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Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives

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The copula technique is an essential quantitative method employed in the assessment of financial and econometric risks. This work will conduct a comparative analysis of the effect of pricing portfolio credit derivatives using various copula models. We shall utilise the Gaussian and the Student as elliptical models, as well as the Clayton, the Gumbel, and the Frank as Archimedean copulas, to model the corresponding default times. The Monte-Carlo simulations will be the benchmark in the estimation of the default times, the payment legs of the derivatives, and finally the cumulative swap premium. The research concludes by analysing inherent model risks through the conduction of some sensitivity analysis on the impact of swap parameters on the fair prices of the nth-to-default swaps. Finally, the numerical experiments which will be presented will clearly show that the choice of the copula model hugely affects the quantitative risk analysis of the portfolio.

Keywords: Copulas, Elliptical copulas, Archimedean copulas, Basket default swaps, Monte-Carlo simulations, Default times, Sensitivity analysis.

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1 Introduction

Over the years, financial institutions and industries have investigated certain avenues for increasing returns and at the same time, diversifying credit risks. To attain this objective, the class of multi-name portfolio credit derivatives (PCD), such as the Collateralized Debt Obligations (CDOs) and the Basket credit default swaps (BCDS) [35] have seemed to be suitable and practical, owing to their increased liquidity and speedy growth. A basket or portfolio of default swaps payoff when there is a credit event, such as restructuring, downgrade, bankruptcy, or any situation which affects the creditworthiness of an entity, in a portfolio of several entities. Depending on the seniority or the rank of the default protection, BCDS can be classified into first-to-default (F2D), second-to-default (S2D), or generally, the nth-to-default (n2D) basket swaps. Investors that trade the BCDS encounter correlation risk, or the likelihood of simultaneous defaults in the basket of entities, and these credit securities help in the transfer of credit risks amongst the participants. The prices of these BCDS are dependent on the default correlations of the entities in the portfolio. The credit default swaps (CDS), on the hand, restrict the default to only one entity. Investors have effectively utilised the market of credit derivatives because of the crucial roles these securities play in hedging and speculation, together with credit risk diversification. Furthermore, the valuation of multi-named credit derivatives entails the knowledge of the joint distribution of the default times, which poses a problem during the modelling, but the introduction of copulas had made it feasible.

With regards to the methodologies, the copula model has been in existence to mitigate the inconsistencies, inefficiencies, as well as, the computational costliness of the widespread structural models and the reduced form models, typically employed in the valuation of multi-name PCD. The default correlation factor, which is a crucial parameter in the PCD modelling, has motivated market participants to model credit events using the factor copula techniques. This default correlation estimates the likelihood of two entities to experience a simultaneous default. In the financial and insurance sector, copula models have been fully utilised in ensuring efficient and flexible modelling of dependence structures. Peng and Qi (2017) stated that the copula and the survival copula models had been recommended in Basel III and Solvency II, for both the banking sectors and the insurance industries, respectively. There are lots of copula models, but not all of them are applicable in modelling PCDs. As one significant drawback, Watts (2016) explained that for independent variables with a correlation of +1 or -1, the corresponding copula used to model the variables are inherently inappropriate for real-life applications.

An increased amount of research has been channelled to the valuation of financial credit derivatives, primarily the CDOs and the basket default swaps, in the field of copula models. The first implementation of copula functions in the modelling of financial credit derivatives was by Li (2000), who used the market prices of CDS to model the default dependency structures. Practically, he focused on pricing the CDS and the F2D swaps using the Gaussian copula model. Fathi and Nader (2007) further focused on the impact of dependency structure and the choice of simulation methodology to value the n2D swap and CDO. Mashal and Naldi (2002) considered the n2D and the CDOs pricing using a hybrid of the structural and the reduced form techniques as default models. They further studied the student-t copula, which is a class of elliptical copulas with larger tail dependency structure. In the presence of stochastic recovery rates, the student-t copula has been employed by Goegebeur et al. (2007) in the modelling of default time dependencies and thus, they applied the concept to the valuation of the synthetic CDOs. Hull and White (2004) and Andersen et al. (2003) both developed analytical methodologies which are based on copula functions, to value credit derivatives. The former incorporated a recursion method to value n2D swaps and the latter introduced an improved recursion-based technique which derives the portfolio loss distribution.

Furthermore, Choe and Jang (2011b) proposed an importance sampling algorithm, in connection with the nested Gumbel copula and the exchangeable Archimedean copulas, to value the BDS. To achieve this, they established a multi-level dependence structure in the case of nested copulas, to model the credit risks of a portfolio which consists of sub-portfolios. Importance sampling techniques were also employed by Schröter and Heider (2013b) to quantify the credit model risk of a given portfolio, and they further presented an analytical formula to value the n2D swaps [31]. Burtschell et al. (2005) compared different copula functions such as, Gaussian, student-t, double-t, Clayton and Marshall-Olkin copulas, to model the structure of the joint default of default times which is based on the factor model. They compared the concept of semi-analytical pricing to the approximation techniques associated with large portfolio securities, and subsequently estimated the premiums of the BCDS and CDOs. However, the appropriate choice of copula model remains an unavoidable question in financial modelling, as Durrleman et al. (2000) presented few methodologies for the right choice within the family of the Archimedean copulas. In each family, they considered the estimation of parameters, like the maximum likelihood, information matrix systems, dependence measures for the parametric families; together with the non-parametric estimation, which are all rooted on the Deheuvels or the empirical copula models. They concluded by proposing an appropriate selection criteria for the optimal copula to be considered.

Our research, nonetheless, is prompted by the interplay of various copula models and the default time modelling with regards to their tangible applications in PCD valuations. It is vital to have substantial knowledge of the effects of different copula modelling to the pricing of the basket credit derivatives. Hence, this research will focus on a comparative methodology on the valuation of the BCDS, and will further discuss the impact of modelling the default time parameter using different elliptical and Archimedean copula methods. To ensure clarity and simplicity in the comparative study, we will focus on a set of homogeneous basket, and further assume that parameters such as the hazard rate, concordance measures (ρ and θ), time to expiration, recovery rate, and interest rate, are all kept constant irrespective of the copula models employed. The study will also present some numerical experiments, in accordance with the same Kendall's tau (τ) concordance structure to estimate the prices of the n2D swaps. We will equally mention the concept of ordered statistic employed by Choe and Jang (2011a), to numerically compute the corresponding payment legs associated with the pricing of BDS. Furthermore, we will conclude the study by estimating the prices of the n2D swaps based on the Monte-Carlo methods, as well as conducting some sensitivity analysis with regards to the default swap parameters.

This structure of this study is as follows: In Section 1, we introduce a brief description of the topic and outline some of the recent related studies on the pricing of basket credit derivatives. Section 2 introduces the structure of the model, which is the copula model and highlights its applicability in the derivatives pricing. Section 3 introduces the concept of the valuation of BCDS, explains how default time can be modelled via different copula methods, and outlines how the swap premium can be obtained. It further introduces the payment legs associated with swap spread, and then discussed the Monte-Carlo method, as the simulation technique employed in the research. Section 4 focuses on results obtained in the numerical experiments of the BCDS pricing, together with some sensitivity analysis on the default swap spreads. Section 5 concludes our research study.

2 Model Structure

In this section, we introduce and employ the copula method, together with the Monte-Carlo simulations to model default time parameters. Copulas are generally classified into one-factor and two-factor models. The former consists of the Gaussian and the Archimedean copula, whereas the latter comprises of student-t, Frechet and Marshall Olkin models. This study will consider five copula models: Gaussian, Archimedean (Clayton, Gumbel and Frank), and the student-t copula.

2.1 Introduction to copula models

A copula function joins the univariate distributions and thus, generates multivariate distribution functions.

Definition 2.1 (d-copula, [14, 34]) *A copula $C = C(u_1, u_2, \dots, u_d)$ is a multivariate probability distribution, possessing uniform marginals in a unit interval. Mathematically, it is defined by $C : [0, 1]^d \to [0, 1]$, having the following properties:*

- $C(u_1, u_2, \dots, u_d) = 0$ *if* $u_j = 0$*, for any* $j \leq d$*.*
- $C(1, 1, \dots, 1, u_j, 1, \dots, 1) = u_j$*, for every* $j \leq d$ *and for all* $u_j \in [0, 1]$*;*
- $C(u_1, u_2, \dots, u_d)$ *is d-increasing.*

In the analysis of financial and insurance risks, the concept of quantifying dependence has been a significant issue, and this had sparked the interest of developing copula models. An essential mathematical theory in connection with the copula models is the Sklar's theorem, which essentially explains that any multivariate probability distribution can be disintegrated into the components of a copula and its univariate margins.

Conversely, a combination of some given margins together with a corresponding copula can be integrated back into a multivariate distribution.

Theorem 2.2 (Sklar's Theorem, [30]) *Let $\mathbf{F} = (F_1, F_2, \dots, F_d)$ be a d-dimensional probability distribution function having marginals $F_1, F_2, \dots, F_d$. Then there exists a d-copula function C, with uniform marginals such that for all $\mathbf{x} \in \mathbb{R}^d$,*

$$\mathbf{F}(x_1, x_2, \dots, x_d) = C\big(F_1(x_1), F_2(x_2), \dots, F_d(x_d)\big)\,.$$

Moreover, if the marginal distribution functions $F_1, F_2, \dots, F_d$ are continuous then the copula function C is unique.Thus, instead of specifying a multivariate distribution $\mathbf{F}$, the dependency structure can be modelled by identifying the marginal function F_d and a copula C. It should be noted, however, that Sklar's theorem only guarantees that the copula exists, but the main issue lies on the specific choice of a suitable copula function.

A consequence of this is that copulas enable multivariate distribution to be expressed in terms of their marginal distributions, and it has contributed immensely in capturing correlation structures. Many copula models are implemented in practice; the following subsection will graphically discuss the structures of some copula models, as well as, their tail dependencies.

2.2 Copula Models

In this section, we introduce a graphical description of the random variables generated by various copula models. This part will also show the effect of using the copula models in analysing the tail dependency structures of random variables. The tail dependency is noteworthy to be included because they are significant factors in measuring the probability of simultaneous extreme losses, specifying the quantity of joint dependence in the tail of the distribution function, and ignoring them will underestimate the default risk premium [26].

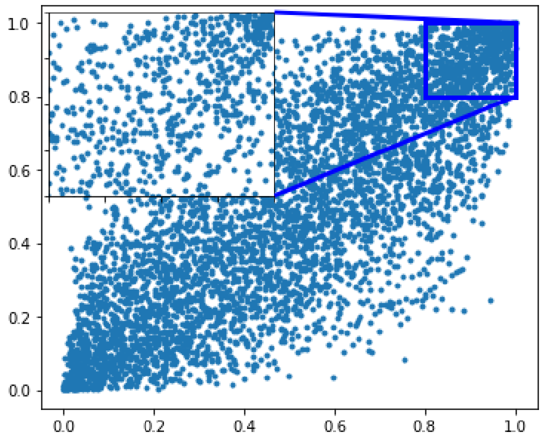
2.2.1 Elliptical copula

Definition 2.3 (Elliptical copula, [11]) *A d-variate family of copula is said to be elliptical if it is written in the form*

$$C(u_1, u_2, \dots, u_d; \Sigma) = \Psi^d_\Sigma\big(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \dots, \Psi^{-1}(u_d)\big)\,,$$

where Ψ^d_Σ is a d-dimensional multivariate ζ distribution with Σ as the correlation matrix; and Ψ^{-1} denotes an inverse of the univariate ζ distribution. The distribution ζ corresponds to either standard normal distribution or the Student-t distribution (t_β), with β degrees of freedom (df). The elliptical copulas are also known as inversion-method copulas.

(a) The Gaussian copula: 5000 random numbers, with 0.8 correlation.

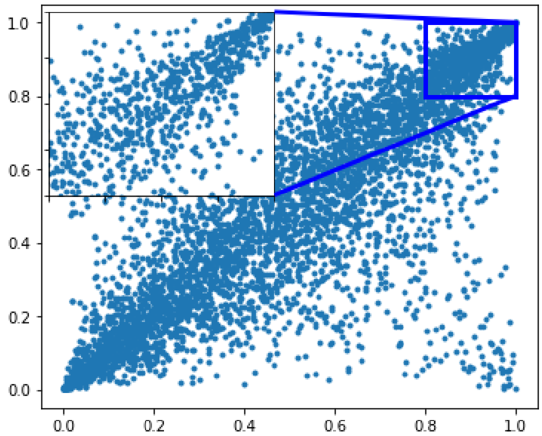


The Gaussian copula is given by $C_\rho^\Sigma(\mathbf{u}) =$

$$\frac{1}{2\pi\sqrt{|\Sigma|}}\int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2|\Sigma|}\right)\mathrm{d}\mathbf{x}\,,$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{x} = (x_1, x_2)$, $\rho \in (-1, 1)$ is the correlation parameter in the 2×2 matrix Σ, and Φ^{-1} is the inverse of a univariate standard normal distribution [34, 28]. For any correlation parameter, the tail dependency is zero [33, p. 176], as can be seen in the corresponding figure.

(b) Student-t copula: 5000 random points with 0.8 correlation, and 1 degree of freedom.



The student-t copula is given by $C_{\rho,\beta}^\Sigma(\mathbf{u}) =$

$$\frac{1}{2\pi\sqrt{|\Sigma|}}\int_{-\infty}^{t_\beta^{-1}(u_1)}\int_{-\infty}^{t_\beta^{-1}(u_2)}\left(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\beta|\Sigma|}\right)^{-\frac{\beta+2}{2}}\mathrm{d}\mathbf{x}\,,$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{x} = (x_1, x_2)$, $\rho \in (-1, 1)$ is the correlation parameter in the 2×2 matrix Σ, and t_β^{-1} is the inverse of a univariate t_β with β df [34, 28]. The upper and lower tail dependency are symmetric, given by $2t_{\beta+1}\left(-\sqrt{(\beta+1)\left(\frac{1-\rho}{1+\rho}\right)}\right)$.

2.2.2 Archimedean copula

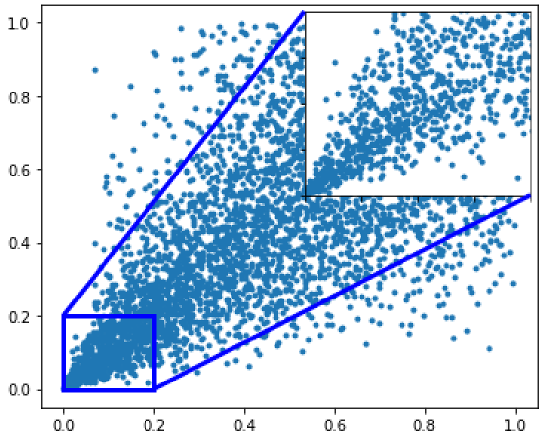
Definition 2.4 (Archimedean copula, [28]) *A d-variate copula family is said to be Archimedean if it is written in the form*

$$C(u_1, u_2, \cdots, u_d) = \psi^{-1}\big(\psi(u_1) + \psi(u_2) + \cdots + \psi(u_d)\big)\,,$$

with the generator of the copula $\psi(x)$, satisfying the following conditions:

- $\psi(0) = \infty$ *and* $\psi(1) = 0$.
- *The inverse function $\psi^{-1}(x)$ corresponds to a probability, such that $\psi^{-1} : [0, \infty] \to [0, 1]$.*
- *The function $\psi(x)$ is convex and decreasing, such that $x \in [0, \infty)$.*

(a) Clayton copula: 5000 Clayton random points with $\theta = 2.8820$ dependence parameter.

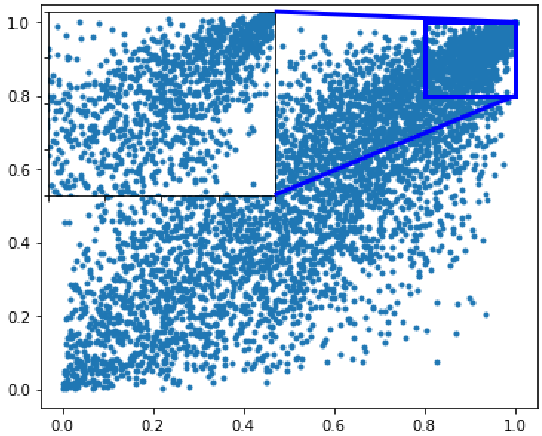


Denote the generator function as $\psi(c) = \theta^{-1}(c^{-\theta} - 1)$, where $\theta \in [-1, \infty) \setminus \{0\}$, then the bivariate Clayton copula is defined as

$$C_\theta(u_1, u_2) = \max\{[u_1^{-\theta} + u_2^{-\theta} - 1]^{-\frac{1}{\theta}}, 0\}\,;$$

for $\theta > 0$, we have $C_\theta = [u_1^{-\theta} + u_2^{-\theta} - 1]^{-\frac{1}{\theta}}$. The upper and lower tail dependencies are given as $\lambda_U = 0$ and $\lambda_L = 2^{-\frac{1}{\theta}}$ respectively. The Clayton model is beneficial for modelling data points, which are strongly correlated at the lower values and weakly correlated at the upper values.

(b) Gumbel copula: 5000 Gumbel random points with $\theta = 2.4410$ dependence parameter.

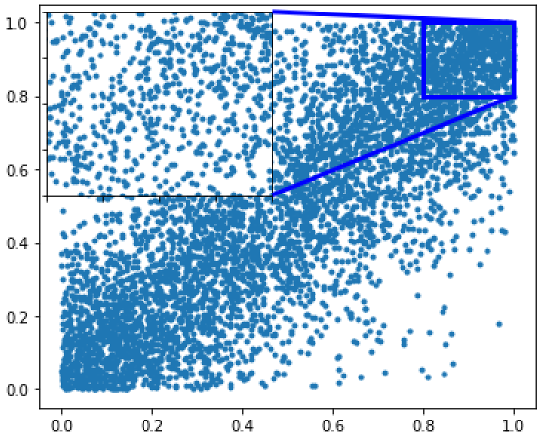


Denote the generator function as $\psi(c) = (-\log c)^\theta$, where $\theta \geq 1$, then the bivariate Gumbel copula is defined as

$$C_\theta(u_1, u_2) = \exp\{-[(-\log u_1)^\theta + (-\log u_2)^\theta]^{\frac{1}{\theta}}\},$$

for $\theta > 0$. The upper and lower tail dependencies are given as $\lambda_U = 2 - 2^{\frac{1}{\theta}}$ and $\lambda_L = 0$ respectively. The Gumbel model is beneficial for modelling data points, which are strongly correlated at the upper values and weakly correlated at the lower values.

(c) Frank copula: 5000 Frank random points with $\theta = 7.6762$ dependence parameter.



Denote the generator function as $\psi(c) = -\log[(\mathrm{e}^{-\theta c} - 1)(\mathrm{e}^{-\theta} - 1)^{-1}]$, where $\theta \in \mathbb{R} \setminus \{0\}$, then the bivariate Frank copula is defined as

$$C_\theta(u_1, u_2) = -\frac{1}{\theta} \log \left(1 + \frac{(\mathrm{e}^{-\theta u_1} - 1)(\mathrm{e}^{-\theta u_2} - 1)}{(\mathrm{e}^{-\theta} - 1)}\right).$$

The upper and lower tail dependencies are zero, and they are the only strict Archimedean copula with the so-called radial symmetry [12]. For further Further on Frank copula as radially symmetric we refer the reader to [16].

2.2.3 Concordance structures

The measures of dependence such as the Kendall's τ, the Spearman's rho, and the tail dependence coefficients are all bivariate concordance measures which are copula-based [1]. We employed the Kendall's τ concordance measure to describe the correlation among the portfolio entities, and this is because they are flexible for non-linear correlations. The Kendall's τ and their corresponding domains are given in Table 1, and the parameter θ for both the Gaussian and the Student-t copulas are the correlation coefficients. The θ parameter in the models influences the dependency structures, as an increase in θ results to the increment of the dependence. Furthermore, it is possible to have the same concordance measure for different copula models, and subsequently different n2D basket prices. In all the numerical results of this study, we used the Kendall's τ concept to estimate the θ-parameters of the Archimedean copulas, which in turn, are employed in valuing the n2D swaps.

Table 1: Concordance structure of different copula models [33, 36].

S/N	Copula type	$\tau = g(\theta)$	Domain $\tau \in \Omega$
1	Gaussian	$\frac{2}{\pi}\sin^{-1}(\theta)$	$[-1, 1]$
2	Student-t	$\frac{2}{\pi}\sin^{-1}(\theta)$	$[-1, 1]$
3	Clayton	$\frac{\theta}{\theta+2}$	$(0, 1]$
4	Gumbel	$1 - \frac{1}{\theta}$	$[0, 1]$
5	Frank	$1 - \frac{4}{\theta}\left(1 - \frac{1}{\theta}\int_0^\theta \frac{q}{\mathrm{e}^q - 1}\mathrm{d}q\right)$	$[-1, 1] \setminus \{0\}$

3 Pricing basket credit default swaps

This section will introduce and discuss the concept of basket credit default swaps (BCDSs) and default time modelling, as well as the Monte-Carlo methods used in the result section of this study.

3.1 Basket credit default swaps

Basket default swap is a financial derivative contract between the protection buyer and the protection seller which promises a payoff in the event of a credit default among the portfolio of entities. This contractual agreement ensures the transfer of credit exposure of securities from one person to another. A periodic payment (spread) is made at specified regular intervals to the protection seller till a credit event occurs, or till the contract's expiration. Basket default swaps are generally classified into all-to-default, n2D, and n-out-of-m-to-default. This study will focus on n2D basket swaps. The intuition behind the construction of BCDS can be linked with the approach of redistributing the financial credit risk of a basket of CDS. The mechanism of both CDS and BCDS are essentially the same, but the difference lies in the trigger of the credit event. Consider for example the F2D swaps, a contingent payment made to the protection buyer by the seller is triggered only when a basket of underlying credit experiences a first credit event, and in this way, the buyer is only protected against the first default.

In the valuation of the spread for the BCDS, defined in a given homogeneous portfolio, the following notations will be used: The portfolio consists of N number of reference entities, with A as the notional value of the contract. Let the time to maturity of the contract be $T = t_k$, with current time $t_0 = 0$, and define the deterministic discount factor by $f(t) = \mathrm{e}^{-r(t-t_0)}$, with r as the risk-free interest rate. $R = R_{(j)}$, for $j = 1, 2, \ldots, N$ is the homogenized recovery rate, and the rank of the swap or the seniority level is defined by n, such that upon nth default at time $\tau_{(n)}$ with $\tau_{(1)} < \cdots, < \tau_{(n)} < \cdots < \tau_{(N)}$, a default payment will be received by the protection buyer. Let DL denote the default leg, which is the payment made by the protection seller in case of a default, and it is calculated by the difference between the default payment (DP) and the accrued premium AP[1]. The frequency of the payment dates[2], payable in units of years is denoted by $\Delta = t_i - t_{i-1}$, where $i = 1, 2, \ldots, N$, and the time interval in which the nth default happened in the case of AP is given by $S = t_{i-1} < \tau_{(n)} \leq t_i$. Furthermore, let PL be the premium leg which involves the series of cash flows paid by the protection buyer until maturity or before maturity, in the case of a credit event, and finally define $\gamma_{(n)}$ as the fair price of the n2D swap.

Hence, according to Galiani (2003), the fair price under the risk-neutral pricing measure of the n2D swap is calculated by solving for $\gamma_{(n)}$ in the expression $\mathbb{E}[\mathrm{DL}]=\mathbb{E}[\mathrm{PL}]$.

[1] The accrued premium is payable from the last date the payment was made prior to default, till the time $\tau_{(n)}$ of the nth default.

[2] The frequency $\Delta = 1$ corresponds to an annual frequency, $\Delta = \frac{1}{2}$ for semi-annual frequency payment, etc, assuming we are considering a $\frac{30}{360}$ day count convention.

The present value and the expected present value for the PL are given respectively by:

$$\mathrm{PL} = \gamma_{(n)} A \sum_{i=1}^{N} \Delta f(t_i) \mathbb{I}_{\tau_{(n)} > t_i} \qquad \text{and} \qquad \mathbb{E}[\mathrm{PL}] = \sum_{i=1}^{N} \gamma_{(n)} A \Delta f(t_i)[1 - F_{(n)}(t_i)]\,.$$

The present value and the expected present value for the DP are given respectively by:

$$\mathrm{DP} = A \sum_{j=1}^{N} (1-R) f(\tau_{(n)}) \mathbb{I}_{\{\tau_{(n)} \leq T\}} \qquad \text{and} \qquad \mathbb{E}[\mathrm{DP}] = A \sum_{j=1}^{N} (1-R) \int_0^{\top} f(t) F_{(n)}^{n^{th}=j}\, \mathrm{d}t\,.$$

The present value and the expected present value for the AP are given respectively by:

$$\mathrm{AP} = \sum_{i=1}^{N} A\gamma_{(n)} \left(\tau_{(n)} - t_{i-1}\right) f(\tau_{(n)}) \mathbb{I}_{\{S\}} \quad \text{and} \quad \mathbb{E}[\mathrm{AP}] = \sum_{i=1}^{N} A\gamma_{(n)} \int_{t_{i-1}}^{t_i} (a - t_{i-1}) f(a) F_{(n)}(\mathrm{d}a)\,.$$

Under the assumptions that credit events occur at discrete dates and the outstanding debts resulting from the DP are cleared promptly upon default, then Theorem 3.1 below gives the price of the n2D swap premium:

Theorem 3.1 ([15, 13, 7]) *The risk neutral pricing measure for the fair price of the annualized n2D swap, in the presence of accrued premium, is given below:*

$$\gamma_{(n)} = \frac{\sum_{j=1}^{N} (1-R) \int_0^{\top} f(t) F_{(n)}^{n^{th}=j} \mathrm{d}t}{\sum_{i=1}^{N} \Delta f(t_i)[1 - F_{(n)}(t_i)] + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (a - t_{i-1}) f(a) F_{(n)}(\mathrm{d}a)}\,,$$

where $F_{(n)}(t) = \mathbb{P}(\tau_{(n)} \leq t)$ is the probability distribution function (PDF) of $\tau_{(n)}$, and $F_{(n)}^{n^{th}=j}(t)$ is the PDF of the nth basket of default times which is relative to the jth default.

It is, however, pertinent to model the probabilities of these default times, and the next subsection will introduce the various concepts of copula models in the generation of the default times.

3.2 Default time modelling

In pricing portfolio derivatives, copulas are essential tools in modelling entities defaults, which in turn facilitates the BCDS valuation. Define $(\Omega, \mathcal{F}, \mathbb{P})$ as a given probability space, where Ω is the sample space of all possible events defined in a horizon of finite time, $\mathcal{F}$ is the σ-algebra consisting of a set of events, and $\mathbb{P}$ is the probability measure, which under the no-arbitrage principle, becomes risk-neutral [4]. Assume that τ_i, the default time, is the time at which each entity i experiences a credit event. Let $F_i(t)$ be the risk-neutral cumulative probability distribution that each i entity will default before a given time t, that is, $F_i(t) = \mathbb{P}(\tau_i \leq t)$, and let $S_i(t) = 1 - F_i(t) = \mathbb{P}(\tau_i > t)$ be the survival probability of the ith entity. According to Kijima (2000), the PDF of default times is given by:

$$F_i(t) = \mathbb{P}(\tau_i \leq t) = 1 - \exp\left(-\int_0^t \lambda_i(u)\mathrm{d}u\right),$$

where λ_i is the instantaneous intensity rate for each entity. Laurent and Gregory (2005) simulated the default time of each reference entities by using the following expression:

$$\tau_i = -\frac{1}{\lambda_i}\ln(1 - X_i), \tag{3.1}$$

where X_i arises from the corresponding copula model.

This study will model default times using the five copula models listed in the introductory part of this work. We refer to Section 4 for some default time points of all the models used in this study. First, Li (1999) considered the case of the Gaussian copula, and he noted that the survival probability or the default probability can be employed to estimate the default times, provided that the copulas are symmetric (Gaussian and student t). Furthermore, the default times for the **Gaussian copula model** are calculated as follows:

$$\tau_i = F_i^{-1}\big(\Phi(s_i)\big),$$

where Φ is defined as the cumulative distribution function of a standard Gaussian random variable s_i, which in turn, is given by $s_i = \rho_i V + \sqrt{1-\rho_i^2}\eta_i$. The correlation term $\rho_i \in [-1, 1]$, and the independent random variables V and η_i are the single common factor and the error term respectively, which are generated from the standard Gaussian distribution.

The corresponding default time for the **student t-copula model**, on the other hand, follows a dual factor copula model, and it is given by [3]

$$\tau_i = F_i^{-1}\big(t_\beta(s_i)\big),$$

where t_β is the cumulative distribution function of a student random variables s_i, which in turn, is given by $s_i = \sqrt{V_2}(\rho_i V_1 + \sqrt{1-\rho_i^2}\eta_i)$, having β degrees of freedom. The factors V_1 and V_2 are independent random variables, together with the error term η_i. The parameter V_1 follows a normal distribution and V_2 follows an inverse gamma distribution, with equal scale and shape parameters of $\beta/2$.

The default times modelled under the Archimedean copulas are quite different from the elliptical copulas. The general form for obtaining their corresponding random variables X_i, is shown in the works of[5, 18, 25] as:

$$X_i = \psi^{-1}\Big(\frac{-\ln(k_i)}{M}\Big), \tag{3.2}$$

where the uniform random variables k_i are independent and identically distributed, i.e. $k_i \sim \mathcal{U}([0, 1])$. The function $\psi^{-1}(\cdot)$ refers to the inverse generator associated with each copula models.

Consider the **Clayton copula model**, with $\theta > 0$ parameter, as example, the random variable X_i returns a uniformly distributed random variables. The positive random variable M follows a standard gamma distribution having a unit scale parameter and $1/\theta$ shape parameter. In other words, $M \sim \Gamma(1/\theta, 1)$, and finally, the inverse generator $\psi^{-1}(c)$ for the Clayton model is calculated as

$$\psi^{-1}(c) = (1 + c)^{-1/\theta}, \qquad \theta > 0\,.$$

For the **Frank copula model**, with parameter θ, the random variable M follows a Logarithmic series on $\mathbb{N}^+$ having a parametric value of $\alpha = 1 - \mathrm{e}^{-\theta} \in (0, 1)$. In other words, $M \sim \text{Logarithmic}(1 - \mathrm{e}^{-\theta})$. The corresponding probability density function can be defined discretely as

$$\mathbb{P}(q) = \frac{-\alpha^q}{q \ln(1 - \alpha)}, \qquad \text{for } q \in \mathbb{N}\,.$$

Furthermore, the inverse generator $\psi^{-1}(c)$ for the Frank model is calculated as

$$\psi^{-1}(c) = -\frac{1}{\theta} \log\big(1 + \mathrm{e}^{-c}(\mathrm{e}^{-\theta} - 1)\big), \qquad \theta \in \mathbb{R} \setminus \{0\}\,.$$

On the other hand, the random variable M for the **Gumbel copula model** follows an $\alpha -$ stable distribution, that is, $M \sim \text{Stable}(\alpha, \beta, \gamma, \delta)$. The distribution is also known as the Lévy alpha-stable distribution, with parameters $\alpha \in (0, 2]$, as the stability parameter; $\beta \in [-1, 1]$, as the skewness parameter; $\gamma \in (0, \infty)$, as the scale parameter; and $\delta \in \mathbb{R}$, as the location parameter. Specifically, the random variables follow

$$M \sim \text{Stable}\Big(\frac{1}{\theta}, 1, \cos\Big(\frac{\pi}{2\theta}\Big)^{\theta}, 0\Big)\,.$$

The inverse generator $\psi^{-1}(c)$ for the Gumbel model is calculated as

$$\psi^{-1}(c) = \mathrm{e}^{-c^{\frac{1}{\theta}}}, \qquad \theta \geq 1\,.$$

3.3 Algorithm for default time modelling

This subsection will outline the algorithms employed in the generation of the default times for all the copula models applied in this study. The Gaussian and the student-t copula models for generating the default times involve the use of the Cholesky decomposition of the specified correlation matrix Σ, so as to obtain a lower triangular matrix L, such that $\Sigma = LL^\top$. We employ the techniques of Scherer and Mai [33]. For the generation of an α-stable distribution used in the Gumbel copula, we employ the methodologies proposed by Nolan[27] in Melchiori (2006). For generating random Logarithmic series utilized in the Frank copula, we used the `ipython numpy` inbuilt function `logseries(p)`, with the parameter $p = 1 - \mathrm{e}^{-\theta}$. Otherwise, consider the full algorithm proposed by Melchiori (2006). Here, random variables, which are logarithmically distributed were obtained from Kemp's second generator with acceleration. For further details, we refer to [9].

(a) Gaussian copula:

- Obtain the Cholesky decomposition L of Σ, such that $\Sigma = LL^\top$.
- Generate independent random variates $Z \sim \mathcal{N}(0,1)$, where $Z = (z_1, z_2, \ldots, z_n)^\top$.
- Compute $s_i = LZ$, where $s_i \sim \mathcal{N}(0, \Sigma)$.
- Return the vector $X_i = \big(\psi(s_i)\big)^\top$, with ψ denoting the distribution function associated with univariate standard normal distribution.
- Finally compute $\tau_i = F_i^{-1}(\Phi(s_i))$ with $X_i = \psi(s_i)$ in equation (3.1).

(b) Student-t copula:

- Obtain the Cholesky decomposition L of Σ, such that $\Sigma = LL^\top$.
- Generate independent random variates $Z \sim \mathcal{N}(0,1)$, where $Z = (z_1, z_2, \ldots, z_n)^\top$.
- Simulate an independent random variate $r_i \sim \chi^2(\beta)$, where β is the degrees of freedom.
- Compute $s_i = \sqrt{\frac{\beta}{r_i}} LZ$.
- Return the vector $X_i = \big(t_\beta(s_i)\big)^\top$, with t_β denoting the distribution function associated with univariate t-distribution, with zero mean.
- Finally compute $\tau_i = F_i^{-1}(\Phi(s_i))$ with $X_i = \psi(s_i)$ in equation (3.1).

(c) Clayton copula:

- Simulate a uniform random variable $k_i \sim \mathcal{U}([0,1])$.
- Generate gamma random variables $M \sim \Gamma\left(\frac{1}{\theta}, 1\right)$.
- Calculate X_i from the equation (3.2) and then, the default time from equation (3.1).

(d) Gumbel copula:

- Set $\alpha = 1/\theta, \beta = 1, \gamma = (\cos(\pi/2\theta))^\theta$ and $\delta = 0$.
- Simulate a uniform random variable $p \sim \mathcal{U}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- Simulate Q from an exponential distribution, with mean 1. Both Q and p must be independent.
- Set $\Psi = 1/\alpha(\arctan(\beta\tan(\pi\alpha)/2))$
- Compute $M \sim \text{Stable}(\alpha, \beta, \gamma, \delta)$. Here, $M = \gamma Y + \delta$, where

$$Y = \frac{\sin\alpha(\Psi + p)}{(\cos\alpha\Psi\cos p)^{\frac{1}{\alpha}}}\left[\frac{\cos(\alpha\Psi + (\alpha-1)p}{Q}\right]^{\frac{1-\alpha}{\alpha}}, \qquad \text{since } \alpha \neq 1\,.$$

- Finally, calculate the random variables X_i from equation (3.2) and then the default time τ_i follows from equations (3.1).

(e) Frank copula:

- Simulate a uniform random variable $k_i \sim \mathcal{U}([0,1])$.
- Generate log-series random variables $M \sim \text{Logarithmic}(1-\mathrm{e}^{-\theta})$. Both M and k_i must be independent.
- Calculate X_i from the equation (3.2) and then, the default time from equation (3.1).

Generally, this study employed the Monte-Carlo method in the generation of the random numbers and the default times, which correspond to different copula models. The present value of an n2D swap depends on the time in which the nth entity defaults, and so, obtaining the nth default time suffices to obtain the default times of all the entities in a given portfolio. In the valuation of an n2D BDS, the initial default time results in an (M,N) matrix, where M is the number of simulations and N, the number of reference entities. For a tabular representation of these default times, see Table 2. We implemented the concept of order statistic (See Definition 3.2) to the unordered default times of the individual entities. The implementation was achieved by categorising the N-dimensional vector of the default times in their increasing sequence of order, and this sorting is in connection to their rank of default. For instance, for the F2D swap pricing, we selected the first coordinate. Using this ordered statistic, the corresponding values for the PL, DP, AL, and finally, the fair spreads of the n2D were all computed.

Definition 3.2 (nth order statistic, [7]) *Denote $\tau_1, \tau_2, \ldots, \tau_n$ as independent and identically distributed (i.i.d.) random variables of sample size n from a continuous distribution. If $\tau_1, \tau_2, \ldots, \tau_n$ are rearranged in ascending order of magnitude, with the minimum $\tau_1 = \min_i(\tau_i)$ and the maximum $\tau_n = \max_i(\tau_i)$, then the nth order statistic is given as $\tau_1 < \tau_2 < \cdots < \tau_N$.*

4 Numerical Experiments and Sensitivity Analysis

This research uses the `ipython notebook` in the numerical computations for the prices of the n2D swaps, with respect to their varied parameters[3].

4.1 Results

Let N be the number of entities; R, the recovery rate of entities; r, the deterministic risk-free interest rate; dt, the frequency payment dates; df, the degree of freedom; M, the Monte-Carlo simulation points; mm, the number of sub-time steps within each dt for integration; ρ, the correlation coefficient; θ_C, the theta parameter for the Clayton model; θ_F, the theta parameter for the Frank model; θ_G, the theta parameter for the Gumbel model; and λ, the intensity rate. For more consistent result, we used Kendall's τ to get a rho-theta equivalent in terms of the model parameters.

[3]The full `ipython notebook` code can be assessed at `https://github.com/NnekaU/Codes/blob/master/BDS%20pricing%20using%20Elliptical%20and%20Archimedean%20copulas.ipynb`.

We first generate 50000 data points of default times which correspond to the applicable models used in this work, and the results were obtained using the algorithms in Section 3.3 above. We consider the other parameters: $\lambda = 0.04$, df $= 1$, $\rho = 0.7$, $\theta_C = 1.9497$, $\theta_F = 5.6212$, and $\theta_G = 1.9749$. Displayed in Table 2 are the outputs of the first 5 simulations, for the first-to-default BCDS consisting of 10 reference entities in each models. The results of the simulated default times for each of the five models are in the form of an (M, N) matrix.

Table 2: Simulation of default times corresponding from the copula models

Gaussian	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	23.10	7.96	16.34	9.92	4.63	21.21	16.32	33.61	3.96	17.25
[2,]	31.92	16.16	11.03	27.22	21.47	24.88	26.95	42.79	17.74	79.85
[3,]	74.10	25.75	8.49	12.13	14.82	33.24	15.22	9.80	20.46	22.05
[4,]	85.77	56.25	46.85	48.93	101.04	32.29	33.28	89.99	56.65	36.64
[5,]	38.37	41.06	95.49	38.03	82.34	28.16	53.00	72.06	23.36	95.13

Student-t	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	8.34	3.33	2.89	16.02	4.96	4.03	0.81	2.24	0.60	4.28
[2,]	18.47	5.26	16.21	14.81	10.45	5.73	0.60	35.43	1.32	8.89
[3,]	16.05	26.23	34.99	40.35	32.36	31.75	83.85	26.34	75.26	35.62
[4,]	42.73	44.37	50.13	42.55	35.64	40.09	50.97	66.74	101.81	15.58
[5,]	9.48	5.35	5.61	5.02	6.75	6.68	1.19	3.18	0.37	4.04

Clayton	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	15.85	11.48	18.98	14.65	6.37	16.47	12.44	43.91	18.75	22.73
[2,]	86.01	40.70	64.92	48.02	118.37	35.74	80.50	80.37	60.87	49.09
[3,]	6.83	0.64	6.32	0.95	1.26	2.97	2.76	1.58	3.34	0.89
[4,]	0.23	1.66	0.36	1.64	0.55	0.80	1.79	0.88	0.55	2.59
[5,]	4.33	6.77	12.43	23.87	4.41	5.49	3.42	6.85	6.74	6.59

Gumbel	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	55.87	39.76	30.02	53.88	54.85	48.49	39.11	50.29	41.25	41.15
[2,]	23.50	9.18	34.41	4.43	11.62	12.57	4.26	22.16	16.55	8.53
[3,]	8.86	6.01	12.80	7.53	1.32	9.68	1.66	27.56	10.66	12.50
[4,]	15.04	17.19	6.18	17.29	9.30	11.66	11.12	8.93	26.62	16.78
[5,]	26.09	29.51	44.83	38.42	28.11	65.95	66.55	45.22	39.59	54.65

Frank	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	1.54	15.53	21.24	1.94	0.82	9.36	1.61	1.79	2.31	0.89
[2,]	16.39	26.54	14.63	2.54	11.37	13.60	7.04	14.84	5.25	18.51
[3,]	7.75	4.81	5.93	30.32	7.59	9.21	30.80	9.25	14.21	16.49
[4,]	37.90	66.18	8.12	10.91	10.48	23.09	11.02	20.73	11.02	8.62
[5,]	14.92	21.95	12.56	32.10	20.22	22.75	19.91	66.28	9.97	25.32

One of the major objectives achieved in this study is to model default times using different copula models, and Table 2 above shows the discrepancies in the results obtained. Default times are first estimated, and then incorporated in the valuation of the n2D swaps.

The parameters in Table 3 will be used to output the values of the premium legs, survival probabilities, as well as the actual premiums for different ranks of n2D swaps.

Table 3: Parameters for the valuation of n2D swap using the same Kendall's τ

N	R	r	dt	df	M	mm	λ	θ_C	θ_F	θ_G	ρ
10	40%	5%	0.5	4	50000	10	5%	0.5895	1.2947	2.1393	0.35

Table 4: Premium legs (PL) & Survival probabilities (SP) for n2D swap using different copulas

Rank	Gaussian		Student-t		Clayton		Gumbel		Frank	
	SP	PL	SP	PL	SP	PL	SP	PL	SP	PL
1	0.2698	2.1940	0.2653	2.1177	0.3575	2.6446	0.2884	2.1666	0.3422	2.2992
2	0.4792	3.0821	0.4887	3.0853	0.5307	3.2160	0.4479	2.9724	0.4733	2.9655
3	0.6361	3.5792	0.5843	3.4343	0.6551	3.5660	0.5897	3.5058	0.5858	3.4620
4	0.7548	3.8857	0.7501	3.8681	0.7434	3.7953	0.7280	3.8888	0.6985	3.8315
5	0.8445	4.0908	0.8240	4.0614	0.8127	3.9614	0.8453	4.1362	0.8177	4.1064
6	0.9026	4.2069	0.8943	4.2083	0.8707	4.0920	0.9253	4.2714	0.9088	4.2614
7	0.9446	4.2834	0.9481	4.2983	0.9114	4.1807	0.9726	4.3369	0.9674	4.3348
8	0.9715	4.3287	0.9791	4.3441	0.9471	4.2588	0.9923	4.3613	0.9925	4.3626
9	0.9888	4.3527	0.9946	4.3635	0.9737	4.3147	0.9981	4.3670	0.9987	4.3679
10	0.9968	4.3646	0.9978	4.3669	0.9905	4.3501	0.9998	4.3688	0.9999	4.3689

Table 4 shows the increasing sequence in terms of the ranks of the n2D swaps. Here, we used five copula models to output the premium legs and the survival probabilities of each of the n2D swaps. Notwithstanding the distinct values that all the models' outputs, they generally exhibit similar characteristics with regards to the observed values. With an increase in the ranks, the survival probabilities become negatively correlated with the premium legs. The survival probability is the likelihood of each entity or entities in a portfolio to survive a credit event before a given time t. Considering the portfolio of 10 entities, for instance, the probability of having one entity out of the whole portfolio to default is highly significant, and this, in turn, implies that the survival probability of other non-defaulting entities becomes very slim. Thus, as the rank increases, there is a higher chance for a few of the entities to survive, and hence, the protection buyer will be entitled to pay higher premiums. Finally, we output the n2D swap premiums using the parameters in Table 2, and the results will be displayed in Table 5 below:

Table 5: Premiums for n2D swap using different copula models

Rank	Gaussian copula	Student-t copula	Clayton copula	Gumbel copula	Frank copula
1	1649.8	1686.7	1251.1	1623.6	1417.0
2	885.5	866.2	762.9	977.9	944.6
3	532.2	638.2	507.5	611.8	630.0
4	327.8	336.5	354.4	359.6	405.2
5	195.6	222.7	247.8	189.4	224.0
6	118.4	123.9	165.7	87.5	106.3
7	65.7	60.7	111.0	31.4	37.0
8	33.1	23.9	65.1	8.7	8.4
9	13.0	6.1	31.9	2.1	1.5
10	3.7	2.4	11.4	0.2	0.0

In Table 5, we observe a sharp decline in the premiums of the n2D swaps using different copula models, as the rank of the n2D swaps increases. For the F2D swap price, the Clayton copula model outputs a significantly least value in comparison with other models. However, the Clayton values far exceed the others when the rank moves from fifth till tenth-to-default. In Burtschell, et al. (2005), it can still be argued that for a specific value of θ, the outputs from the Clayton copula model are almost the same as the Gaussian copula, but the reverse is the case when the Kendall's τ equivalent is being used. The degree of freedom for the student-t copula significantly affects the swap prices, but when the degrees are increased, the student-t copula values tend to the Gaussian copula values. Consider for instance, when the degree of freedom increases from 4 to 15, the student-t n2D swap premiums become 1650.7, 872.3, 534.9, 322.7, 191.5, 117.1, 66.3, 33.8, 15.1 and 3.7, as the rank increases.

Furthermore, we observe from Table 5, that the choice of the copula model has a considerable effect on the value of the n2D swap. The swap prices differ using different copula models, even if the modelling assumes the same concordance structure. We note that in our case, we used the same Kendall's τ equivalent. The θ parameters of the Archimedean copula are equivalent to the ρ value in the elliptical copula. Thus, in connection to this, Schröter and Heider (2013) asserted that the choice of the copula function should be considered as part of the modelling since the function plays a determinant role in the risk profile of the BDS, and they should be chosen wisely in connection to the regulatory requirements.

4.2 Sensitivity Studies

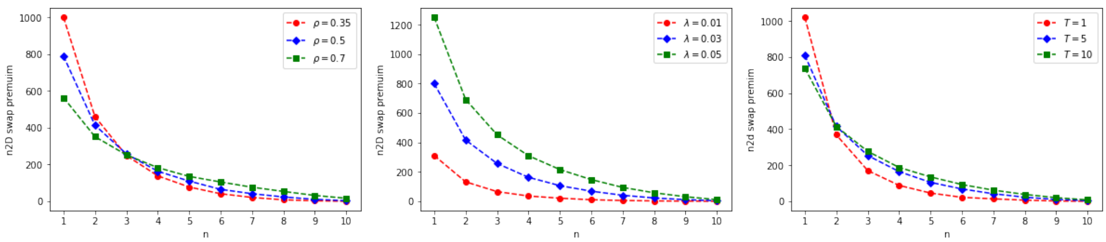
Sensitivity analysis plays a huge role in any model of derivative pricing, as they specify which trading strategy hedges the security effectively. Several model factors affect the riskiness of the portfolio and consider the F2D as an example. However, we note that the following analysis depends on the rank to default in the portfolio. That is, whether it is F2D, second-to-default (S2D), etc., The model factors include: concordance structure

(ρ and θ) - the higher the concordance rate, the lower the probability of default; hazard rates - the lesser the rates, the smaller the likelihood of default; time to maturity - the longer the maturity time, the lesser the probability of default; recovery rate - higher rate of recovery results to lower default probability; and number of entities - the bigger the basket, the more the chances of default. In this work, we shall numerically consider the variations of the $F2D$ till tenth-to-default swap premiums against their respective concordance structures, default rates and the times to expiration. Table 6 lists the parameters used in the results:

Table 6: Parameters for the sensitivity analysis of the n2D swap using the same Kendall's tau

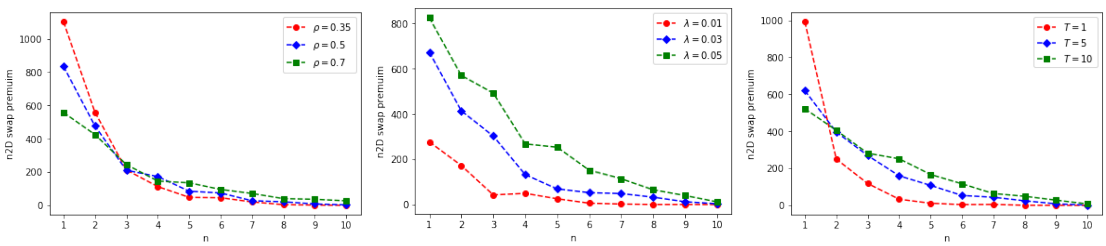
N	R	r	dt	df	M	mm	λ	θ_C	θ_F	θ_G	ρ
10	45%	3%	0.25	7	50000	10	3%	1.0000	3.3057	1.5000	0.50

The graphical description of the sensitivity analysis are depicted in Figures 1 till 5. The analysis are conducted with different default times arising from the corresponding models been used, and the explanations follow in Sections 4.2.1, 4.2.2, and 4.2.3 below.



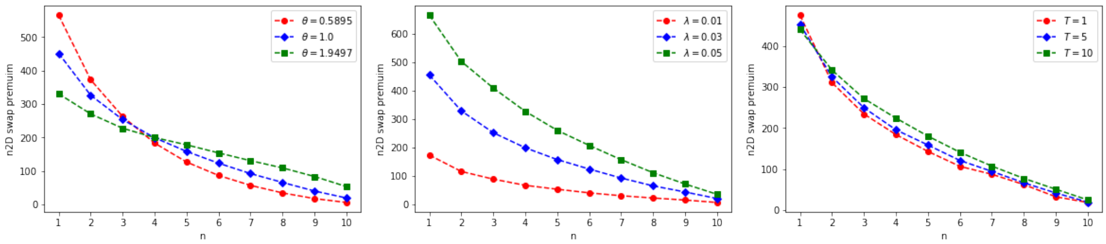
(a) ρ variations: n2D swaps vs n (b) λ variations: n2D swaps vs n (c) T variations: n2D swaps vs n

Figure 1: Gaussian copula: n2D swap prices versus the parameters ρ, λ and T respectively.



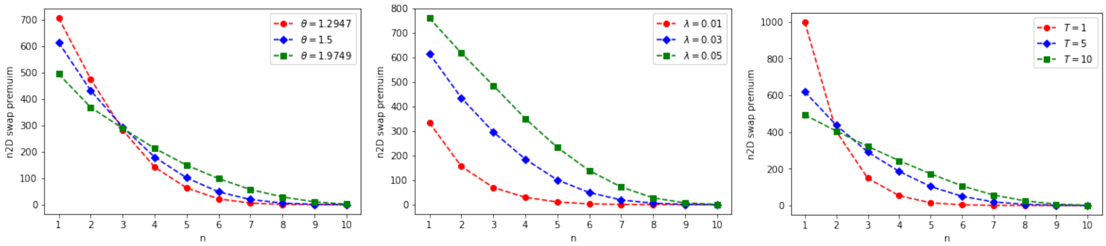
(a) ρ variations: n2D swaps vs n (b) λ variations: n2D swaps vs n (c) T variations: n2D swaps vs n

Figure 2: Student-t copula: n2D swap prices versus the parameters ρ, λ and T respectively.



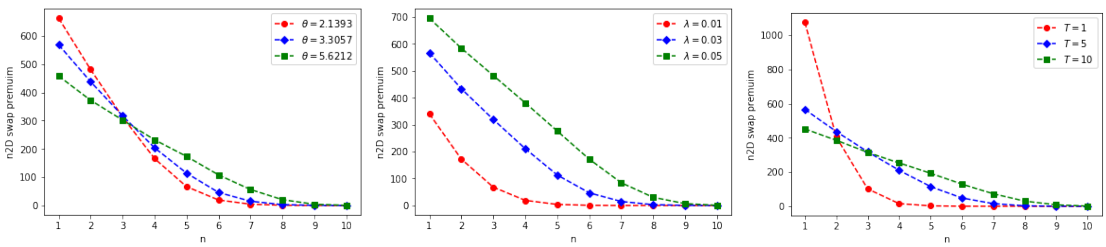
(a) θ variations: n2D swaps vs n (b) λ variations: n2D swaps vs n (c) T variations: n2D swaps vs n

Figure 3: Clayton copula: n2D swap prices versus the parameters ρ, λ and T respectively.



(a) θ variations: n2D swaps vs n (b) λ variations: n2D swaps vs n (c) T variations: n2D swaps vs n

Figure 4: Gumbel copula: n2D swap prices versus the parameters ρ, λ and T respectively.



(a) θ variations: n2D swaps vs n (b) λ variations: n2D swaps vs n (c) T variations: n2D swaps vs n

Figure 5: Frank copula: n2D swap prices versus the parameters ρ, λ and T respectively.

4.2.1 Sensitivity Analysis (w.r.t. concordance parameters)

The concordance structure is one of the essential parameters which affect the sensitivity of the n2D swap prices. For the correlation factor, we observe that higher correlation results in decline in the swap spread, and this is evident for the F2D and the S2D spread. It could be said that if an investor takes a position of a short protection, then it will be a long correlation; which obviously implies that the investor will stand to gain from an equivalent increase in the probability of no default. Furthermore, as the rank of the default increases (as from the third-to-default (T2D)), we observe that an increase in the correlation factor subsequently led to an increase in the swap premium.

The θ-parameters of the Archimedean copulas measure the tail dependency, as both the parameters and the tail dependencies are positively correlated. When the default time is modelled using the Clayton copula, we observe that as from the fourth-to-default (Fo2D) upwards, the swap prices are increased owing to a consistent rise in the θ-parameters. Similar results are seen from the T2D swap spreads when the default times are modelled using both Frank and Gumbel copulas. However, as we increase the rank further on, the swap spreads coincide at the tenth-to-default, since we considered a portfolio of 10 entities. Hence, increasing the dependency structures in the tail of the distribution, that is θ, leads to a remarkable increase in the probability of joint defaults, and in turn, the default correlation.

In general, the concordance structures affect the tail dependencies of the models, and hence, investors seeking to gain from this derivative security must choose the basket swap type and the position strategically. For instance, the correlation sensitivity has resulted in speculating the direction of the correlation in the credit derivatives market. If the investor perceives that the default correlation of entities in a portfolio will rise, then the investor can sell F2D default protection with the sole aim of making a profit should default correlation eventually increase.

4.2.2 Sensitivity Analysis (w.r.t. hazard rate)

We observed similar trends in all the models, as default time values are estimated using various copula models. A steep decline in the rank of default is observed with respect to the n2D swap prices, and the swap prices are seen to coincide if the rank is increased further to 10. There is a consistent increase in the swap spread as the default intensities of all the copula models increase. Thus, this is intuitively right because the default intensities measure the loss expectation for a greater specified dependency structure, and this requires a higher premium [20]. Furthermore, as the hazard rate increases, the chances of the entities to default become more likely, and this accounts to a more substantial premium. The behaviour of the hazard rate and the correlation coefficient are in sharp contrast because increasing both parameters result to a monotone decrease and monotone increase of the swap spreads respectively when the F2D swaps are considered.

In general, when the default intensity is high, the likelihood of joint defaults become exceedingly vast, resulting in larger spreads. On the other hand, at low default intensity or as the limit of default intensity tends to zero, the dependency structure becomes irrelevant, the spreads gradually decline, and no default will be experienced in the given portfolio.

4.2.3 Sensitivity Analysis (w.r.t. time to expiration)

We finally showed the dependence of the n2D swap spread on time to maturity of the basket. While the time to expiration of the contract increases, we observe a gradual reduction in the quarter-annual swap premium payments. This observation is evident for the F2D basket swap only, and the reverse becomes the case when the seniority of the default payment increases. Thus, for each of the models, the payments made by the protection seller upon default and the regular premium payments by the protection buyer are seen to increase simultaneously, with an increase in the contract's expiration time. This increment, in turn, lowers the cumulative swap value.

From the S2D swap and above, there will be more likelihood of joint defaults by the entities in the portfolio, provided the time horizon is extended further. The results obtained from the copula models equally showed some discrepancies, especially with regards to the swap prices when the rank of default is between 2 and 6. For a year n2D swap, the student-t copula, the Gumbel copula and the Frank copula exhibited a sharp decline in the swap premium prices, whereas, the Gaussian and the Clayton copula showed a steady decrease in the swap prices.

5 Conclusion

In this paper, we considered a comprehensive analysis of the effect of valuing portfolio credit derivatives, especially the n2D basket swaps, using both elliptical and Archimedean copula models. Copula models are essential tools in the modelling of correlated defaults, which are evident in the pricing of a portfolio of credit derivatives. We employed the Monte-Carlo simulation in connection to the swap premium prices using various copula models, such as Gaussian, student-t, Clayton, Gumbel, and the Frank copula models to estimate the default times first. Most credit default events are captured as tail events by nature; we further compared the results obtained using both non-tail distributions (Gaussian and Frank) and fat tail distributions (Gumbel, Clayton, and student-t). The study summed up analysing the inherent model risks, and this was achieved by conducting some sensitivity analysis, with regards to the concordance structures, hazard rates and time to expiration of the contract. We further showed some graphical illustrations on how these model parameters hugely affect the prices of the swap contracts.

The significant contribution of this study lies in conducting comparative statics on the estimation of n2D basket swap prices, using various copula models. Furthermore, to ensure efficient comparative statics, we introduced the Kendall's τ concordance measures to model the dependency structures of the underlying entities, even though the corresponding results produced different swap values for the models used. Additionally, having an in-depth knowledge of the sensitivity studies can assist investors and financial participants in hedging and speculating the direction of the market. The concordance measures of the portfolio should be utilised to determine the risk associated with the choice of a specific copula, as this ensures the dependency or the joint dependencies associated within the portfolio. In conclusion, we can infer that the choice of copula used to model default time contributes immensely in the riskiness of a credit portfolio of multiple entities.

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