix} e_{k1}y_1 \\ \vdots \\ e_{kN}y_N\end{bmatrix}\right\|^2 \\
&\leq \|\mathbf{G}_k(i\omega)\|^2\|\Delta_k\|^2\sum_{j=1}^N \|e_{kj}y_j\|^2 \\
&\leq \|\Delta\|^2\|\mathbf{G}_k(i\omega)\|^2\sum_{j=1}^N e_{kj}^2\|y_j\|^2.
\end{aligned}$$

With $z = \left(\|y_k\|^2\right)_{k=1,\dots,N}$ it follows that

$$|\lambda|^2 z \leq \|\Delta\|^2 \operatorname{diag}(\|\mathbf{G}_k(i\omega)\|^2)\mathcal{E}^{\circ 2}z.$$

Lemma 2.1 now implies

$$\frac{|\lambda|^2}{\|\Delta\|^2} \leq \rho\left(\operatorname{diag}(\|\mathbf{G}_k(i\omega)\|^2)\mathcal{E}^{\circ 2}\right) \leq \Theta^2.$$

Thus, $|\lambda| \leq \|\Delta\|\Theta$ and hence (8) holds with $r = \|\Delta\|\Theta < 1$.
Therefore $I - \mathbb{L}_\infty(\Delta\circ\mathcal{E})$ is invertible. In particular, it follows that $\Delta\circ\mathcal{E}$ is an admissible feedback operator. Moreover, the exponential stability of $(T(t))_{t\geq 0}$

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implies $\Psi_\infty \in \mathcal{L}(X, L^2([0,\infty);Y))$. Thus for every $x_0 \in X$ the equation

$$y = \Psi_\infty x_0 + \mathbb{L}_\infty(\Delta \circ \mathcal{E})y \tag{9}$$

has a unique solution $y = y_{x_0} \in L^2([0,\infty);Y)$ that fulfills

$$\|y_{x_0}\| \leq M\|x_0\| \tag{10}$$

for some $M > 0$, M independent of x_0. Thanks to (4), we have

$$y_{x_0} = \Psi_\infty^{\Delta\circ\mathcal{E}} x_0$$

for every $x_0 \in X$. Since $\Sigma^{\Delta\circ\mathcal{E}}$ is well-posed, the C_0-semigroup $(T^{\Delta\circ\mathcal{E}}(t))_{t\geq 0}$ of $\Sigma^{\Delta\circ\mathcal{E}}$ satisfies, see (4),

$$\begin{aligned} T^{\Delta\circ\mathcal{E}}(t)x_0 &= T(t)x_0 + \Phi_t((\Delta\circ\mathcal{E})\Psi_\infty^{\Delta\circ\mathcal{E}}x_0) \\ &= T(t)x_0 + \Phi_t((\Delta\circ\mathcal{E})y_{x_0}) \end{aligned}$$

$t \geq 0,\ x_0 \in X$.
Since $\mathbb{T}$ is exponentially stable, Datko's theorem (cf. [1, Thm.V.5.8.]) yields a constant $M_0 > 0$ such that

$$\int_0^\infty \|T(t)x_0\|^2 dt \leq M_0\|x_0\|^2 \tag{11}$$

for every $x_0 \in X$. Now inequality (10), Remark 4.2 and inequality (11) imply

$$\begin{aligned} &\int_0^\infty \|T^{\Delta\circ\mathcal{E}}(t)x_0\|^2 dt \\ &\leq 2\int_0^\infty \|T(t)x_0\|^2 dt + 2\int_0^\infty \|\Phi_t((\Delta\circ\mathcal{E})y_{x_0})\|^2 dt \\ &\leq 2M_0\|x_0\|^2 + 2\int_0^\infty \|\Phi_\infty((\Delta\circ\mathcal{E})y_{x_0})(t)\|^2 dt \\ &\leq 2M_0\|x_0\|^2 + 2M_1\int_0^\infty \|y_{x_0}(t)\|^2 dt \\ &\leq 2(M_0 + M_1M^2)\|x_0\|^2. \end{aligned}$$

By Datko's theorem, it follows that $(T^{\Delta\circ\mathcal{E}}(t))_{t\geq 0}$ is exponentially stable. Thus $r(\Sigma,\mathcal{E}) \geq \frac{1}{\Theta}$. □

6. Stability radius for regular systems

In this section we give an upper bound for the stability radius of N well-posed linear systems under the additional assumption that every subsystem $(\mathbb{T}_i, \Phi_i, \Psi_i, \mathbb{L}_i)$ is a regular linear system with feedthrough operator $D_i = 0$. More precisely, we consider N regular linear systems $\Sigma_j = (A_j, B_j, C_j, 0)$,

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$j = 1 \ldots, N$, of the form

$$\begin{aligned} \dot{x}_j(t) &= A_j x_j(t) + B_j u_j(t), \\ y_j(t) &= (C_j)_L x_j(t) \end{aligned} \tag{12}$$

and assume that A_j generates an exponentially stable C_0-semigroup $(T_j(t))_{t\geq 0}$ on X_j, $B_j \in \mathcal{L}(U_j, (X_j)_{-1})$ and $C_j \in \mathcal{L}(D(A_j), Y_j)$. Clearly,

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_N \end{bmatrix} : D(A) := \bigoplus_{i=1}^{N} D(A_i) \subseteq X \to X$$

generates an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on X, where X is defined by (6). Additionally we define

$$B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_N \end{bmatrix} \in L(U, X_{-1}),$$

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_N \end{bmatrix} \in \mathcal{L}(D(A), Y),$$

where U and Y are defined by (6). Note, that $X_{-1} = \bigoplus_{i=1}^{N} (X_i)_{-1}$. Then $\Sigma = (A, B, C, 0)$ is a regular linear system. With Δ, $\mathcal{E}$ and the couplings of the systems defined as in Section 5 the entire interconnected system can be represented via

$$\dot{x}(t) = A^{\Delta\circ\mathcal{E}} x(t) = (A_{-1} + B\,(\Delta \circ \mathcal{E})\, C_L)\, x(t) \tag{13}$$

with $C_L = \operatorname{diag}((C_1)_L, \ldots, (C_N)_L)$.

Theorem 6.1. *If $\Sigma = (A, B, C, 0)$ is a regular exponentially stable linear system, then the stability radii satisfy*

$$r(\Sigma, \mathcal{E}) \leq r_{2,\infty}(\Sigma, \mathcal{E}) \leq \frac{1}{\Theta},$$

with Θ defined as in Theorem 5.3. If additionally, Y is finite-dimensional, then

$$r(\Sigma, \mathcal{E}) = r_{2,\infty}(\Sigma, \mathcal{E}) = \frac{1}{\Theta}.$$

Proof. By the definition of the stability radii it suffices to show that for every $\delta \in (0, \Theta)$ there exists $\Delta \in \mathcal{L}(Y, U)$ with $\|\Delta\|_{2,\infty} \leq \frac{1}{\Theta - \delta}$ such that either $\Delta \circ \mathcal{E}$ is not an admissible feedback or $A^{\Delta\circ\mathcal{E}}$ does not generate an exponentially stable C_0-semigroup.
Let $\delta \in (0, \Theta)$ be arbitrary. Choose $\alpha \in (0, 1)$ with $\frac{\Theta - \delta}{\alpha} < \Theta$. By the definition

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of Θ there exist $\omega_0 \in \mathbb{R}$ such that

$$\begin{aligned}\lambda &:= \rho\left(\begin{bmatrix}\|\mathbf{G}_1(i\omega_0)\|^2 & & 0\\ & \ddots & \\ 0 & & \|\mathbf{G}_N(i\omega_0)\|^2\end{bmatrix}\mathcal{E}^{\circ 2}\right)\\ &\geq \left(\frac{\Theta-\delta}{\alpha}\right)^2.\end{aligned}$$

In particular $\lambda > 0$. By Lemma 2.1 there exists a vector $z \in \mathbb{R}^N_{\geq 0}, z \neq 0$, such that

$$\begin{bmatrix}\|\mathbf{G}_1(i\omega_0)\|^2 & & 0\\ & \ddots & \\ 0 & & \|\mathbf{G}_N(i\omega_0)\|^2\end{bmatrix}\mathcal{E}^{\circ 2} z = \lambda z.$$

Componentwise this reads

$$\|\mathbf{G}_k(i\omega_0)\|^2 \sum_{l=1}^{N} e_{kl}^2 z_l = \lambda z_k. \tag{14}$$

Now for every $k \in \{1,\dots,N\}$ we choose $u_k \in U_k$ such that $\|\mathbf{G}_k(i\omega_0)u_k\| \geq \alpha\|\mathbf{G}_k(i\omega_0)\|\|u_k\|$ and set $y_k = \mathbf{G}_k(i\omega_0)u_k$. If $z_k \neq 0$ then also $\|\mathbf{G}_k(i\omega_0)\| \neq 0$ by (14), and we can choose the norm of u_k such that $z_k = \|y_k\|^2$. If $z_k = 0$ then we set $u_k = y_k = 0$.
For $k, l \in \{1,\dots,N\}$ we define

$$\Delta_{kl} = \begin{cases}\frac{u_k e_{kl}\langle\cdot, y_l\rangle}{\sum_{j=1}^N e_{kj}^2\|y_j\|^2} & \text{if } \sum_{j=1}^N e_{kj}^2\|y_j\|^2 \neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

Next we set $x_k = (i\omega_0 I - (A_k)_{-1})^{-1} B_k u_k$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_N\end{bmatrix}$. Note that $x_k \in D((C_k)_L)$ and $(C_k)_L x_k = \mathbf{G}_k(i\omega_0)u_k = y_k$. Hence $x_k \neq 0$ if $z_k \neq 0$ and thus $x \neq 0$.
Now we show that

$$(A_k)_{-1}x_k + B_k \sum_{l=1}^{N} e_{kl}\Delta_{kl}\,(C_l)_L\, x_l = i\omega_0 x_k \tag{16}$$

for all $k \in \{1,\dots,N\}$. If $\sum_{j=1}^N e_{kj}^2\|y_j\|^2 \neq 0$, then indeed

$$\begin{aligned}&(A_k)_{-1}x_k + B_k \sum_{l=1}^{N} e_{kl}\Delta_{kl}\,(C_l)_L\, x_l\\ &= (A_k)_{-1}x_k + B_k \sum_{l=1}^{N} e_{kl}\frac{u_k e_{kl}\|y_l\|^2}{\sum_{j=1}^N e_{kj}^2\|y_j\|^2}\end{aligned}$$

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$$= (A_k)_{-1}x_k + B_k u_k = i\omega_0 x_k.$$

If $\sum_{j=1}^N e_{kj}^2 \|y_j\|^2 = 0$, then $\Delta_{kl} = 0$ for all $l \in \{1, \dots, N\}$. Moreover $z_k = 0$ by (14), which in turn implies $x_k = 0$ by construction. Hence (16) holds too. We have thus shown that

$$(A_{-1} + B(\Delta \circ \mathcal{E})C_L)x = i\omega_0 x. \tag{17}$$

Since $i\omega_0 x \in X$ it follows that $x \in D(A^{\Delta\circ\mathcal{E}})$. If $\Delta \circ \mathcal{E}$ is admissible, equation (17) implies that $A^{\Delta\circ\mathcal{E}}$ does not generate an exponentially stable C_0-semigroup. It remains to show that $\|\Delta\|_{2,\infty} \leq \frac{1}{\Theta-\delta}$.

If $z_k \neq 0$, then (14) yields $\sum_{l=1}^N e_{kl}^2 \|y_l\|^2 \neq 0$ and $\|\mathbf{G}_k(i\omega_0)\| \neq 0$, and we compute

$$\begin{aligned}
\sum_{l=1}^N \|\Delta_{kl}\|^2 &= \sum_{l=1}^N \left(\frac{\|u_k\| \cdot e_{kl}\|y_l\|}{\sum_{j=1}^N e_{kj}^2 \|y_j\|^2} \right)^2 \\
&= \sum_{l=1}^N \frac{\|u_k\|^2 \cdot e_{kl}^2 \|y_l\|^2}{\left(\sum_{j=1}^N e_{kj}^2 \|y_j\|^2\right)^2} \\
&= \frac{\|u_k\|^2}{\sum_{j=1}^N e_{kj}^2 \|y_j\|^2} \\
&\leq \frac{\|\mathbf{G}_k(i\omega_0)u_k\|^2}{\alpha^2 \|\mathbf{G}_k(i\omega_0)\|^2 \sum_{j=1}^N e_{kj}^2 z_j} \\
&= \frac{z_k}{\alpha^2 \|\mathbf{G}_k(i\omega_0)\|^2 \sum_{j=1}^N e_{kj}^2 z_j} \\
&= \frac{1}{\alpha^2 \lambda} \leq \frac{1}{(\Theta-\delta)^2}.
\end{aligned}$$

If $z_k = 0$, then $u_k = 0$ and therefore $\Delta_{kl} = 0$ for $l \in \{1, \dots, N\}$, i.e. $\sum_{l=1}^N \|\Delta_{kl}\|^2 = 0$. Consequently,

$$\|\Delta\|_{2,\infty} = \max_{k=1,\dots,N} \left(\sum_{l=1}^N \|\Delta_{kl}\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\Theta-\delta}.$$

The statement for finite-dimensional output spaces Y now follows from Theorem 5.3. □

7. Conclusion

In this paper we studied the stability radius for finitely many interconnected linear exponentially stable well-posed systems with respect to static perturbations. If the output space of each system is finite-dimensional, then we were

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able to show a sharp lower bound. Moreover, for regular linear systems with zero feedthrough operator and finite-dimensional output spaces a formula for the stability radius has been developed. An interesting problem for future research is the characterization of the real stability radius and the stability radius with respect to dynamic perturbations.

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