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ERGODICITY OF AFFINE PROCESSES ON THE CONE OF SYMMETRIC POSITIVE SEMIDEFINITE MATRICES

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ABSTRACT. This article investigates the long-time behavior of conservative affine processes on the cone of symmetric positive semidefinite $d \times d$-matrices. In particular, for conservative and subcritical affine processes on this cone we show that a finite log-moment of the state-independent jump measure is sufficient for the existence of a unique limit distribution. Moreover, we study the convergence rate of the underlying transition kernel to the limit distribution: firstly, in a specific metric induced by the Laplace transform and secondly, in the Wasserstein distance under a first moment assumption imposed on the state-independent jump measure and an additional condition on the diffusion parameter.

1. INTRODUCTION

An affine process on the cone of symmetric positive semidefinite $d \times d$-matrices $\mathbb{S}_d^+$ is a stochastically continuous Markov process taking values in $\mathbb{S}_d^+$, whose log-Laplace transform depends in an affine way on the initial state of the process. *Affine processes* on the state space $\mathbb{S}_d^+$ are first systematically studied in the seminal article of Cuchiero *et al.* [11]. In their work, the generator of an $\mathbb{S}_d^+$-valued affine process is completely characterized through a set of *admissible parameters*, and the related *generalized Ricccati equations* are investigated. Subsequent developments complementing the results of [11] can be found in [30, 36, 37, 38]. Note that the notion of affine processes is not restricted to the state space $\mathbb{S}_d^+$. For affine processes on other finite-dimensional cones, particularly the canonical one $\mathbb{R}_+^m \times \mathbb{R}^n$, we refer to [2, 5, 6, 12, 13, 14, 26, 30, 32]. We remark that the above list is, by far, not complete.

The importance of $\mathbb{S}_d^+$-valued affine processes has been demonstrated by their rapidly growing applications in mathematical finance. In particular, they provide natural models for the evolution of the covariance matrix of multi-asset prices that exhibit random dependence, for instance, the Wishart process [9], the jump-type Wishart process [34], and a certain class of matrix-valued Ornstein-Uhlenbeck processes driven by Lévy subordinators [7]. Among them, the Wishart process is the most popular one, and it has been successfully applied to generalize the well-known Heston model [24] to multi-asset setting, see also [3, 8, 10, 15, 19, 20, 21, 22, 23]. The jump-type Wishar process as introduced by Leippold and Trojani [34] allows jumps which help the model to fit better to real world interest rates or volatility of multi-asset prices. In [34] the jump-type Wishart process is used in multi-variate option pricing, fixed-income models and dynamic portfolio choice. For a more detailed review on financial application of affine processes on $\mathbb{S}_d^+$ we refer to the introduction of [11], see also the references therein.

In this article we investigate the long-time behavior of affine processes on $\mathbb{S}_d^+$. First, we study the existence of limit distributions for these processes. This problem was studied for particular $\mathbb{S}_d^+$-valued affine models by Alfonsi *et al.* [1] in the case of Wishart processes, while Barndorff-Nielsen and Stelzer [7] studied matrix-valued Ornstein-Uhlenbeck processes driven by

Lévy subordinators. Our main result (see Theorem 2.5 below) is applicable to general conservative, subcritical affine processes on $\mathbb{S}_d^+$, and therefore covers the aforementioned results. Having established the existence of a unique limit distribution for affine processes on $\mathbb{S}_d^+$, our next aim is to study the convergence rate of the underlying transition probability to the limit distribution in a suitably chosen metric, for instance, the Wasserstein or total variation distance. While exponential ergodicity in total variation has been investigated very recently by Mayerhofer *et al.* [38], we use two other metrics in the present article: the Wasserstein-1-distance[1] and a metric induced by the Laplace transform. We also provide sufficient conditions for exponential ergodicity with respect to these two metrics.

The long-time behavior of general affine processes has previously been studied in many different settings, see, e.g., [4, 18, 27, 29, 31, 35, 40]. One application of such a study is towards the calibration of affine models. In the case of the Wishart process, the maximum-likehood estimator for the drift parameter was recently studied by Alfonsi *et al.* [1]. As demonstrated in their article, ergodicity helps to derive strong consistency and asymptotic normality of the estimator.

This paper is organized as follows: In Section 2, we introduce $\mathbb{S}_d^+$-valued affine processes, formulate and discuss our main results. The proofs are then given in Sections 3 – 7. Finally, Section 8 is dedicated to applications of our results to specific affine models often used in finance.

2. MAIN RESULTS

In terms of terminology, we mainly follow the coordinate free notation used in Mayerhofer [36] and Keller-Ressel and Mayerhofer [30].

Let $d \geq 2$ and denote by $\mathbb{S}_d$ the space of symmetric $d \times d$ matrices equipped with the scalar product $\langle x, y \rangle = \operatorname{tr}(xy)$, where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. Accordingly, $\| \cdot \|$ is the induced norm on $\mathbb{S}_d$, that is, $\|x\| := \langle x, x \rangle^{1/2}$. Note that $\| \cdot \|$ is the well-known Frobenius norm. We list some properties of the trace and its induced norm in Appendix A which are repeatedly used in the remainder of the article. Denote by $\mathbb{S}_d^+$ (resp. $\mathbb{S}_d^{++}$) the cone of symmetric and positive semidefinite (resp. positive definite) real $d \times d$ matrices. We write $x \preceq y$ if $y - x \in \mathbb{S}_d^+$ and $x \prec y$ if $y - x \in \mathbb{S}_d^{++}$ for the natural partial and strict order relation introduced respectively by the cones $\mathbb{S}_d^+$ and $\mathbb{S}_d^{++}$. Let $\mathcal{B}(\mathbb{S}_d^+ \backslash \{0\})$ be the Borel-σ-algebra on $\mathbb{S}_d^+ \backslash \{0\}$. An $\mathbb{S}_d^+$-*valued measure* η on $\mathbb{S}_d^+ \backslash \{0\}$ is a $d \times d$-matrix of signed measures on $\mathbb{S}_d^+ \backslash \{0\}$ such that $\eta(A) \in \mathbb{S}_d^+$ whenever $A \in \mathcal{B}(\mathbb{S}_d^+ \backslash \{0\})$ with $0 \notin \overline{A}$.

In the following we introduce the notion of admissible parameters first introduced in Cuchiero *et al.* [11, Definition 2.3]. Here we mainly follow the one given in Mayerhofer [36, Definition 3.1], with a slightly stronger condition on the linear jump coefficient.

Definition 2.1. Let $d \geq 2$. An admissible parameter set (α, b, B, m, μ) consists of:
  (i) a linear diffusion coefficient $\alpha \in \mathbb{S}_d^+$;
  (ii) a constant drift $b \in \mathbb{S}_d^+$ satisfying $b \succeq (d-1)\alpha$;
  (iii) a constant jump term: a Borel measure m on $\mathbb{S}_d^+ \backslash \{0\}$ satisfying

$$\int_{\mathbb{S}_d^+ \backslash \{0\}} (\|\xi\| \wedge 1) \, m \, (\mathrm{d}\xi) < \infty;$$

  (iv) a linear jump coefficient μ which is an $\mathbb{S}_d^+$-valued, sigma-finite measure on $\mathbb{S}_d^+ \backslash \{0\}$ satisfying

$$\int_{\mathbb{S}_d^+ \backslash \{0\}} \|\xi\| \operatorname{tr}(\mu) \, (\mathrm{d}\xi) < \infty,$$

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[1]Also known as the Kantorovich-Rubinstein distance.

where $\mathrm{tr}(\mu)$ denotes the measure induced by the relation $\mathrm{tr}(\mu)(A) := \mathrm{tr}(\mu(A))$ for all $A \in \mathcal{B}(\mathbb{S}_d^+ \backslash \{0\})$ with $0 \notin \bar{A}$;

(v) a linear drift B, which is a linear map $B : \mathbb{S}_d \to \mathbb{S}_d$ satisfying

$$\langle B(x), u \rangle \geq 0 \quad \text{for all } x, u \in \mathbb{S}_d^+ \text{ with } \langle x, u \rangle = 0.$$

According to our definition, a set of admissible parameters does not contain parameters corresponding to killing. In addition, our definition involves a first moment assumption on the linear jump coefficient μ.

Theorem 2.1 ([11]). *Let (α, b, B, m, μ) be admissible parameters in the sense of Definition 2.1. Then there exists a unique stochastically continuous transition kernel $p_t(x, \mathrm{d}\xi)$ such that $p_t(x, \mathbb{S}_d^+) = 1$ and*

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) = \exp\left(-\phi(t, u) - \langle \psi(t, u), x \rangle\right), \quad t \geq 0, \quad x, u \in \mathbb{S}_d^+, \tag{2.1}$$

where $\phi(t, u)$ and $\psi(t, u)$ in (2.1) are the unique solutions to the generalized Riccati differential equations, that is, for $u \in \mathbb{S}_d^+$,

$$\frac{\partial \phi(t, u)}{\partial t} = F\left(\psi(t, u)\right), \quad \phi(0, u) = 0, \tag{2.2}$$

$$\frac{\partial \psi(t, u)}{\partial t} = R\left(\psi(t, u)\right), \quad \psi(0, u) = u, \tag{2.3}$$

and the functions F and R are given by

$$F(u) = \langle b, u \rangle - \int_{\mathbb{S}_d^+ \backslash \{0\}} \left(\mathrm{e}^{-\langle u, \xi \rangle} - 1\right) m\left(\mathrm{d}\xi\right),$$

$$R(u) = -2u\alpha u + B^\top(u) - \int_{\mathbb{S}_d^+ \backslash \{0\}} \left(\mathrm{e}^{-\langle u, \xi \rangle} - 1\right) \mu\left(\mathrm{d}\xi\right).$$

Here, $B^\top$ denotes the adjoint operator on $\mathbb{S}_d$ defined by the relation $\langle u, B(\xi) \rangle = \langle B^\top(u), \xi \rangle$ for $u, \xi \in \mathbb{S}_d$. Under the additional moment condition (iv) of Definition 2.1, we will show in Lemma 3.2 below that $R(u)$ is continuously differentiable and thus locally Lipschitz continuous on $\mathbb{S}_d^+$. This fact, together with the absence of parameters according to killing, implies that the affine process under consideration is indeed conservative (see [11, Remark 2.5]).

2.1. **First moment.** Our first result provides existence and a precise formula for the first moment of conservative affine processes on $\mathbb{S}_d^+$. For this purpose, we define the effective drift

$$\widetilde{B}(u) := B(u) + \int_{\mathbb{S}_d^+ \backslash \{0\}} \langle \xi, u \rangle \mu\left(\mathrm{d}\xi\right), \quad \text{for all } u \in \mathbb{S}_d.$$

Then note that $\widetilde{B} : \mathbb{S}_d \to \mathbb{S}_d$ is a linear map. We define the corresponding semigroup $(\exp(t\widetilde{B}))_{t \geq 0}$ by its Taylor series $\exp(t\widetilde{B})(u) = \sum_{n=0}^\infty t^n/n! \widetilde{B}^{\circ n}(u)$, where $\widetilde{B}^{\circ n}$ denotes the n-times composition of $\widetilde{B}$. For the remainder of the article we write $\mathbb{1}$ without an index for the $d \times d$-identity matrix, while $\mathbb{1}_A$ denotes the standard indicator function of a set A.

Theorem 2.2. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of an affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ) satisfying*

$$\int_{\{\|\xi\| > 1\}} \|\xi\| m\left(\mathrm{d}\xi\right) < \infty. \tag{2.4}$$

Then, for each $t \geq 0$ and $x \in \mathbb{S}_d^+$,

$$\int_{\mathbb{S}_d^+} \xi p_t(x, \mathrm{d}\xi) = \mathrm{e}^{t\widetilde{B}} x + \int_0^t \mathrm{e}^{s\widetilde{B}} \left(b + \int_{\mathbb{S}_d^+ \backslash \{0\}} \xi m(\mathrm{d}\xi)\right) \mathrm{d}s. \tag{2.5}$$

In particular, the first moment exists.

Based on methods of stochastic calculus similar results were obtained for affine processes with state space $\mathbb{R}_{\geq 0}^m$ in [5, Lemma 3.4] and on the canonical state space $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ in [17, Lemma 5.2]. For affine processes on $\mathbb{R}_{\geq 0}$, i.e., continuous-state branching processes with immigration, and also for the more general class of Dawson-Watanabe superprocesses an alternative approach based on a fine analysis of the Laplace transform is provided in [35]. The latter approach has clearly the advantage that it is purely analytical and does not rely on the use of stochastic equations and semimartingale representations for these processes. We provide in Section 3 a purely analytic proof for Theorem 2.2 as well.

Remark 2.3. *Note that the transition kernel $p_t(x, \cdot)$ with admissible parameters (α, b, B, m, μ) is Feller by virtue of [11, Theorem 2.4]. Therefore, there exists a canonical realization $(X, (\mathbb{P}_x)_{x \in \mathbb{S}_d^+})$ of the corresponding Markov process on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, where $\Omega = \mathbb{D}(\mathbb{S}_d^+)$ is the set of all càdlàg paths $\omega : \mathbb{R}_{\geq 0} \to \mathbb{S}_d^+$ and $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Here $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by X and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. For $x \in \mathbb{S}_d^+$, the probability measure $\mathbb{P}_x$ on Ω represents the law of the Markov process X given $X_0 = x$. With this notation, under the conditions of Theorem 2.2, formula (2.5) reads*

$$\mathbb{E}_x[X_t] = \mathrm{e}^{t\widetilde{B}}x + \int_0^t \mathrm{e}^{s\widetilde{B}}\left(b + \int_{\mathbb{S}_d^+ \setminus \{0\}} \xi m(\mathrm{d}\xi)\right)\mathrm{d}s,$$

where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

2.2. **Existence and convergence to the invariant distribution.** In this subsection we formulate our main result. Let $p_t(x, \cdot)$ be the transition kernel of an affine process on $\mathbb{S}_d^+$. Motivated by Theorem 2.2 it is reasonable to relate the long-time behavior of the process with the spectrum $\sigma(\widetilde{B})$ of $\widetilde{B}$. More precisely, an affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ) is said to be *subcritical*, if

$$\sup\left\{\mathrm{Re}\,\lambda \in \mathbb{C} \, : \, \lambda \in \sigma\left(\widetilde{B}\right)\right\} < 0. \tag{2.6}$$

Under condition (2.6), it is well-known that there exist constants $M \geq 1$ and $\delta > 0$ such that

$$\left\|\mathrm{e}^{t\widetilde{B}}\right\| \leq M\mathrm{e}^{-\delta t}, \quad t \geq 0. \tag{2.7}$$

The next remark provides a sufficient condition for (2.7).

Remark 2.4. *According to [38, Theorem 2.7], (2.7) is satisfied if and only if there exists a $v \in \mathbb{S}_d^{++}$ such that $-\widetilde{B}^\top(v) \in \mathbb{S}_d^{++}$. However, in many application the linear drift is of the form $\widetilde{B}(x) = \beta x + x\beta^\top$, where β is a real-valued $d \times d$-matrix, see Section 8. In this case, it follows from [38, Corollary 5.1] that (2.7) is satisfied if and only if*

$$\sup\{\mathrm{Re}\,\lambda \in \mathbb{C} \, : \, \lambda \in \sigma(\beta)\} < 0,$$

which in turn holds true if and only if there exists one $v \in \mathbb{S}_d^{++}$ such that $-(\beta^\top v + v\beta) \in \mathbb{S}_d^{++}$.

Let $\mathcal{P}(\mathbb{S}_d^+)$ be the space of all Borel probability measures on $\mathbb{S}_d^+$. We call $\pi \in \mathcal{P}(\mathbb{S}_d^+)$ an invariant distribution, if

$$\int_{\mathbb{S}_d^+} p_t(x, \mathrm{d}\xi)\pi(\mathrm{d}x) = \pi(\mathrm{d}\xi), \qquad t \geq 0.$$

The following is our main result.

Theorem 2.5. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ). Suppose that the measure m satisfies*

$$\int_{\{\|\xi\|>1\}} \log \|\xi\| \, m\,(\mathrm{d}\xi) < \infty. \tag{2.8}$$

Then there exists a unique invariant distribution π. Moreover, $p_t(x, \cdot) \to \pi$ weakly as $t \to \infty$ for each $x \in \mathbb{S}_d^+$ and π has Laplace transform

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u,x\rangle} \pi\,(\mathrm{d}x) = \exp\left(-\int_0^\infty F\left(\psi(s,u)\right)\mathrm{d}s\right), \quad u \in \mathbb{S}_d^+. \tag{2.9}$$

The proof of Theorem 2.5 is postponed to Section 5. Let us make a few comments. Note that in dimension $d = 1$ it holds $\mathbb{S}_1^+ = \mathbb{R}_{\geq 0}$ and affine processes on this state space coincide with the class of continuous-state branching processes with immigration introduced by Kawazu and Watanbe [28]. In this case, the long-time behavior has been extensively studied in the articles [33, Theorem 3.16], [31, Theorem 2.6], and the monograph [35, Theorem 3.20 and Corollary 3.21]. This is why we restrict ourselves to the case $d \geq 2$. Theorem 2.5 establishes sufficient conditions for the existence, uniqueness, and convergence to the invariant distribution. For affine processes on the canonical state space $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ a similar statement was recently shown in [27].

For dimension $d = 1$ it is known that (2.8) is not only sufficient, but also necessary for the convergence to some limiting distribution, see, e.g., [35, Theorem 3.20 and Corollary 3.21]. To our knowledge, extensions of this result to higher dimensional state space has not yet been obtained. In this context, we have the following partial result for subcritical affine processes on $\mathbb{S}_d^+$.

Proposition 2.6. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ). Suppose that there exists $x \in \mathbb{S}_d^+$ and $\pi \in \mathcal{P}(\mathbb{S}_d^+)$ such that $p_t(x, \cdot) \to \pi$ weakly as $t \to \infty$. If $\alpha = 0$ and there exists a constant $K > 0$ satisfying*

$$K\xi + B(\xi) \succeq 0, \quad \xi \in \mathbb{S}_d^+, \tag{2.10}$$

then (2.8) holds.

We note that any linear map $B : \mathbb{S}_d \to \mathbb{S}_d$ which leaves $\mathbb{S}_d^+$ invariant satisfies condition (2.10) for each $K > 0$. As an example of such a map, let $B(x) = \beta x \beta^\top$ for $x \in \mathbb{S}_d$, where β is a real-valued invertible $d \times d$-matrix. Obviously, B defined in this way is admissible in the sense of Definition 2.1 and $B(\mathbb{S}_d^+) = \mathbb{S}_d^+$. Moreover, in view of [41, Theorem 2], any linear map that leaves $\mathbb{S}_d^+$ invariant must be of this form.

In order to prove Theorem 2.5 and Proposition 2.6 we first establish in Section 4 precise lower and upper bounds for $\psi(t,u)$. Since in dimension $d \geq 2$ different components of the process interact through the drift B in a nontrivial manner on $\mathbb{S}_d^+$, the proof of the lower bound is deduced from the additional conditions $\alpha = 0$ and (2.10), which guarantees that these components are coupled in a *well-behaved way*.

We close this section with a useful moment result regarding the invariant distribution.

Corollary 2.7. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ) satisfying (2.4). Let π be the unique invariant distribution. Then*

$$\lim_{t\to\infty} \int_{\mathbb{S}_d^+} y p_t(x, \mathrm{d}y) = \int_{\mathbb{S}_d^+} y \pi(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{s\widetilde{B}}\left(b + \int_{\mathbb{S}_d^+\setminus\{0\}} \xi m(\mathrm{d}\xi)\right)\mathrm{d}s.$$

2.3. **Study of convergence rate.** Noting that δ defined by (2.7) is supposed to be strictly positive, we will see that it appears naturally in the rate of convergence towards the invariant distribution. In order to measure this rate of convergence we introduce

$$d_L(\eta, \nu) := \sup_{u \in \mathbb{S}_d^+ \setminus \{0\}} \frac{1}{\|u\|} \left| \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, x \rangle} \eta(\mathrm{d}x) - \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, x \rangle} \nu(\mathrm{d}x) \right|, \quad \eta, \nu \in \mathcal{P}(\mathbb{S}_d^+).$$

Note that this supremum is not necessarily finite. However, it is finite for elements of

$$\mathcal{P}_1(\mathbb{S}_d^+) = \left\{ \varrho \in \mathcal{P}(\mathbb{S}_d^+) \,:\, \int_{\mathbb{S}_d^+} \|x\| \varrho(\mathrm{d}x) < \infty \right\}.$$

Then it is easy to see that d_L is a metric on $\mathcal{P}_1(\mathbb{S}_d^+)$; moreover, $\left(\mathcal{P}_1(\mathbb{S}_d^+), d_L \right)$ is complete. Using well-known properties of Laplace transforms, it can be shown that convergence with respect to d_L is stronger than weak convergence. The next result provides an exponential rate in d_L distance.

Theorem 2.8. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ). Suppose that (2.8) holds and denote by π the unique invariant distribution. Then there exists a constant $C > 0$ such that*

$$d_L\left(p_t(x, \cdot), \pi \right) \leq C \left(1 + \|x\| \right) \mathrm{e}^{-\delta t}, \quad t \geq 0, \quad x \in \mathbb{S}_d^+. \tag{2.11}$$

The proof of this result is given in Section 6. Although under the given conditions $p_t(x, \cdot)$ and π do not necessarily belong to $\mathcal{P}_1(\mathbb{S}_d^+)$, the proof of (2.11) implies that $d_L(p_t(x, \cdot), \pi)$ is well-defined.

We turn to investigate the convergence rate from the affine transition kernel to the invariant distribution in the Wasserstein-1-distance introduced below. Given $\varrho, \widetilde{\varrho} \in \mathcal{P}_1(\mathbb{S}_d^+)$, a coupling H of $(\varrho, \widetilde{\varrho})$ is a Borel probability measure on $\mathbb{S}_d^+ \times \mathbb{S}_d^+$ which has marginals ϱ and $\widetilde{\varrho}$, respectively. We denote by $\mathcal{H}(\varrho, \widetilde{\varrho})$ the collection of all such couplings. We define the *Wasserstein distance* on $\mathcal{P}_1(\mathbb{S}_d^+)$ by

$$W_1(\varrho, \widetilde{\varrho}) = \inf \left\{ \int_{\mathbb{S}_d^+ \times \mathbb{S}_d^+} \|x - y\| H(\mathrm{d}x, \mathrm{d}y) \,:\, H \in \mathcal{H}(\varrho, \widetilde{\varrho}) \right\}.$$

Since ϱ and $\widetilde{\varrho}$ belong to $\mathcal{P}_1(\mathbb{S}_d^+)$, it holds that $W_1(\varrho, \widetilde{\varrho})$ is finite. According to [42, Theorem 6.16], we have that $(\mathcal{P}(\mathbb{S}_d^+), W_1)$ is a complete separable metric space. Exponential ergodicity in different Wasserstein distances for affine processes on the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$ was very recently studied in [17]. Below we provide a corresponding result for affine processes on $\mathbb{S}_d^+$.

Theorem 2.9. *Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ) satisfying (2.4). If $\alpha = 0$, then*

$$W_1\left(p_t(x, \cdot), \pi \right) \leq \sqrt{d} M e^{-\delta t} \left(\|x\| + \int_{\mathbb{S}_d^+} \|y\| \pi(\mathrm{d}y) \right), \quad t \geq 0, \quad x \in \mathbb{S}_d^+. \tag{2.12}$$

The proof of Theorem 2.9 is given in Section 7 which largely follows some ideas of [17]. In contrast to the latter work, for the study of affine processes on $\mathbb{S}_d^+$ we encounter two additional difficulties:

- It is still an open problem whether each affine process on $\mathbb{S}_d^+$ can be obtained as a strong solution to a certain stochastic equation driven by Brownian motions and Poisson random measures. We refer the reader to [37] for some related results. In addition, we do not know if a comparison principle for such processes would be available.

- Following [17], one important step in the proof of Theorem 2.7 therein is based on the decomposition $p_t(x, \cdot) = r_t(x, \cdot) * p_t(0, \cdot)$, where $r_t(x, \cdot)$ is the transition kernel of an affine process on $\mathbb{S}_d^+$ whose Laplace transform is given by

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} r_t(x, \mathrm{d}\xi) = \exp\left(-\langle \psi(t, u), x \rangle\right),$$

that is, $r_t(x, \cdot)$ should have admissible parameters $(\alpha, b = 0, B, m = 0, \mu)$. Unfortunately, such transition kernel $r_t(x, \cdot)$ is well-defined if and only if $(\alpha, b = 0, B, m = 0, \mu)$ are admissible parameters in the sense of Definition 2.1. This in turn is true if and only if $\alpha = 0$ which is a consequence of the particular structure of the boundary $\mathbb{S}_d^+ \backslash \mathbb{S}_d^{++}$.

3. Proof of Theorem 2.2

In this section we study the first moment of a conservative affine process on $\mathbb{S}_d^+$. In particular, we prove Theorem 2.2. Essential to the proof is the space-differentiability of the functions F and R as well as ϕ and ψ. To simplify the notation we introduce $L(\mathbb{S}_d, \mathbb{S}_d)$ as the space of all linear operators $\mathbb{S}_d \to \mathbb{S}_d$, and similarly $L(\mathbb{S}_d, \mathbb{R})$ stands for the space of all linear functionals $\mathbb{S}_d \to \mathbb{R}$. For a function $G : \mathbb{S}_d \to \mathbb{S}_d$ we denote its derivative at $u \in \mathbb{S}_d$, if it exists, by $DG(u) \in L(\mathbb{S}_d, \mathbb{S}_d)$. Similarly, we denote the derivative of $H : \mathbb{S}_d \to \mathbb{R}$ by $DH(u) \in L(\mathbb{S}_d, \mathbb{R})$. We equip $L(\mathbb{S}_d, \mathbb{S}_d)$ and $L(\mathbb{S}_d, \mathbb{R})$ with the corresponding norm

$$\|DG(u)\| = \sup_{\|x\|=1} \|DG(u)(x)\| \quad \text{and} \quad \|DH(u)\| = \sup_{\|x\|=1} \|DH(u)(x)\|.$$

Let F and R be as in Theorem 2.1. According to [11, Lemma 5.1] the function R is analytic on $\mathbb{S}_d^{++}$. Below we study the differentiability of F and R on the entire cone $\mathbb{S}_d^+$.

We first give a lemma that slightly extends [36, Lemma 3.3].

Lemma 3.1. *Let g be a measurable function on $\mathbb{S}_d^+$ with $\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \operatorname{tr}(\mu)(\mathrm{d}\xi) < \infty$. Then $\int_{\mathbb{S}_d^+ \backslash \{0\}} g(\xi) \mu(\mathrm{d}\xi)$ is finite and*

$$\left\| \int_{\mathbb{S}_d^+ \backslash \{0\}} g(\xi) \mu(\mathrm{d}\xi) \right\| \leq \int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \operatorname{tr}(\mu)(\mathrm{d}\xi).$$

Proof. Let $\mu = (\mu_{ij})$ and $\mu_{ij} = \mu_{ij}^+ - \mu_{ij}^-$ be the Jordan decomposition of μ_{ij}. Suppose $\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \operatorname{tr}(\mu)(\mathrm{d}\xi) < \infty$. Then [36, Lemma 3.3] implies that $\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu(\mathrm{d}\xi)$ is finite and

$$\left\| \int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu(\mathrm{d}\xi) \right\| \leq \int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \operatorname{tr}(\mu)(\mathrm{d}\xi).$$

Since the ij-th entry of $\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu(\mathrm{d}\xi)$ is given by

$$\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu_{ij}^+(\mathrm{d}\xi) - \int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu_{ij}^-(\mathrm{d}\xi),$$

which is finite, we must have

$$\int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu_{ij}^+(\mathrm{d}\xi) < \infty \quad \text{and} \quad \int_{\mathbb{S}_d^+ \backslash \{0\}} |g(\xi)| \mu_{ij}^-(\mathrm{d}\xi) < \infty, \quad \forall i, j \in \{1, \dots d\}.$$

So $\int_{\mathbb{S}_d^+ \backslash \{0\}} g(\xi) \mu(\mathrm{d}\xi)$ is finite. Again by [36, Lemma 3.3],

$$\left\| \int_{\mathbb{S}_d^+ \backslash \{0\}} g(\xi) \mu(\mathrm{d}\xi) \right\| = \left\| \int_{\mathbb{S}_d^+ \backslash \{0\}} g^+(\xi) \mu(\mathrm{d}\xi) - \int_{\mathbb{S}_d^+ \backslash \{0\}} g^-(\xi) \mu(\mathrm{d}\xi) \right\|$$

$$\leq \left\| \int_{\mathbb{S}_d^+ \setminus \{0\}} g^+(\xi) \mu(\mathrm{d}\xi) \right\| + \left\| \int_{\mathbb{S}_d^+ \setminus \{0\}} g^-(\xi) \mu(\mathrm{d}\xi) \right\|$$

$$\leq \int_{\mathbb{S}_d^+ \setminus \{0\}} g^+(\xi) \mathrm{tr}(\mu)(\mathrm{d}\xi) + \int_{\mathbb{S}_d^+ \setminus \{0\}} g^-(\xi) \mathrm{tr}(\mu)(\mathrm{d}\xi)$$

$$\leq \int_{\mathbb{S}_d^+ \setminus \{0\}} |g(\xi)| \, \mathrm{tr}(\mu)(\mathrm{d}\xi).$$

The lemma is proved. $\hfill\square$

Lemma 3.2. *The following statements hold:*

(a) *For $u \in \mathbb{S}_d^{++}$, $h \in \mathbb{S}_d$, we have*

$$DR(u)(h) = -2 \left(u\alpha h + h\alpha u \right) + B^\top(h) + \int_{\mathbb{S}_d^+ \setminus \{0\}} \langle h, \xi \rangle \mathrm{e}^{-\langle u, \xi \rangle} \mu(\mathrm{d}\xi). \tag{3.1}$$

Moreover, through (3.1) *$DR(u)$ is continuously extended to $u \in \mathbb{S}_d^+$. In particular, $R \in C^1(\mathbb{S}_d^+)$ and* (3.1) *holds true for all $u \in \mathbb{S}_d^+, h \in \mathbb{S}_d$.*

(b) *If* (2.4) *is satisfied, then for $u \in \mathbb{S}_d^{++}$, $h \in \mathbb{S}_d$,*

$$DF(u)(h) = \langle b, h \rangle + \int_{\mathbb{S}_d^+ \setminus \{0\}} \langle h, \xi \rangle \mathrm{e}^{-\langle u, \xi \rangle} m(\mathrm{d}\xi). \tag{3.2}$$

Moreover, through (3.1) *$DF(u)$ is continuously extended to $u \in \mathbb{S}_d^+$. In particular, $F \in C^1(\mathbb{S}_d^+)$ and* (3.2) *holds true for all $u \in \mathbb{S}_d^+, h \in \mathbb{S}_d$.*

Proof. (a) Let $u \in \mathbb{S}_d^{++}$. Consider $h \in \mathbb{S}_d$ with sufficiently small $\|h\|$ such that $u + h \in \mathbb{S}_d^+$. An easy calculation shows that

$$R(u + h) - R(u) = DR(u)(h) + r(u, h),$$

where

$$r(u, h) := -2h\alpha h + \int_{\mathbb{S}_d^+ \setminus \{0\}} \mathrm{e}^{-\langle u, \xi \rangle} \left(1 - \mathrm{e}^{-\langle h, \xi \rangle} - \langle h, \xi \rangle \right) \mu(\mathrm{d}\xi).$$

Let us prove that $\lim_{0 \neq \|h\| \to 0} \|r(u, h)\| / \|h\| = 0$. Assume $\|h\| \neq 0$. First, note that

$$\frac{\|2h\alpha h\|}{\|h\|} \leq 2\|\alpha\| \frac{\|h\|^2}{\|h\|} \leq 2\|\alpha\| \|h\|.$$

Let $M > 0$. For $\|\xi\| \leq M$, we have

$$\left| \mathrm{e}^{-\langle u, \xi \rangle} \left(1 - \mathrm{e}^{-\langle h, \xi \rangle} - \langle h, \xi \rangle \right) \right| = \left| \langle h, \xi \rangle \left(\int_0^1 \mathrm{e}^{-\langle u + sh, \xi \rangle} \mathrm{d}s - \mathrm{e}^{-\langle u, \xi \rangle} \right) \right|$$

$$= |\langle h, \xi \rangle| \cdot \left| \int_0^1 \left(\mathrm{e}^{-\langle u + sh, \xi \rangle} - \mathrm{e}^{-\langle u, \xi \rangle} \right) \mathrm{d}s \right|$$

$$\leq |\langle h, \xi \rangle|^2, \tag{3.3}$$

where we used that $\langle u + sh, \xi \rangle \geq 0$ and the Lipschitz continuity of $[0, \infty) \in x \mapsto \exp(-x)$ to get the last inequality. Similarly, for $\|\xi\| > M$,

$$\left| \mathrm{e}^{-\langle u, \xi \rangle} \left(1 - \mathrm{e}^{-\langle h, \xi \rangle} - \langle h, \xi \rangle \right) \right| \leq \left| \mathrm{e}^{-\langle u, \xi \rangle} - \mathrm{e}^{-\langle u + h, \xi \rangle} \right| + \left| \mathrm{e}^{-\langle u, \xi \rangle} \langle h, \xi \rangle \right| \leq 2 \left| \langle h, \xi \rangle \right|. \tag{3.4}$$

Combining (3.3), (3.4) and applying Lemma 3.1, we get

$$\frac{1}{\|h\|} \left\| \int_{\mathbb{S}_d^+ \setminus \{0\}} \mathrm{e}^{-\langle u, \xi \rangle} \left(1 - \mathrm{e}^{-\langle h, \xi \rangle} - \langle h, \xi \rangle \right) \mu(\mathrm{d}\xi) \right\|$$

$$\leq \frac{1}{\|h\|} \int_{\mathbb{S}_d^+\setminus\{0\}} \left| e^{-\langle u,\xi\rangle} \left(1 - e^{-\langle h,\xi\rangle} - \langle h,\xi\rangle\right)\right| \mathrm{tr}(\mu)(\mathrm{d}\xi)$$

$$\leq \|h\| \int_{\{\|\xi\|\leq M\}} \|\xi\|^2 \mathrm{tr}(\mu)(\mathrm{d}\xi) + 2 \int_{\{\|\xi\|>M\}} \|\xi\| \mathrm{tr}(\mu)(\mathrm{d}\xi),$$

So

$$\frac{\|r(u,h)\|}{\|h\|} \leq \left(2\|\alpha\| + \int_{\{\|\xi\|\leq M\}} \|\xi\|^2 \mathrm{tr}(\mu)(\mathrm{d}\xi)\right)\|h\| + 2\int_{\{\|\xi\|>M\}} \|\xi\| \mathrm{tr}(\mu)(\mathrm{d}\xi).$$

Note that $\int_{\mathbb{S}_d^+\setminus\{0\}} \|\xi\|\mathrm{tr}(\mu)(\mathrm{d}\xi) < \infty$ by virtue of Definition 2.1 (iv). Let $\varepsilon > 0$ be arbitrary and fix some $M = M(\varepsilon) > 0$ large enough so that $\int_{\{\|\xi\|>M\}} \|\xi\|\mathrm{tr}(\mu)(\mathrm{d}\xi) < \varepsilon/4$. Define

$$\delta = \delta(\varepsilon) := \left(1 + 2\|\alpha\| + \int_{\{\|\xi\|\leq M\}} \|\xi\|^2 \mathrm{tr}(\mu)(\mathrm{d}\xi)\right)^{-1} \frac{\varepsilon}{2}.$$

Then, for $\|h\| \leq \delta$, we see that

$$\frac{\|r(u,h)\|}{\|h\|} \leq \left(2\|\alpha\| + \int_{\{\|\xi\|\leq M\}} \|\xi\|^2 \mathrm{tr}(\mu)(\mathrm{d}\xi)\right)\delta + \frac{\varepsilon}{2} \leq \varepsilon.$$

This proves (3.1) for $u \in \mathbb{S}_d^{++}$. Finally, the continuity of $u \mapsto DR(u)$ in $\mathbb{S}_d^+$ can be easily obtained from the dominated convergence theorem.

(b) Similarly as before, we derive $F(u + h) - F(u) = DF(u)(h) + r(u,h)$ with $r(u,h) := \int_{\mathbb{S}_d^+\setminus\{0\}} \exp(-\langle u,\xi\rangle)(1 - \exp(\langle h,\xi\rangle) - \langle h,\xi\rangle)m(\mathrm{d}\xi)$. Let $\|h\| \neq 0$. By essentially the same reasoning as in (a), we obtain that

$$\frac{\|r(u,h)\|}{\|h\|} \leq \|h\| \int_{\{\|\xi\|\leq M\}} \|\xi\|^2 m(\mathrm{d}\xi) + 2\int_{\{\|\xi\|>M\}} \|\xi\| m(\mathrm{d}\xi),$$

and the second integral on the right-hand side is now finite by (2.4). Hence, we may follow the same steps as in (a) to see that $\|r(u,h)\|/\|h\| \to 0$ as $\|h\| \to 0$ and the continuity of $DF(u)$ in $\mathbb{S}_d^+$. □

Let ϕ and ψ be as in Theorem 2.1. We know from [11, Lemma 3.2 (iii)] that $\phi(t,u)$ and $\psi(t,u)$ are jointly continuous on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$ and, moreover, $u \mapsto \phi(t,u)$ and $u \mapsto \psi(t,u)$ are analytic on $\mathbb{S}_d^{++}$ for $t \geq 0$.

Proposition 3.3. *The following statements hold:*
  (a) *$D\psi$ has a jointly continuous extension on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$.*
  (b) *If (2.4) is satisfied, then $D\phi$ has a jointly continuous extension on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$.*

Proof. (a) Noting that $s \mapsto DR(\psi(s,u)) \in L(\mathbb{S}_d,\mathbb{S}_d)$ is continuous, we may define $f_u(t)$ as the unique solution in $L(\mathbb{S}_d,\mathbb{S}_d)$ to

$$f_u(t) = \mathbb{1} + \int_0^t DR(\psi(s,u)) f_u(s)\mathrm{d}s.$$

Further, we then define the extension of $D\psi$ onto $\mathbb{R}_{\geq 0} \times \partial\mathbb{S}_d^+$ simply by

$$D\psi(t,u) = f_u(t), \quad (t,u) \in \mathbb{R}_{\geq 0} \times \partial\mathbb{S}_d^+.$$

It remains to verify the joint continuity of $D\psi(t,u)$ on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$ extended in this way. By the Riccati differential equation (2.3) we have

$$D\psi(t,u) = \mathbb{1} + \int_0^t DR(\psi(s,u)) D\psi(s,u)\mathrm{d}s, \quad t \geq 0, u \in \mathbb{S}_d^+.$$

Using that $u \mapsto R(u)$ is continuous on $\mathbb{S}_d^+$ and ψ is jointly continuous on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$, for all $T > 0$ and $M > 0$, there exists a constant $C(T, M) > 0$ such that

$$\sup_{s \in [0,T],\, u \in \mathbb{S}_d^+,\, \|u\| \leq M} \|DR\left(\psi(s,u)\right)\| =: C(T, M) < \infty.$$

Hence, for each $u \in \mathbb{S}_d^+$ with $\|u\| \leq M$, we obtain

$$\|D\psi(t,u)\| \leq 1 + C(T, M) \int_0^t \|D\psi(s,u)\| \mathrm{d}s.$$

Applying Gronwall's inequality yields

$$\|D\psi(t,u)\| \leq \mathrm{e}^{C(T,M)T} =: K(T, M) < \infty,$$

for all $t \in [0, T]$ and $u \in \mathbb{S}_d^+$ with $\|u\| \leq M$. Because $D\psi$ is jointly continuous in $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^{++}$, it is enough to prove continuity at some fixed point $(t, u) \in \mathbb{R}_{\geq 0} \times \partial\mathbb{S}_d^+$, where $\partial\mathbb{S}_d^+ := \mathbb{S}_d^+ \backslash \mathbb{S}_d^{++}$.

Without loss of generality we assume $t \in [0, T]$ and $u \in \partial\mathbb{S}_d^+$ with $\|u\| \leq M$. Let $s \in \mathbb{R}_{\geq 0}$ and $v \in \mathbb{S}_d^+$ with $s \in [0, T]$ and $\|v\| \leq M$. We have

$$\|D\psi(t,u) - D\psi(s,v)\| \leq \|D\psi(t,u) - D\psi(s,u)\| + \|D\psi(s,u) - D\psi(s,v)\|. \qquad (3.5)$$

We estimate the first term on the right-hand side of (3.5) by

$$
\begin{aligned}
\|D\psi(t,u) - D\psi(s,u)\| &\leq \left\| \int_0^t DR\left(\psi(r,u)\right) D\psi(r,u)\mathrm{d}r - \int_0^s DR\left(\psi(r,u)\right) D\psi(r,u)\mathrm{d}r \right\| \\
&\leq C(T, M) \int_{[s,t] \cup [t,s]} \|D\psi(r,u)\| \, \mathrm{d}r \\
&\leq C(T, M)K(T, M)|t - s|. \qquad (3.6)
\end{aligned}
$$

Turning to the second term, for $v \in \mathbb{S}_d^{++}$ with $\|v\| \leq M$, $D\psi(s,u) = f_u(s)$, and $D\psi(r,u) = f_u(r)$, we obtain

$$
\begin{aligned}
\|D\psi(s,u) - D\psi(s,v)\| &\leq \int_0^s \|DR\left(\psi(r,u)\right) D\psi(r,u) - DR\left(\psi(r,v)\right) D\psi(r,v)\| \, \mathrm{d}r \\
&\leq \int_0^s \|DR(\psi(r,u)) - DR\left(\psi(r,v)\right)\| \, \|D\psi(r,v)\| \, \mathrm{d}r \\
&\quad + \int_0^s \|DR\left(\psi(r,u)\right)\| \, \|D\psi(r,u) - D\psi(r,v)\| \, \mathrm{d}r \\
&\leq K(T, M) \int_0^T \|DR(\psi(r,u)) - DR\left(\psi(r,v)\right)\| \, \mathrm{d}r \\
&\quad + C(T, M) \int_0^s \|D\psi(r,u) - D\psi(r,v)\| \, \mathrm{d}r \\
&= K(T, M)a_T(v,u) + C(T, M) \int_0^s \|D\psi(r,u) - D\psi(r,v)\| \, \mathrm{d}r,
\end{aligned}
$$

where $a_T(v,u) := \int_0^T \|DR(\psi(r,u)) - DR(\psi(r,v))\|\mathrm{d}r$. Using once again Gronwall's inequality, we deduce

$$\|D\psi(s,u) - D\psi(s,v)\| \leq K(T, M)a_T(v,u)\mathrm{e}^{C(T,M)T}. \qquad (3.7)$$

Noting that $R \in C^1(\mathbb{S}_d^+)$ and $\psi(r,0) = 0$ by [11, Remark 2.5], by dominated convergence theorem, we see that $a_T(v,u)$ tends to zero as $v \to u$. Consequently, the right-hand side of (3.7) tends to zero as $v \to u$. Combining (3.5) with (3.6) and (3.7), we conclude that $D\psi$ extended in this way is jointly continuous in $(t, u) \in \mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$.

(b) We know from the generalized Riccati equation (2.2) that $\phi(t,u) = \int_0^t F(\psi(s,u))\mathrm{d}s$. Noting that $F \in C^1(\mathbb{S}_d^+)$ due to (2.4), the chain rule combined with the dominated convergence theorem implies the assertion. $\qquad\square$

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let $\varepsilon > 0$. We have

$$\frac{\partial}{\partial\varepsilon}\int_{\mathbb{S}_d^+}\mathrm{e}^{-\langle\varepsilon u,\xi\rangle}p_t(x,\mathrm{d}\xi) = \frac{\partial}{\partial\varepsilon}\mathrm{e}^{-\phi(t,\varepsilon u)-\langle x,\psi(t,\varepsilon u)\rangle}$$

$$= -\left(D\phi(t,\varepsilon u)(u) + \langle x, D\psi(t,\varepsilon u)(u)\rangle\right)\mathrm{e}^{-\phi(t,\varepsilon u)-\langle x,\psi(t,\varepsilon u)\rangle}$$

$$\to -\left(D\phi(t,0)(u) + \langle x, D\psi(t,0)(u)\rangle\right) \quad \text{as } \varepsilon \to 0,$$

where we used that the functions $D\phi$ and $D\psi$ have a jointly continuous extension on $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$ in accordance with Proposition 3.3. On the other hand, noting $|\langle u,\xi\rangle\exp(-\langle\varepsilon u,\xi\rangle)| \leq \varepsilon^{-1}\mathrm{e}^{-1}$ and applying dominated convergence theorem, we get

$$\frac{\partial}{\partial\varepsilon}\int_{\mathbb{S}_d^+}\mathrm{e}^{-\langle\varepsilon u,\xi\rangle}p_t(x,\mathrm{d}\xi) = -\int_{\mathbb{S}_d^+}\langle u,\xi\rangle\mathrm{e}^{-\langle\varepsilon u,\xi\rangle}p_t(x,\mathrm{d}\xi) \to -\int_{\mathbb{S}_d^+}\langle u,\xi\rangle p_t(x,\mathrm{d}\xi) \quad \text{as } \varepsilon \to 0.$$

Note that the limit on the right-hand side is finite. Indeed, using Fatou's lemma, we obtain

$$\int_{\mathbb{S}_d^+}\langle u,\xi\rangle p_t(x,\mathrm{d}\xi) \leq \liminf_{\varepsilon\to 0}\int_{\mathbb{S}_d^+}\langle u,\xi\rangle\mathrm{e}^{-\langle\varepsilon u,\xi\rangle}p_t(x,\mathrm{d}\xi) = D\phi(t,0)(u) + \langle x, D\psi(t,0)(u)\rangle < \infty$$

for all $u \in \mathbb{S}_d^+$. So

$$\int_{\mathbb{S}_d^+}\langle u,\xi\rangle p_t(x,\mathrm{d}\xi) = D\phi(t,0)(u) + \langle x, D\psi(t,0)(u)\rangle. \tag{3.8}$$

In what follows, we compute the derivatives $D\phi(t,0)$ and $D\psi(t,0)$ explicitly. By means of the generalized Riccati equation (2.3), we have

$$\psi(t,u) - u = \int_0^t R\left(\psi(s,u)\right)ds, \quad t \geq 0, \ u \in \mathbb{S}_d^+.$$

According to Lemma 3.2 and Proposition 3.3 we are allowed to differentiate both sides of the latter equation with respect to $u \in \mathbb{S}_d^+$ and evaluate at $u = 0$, thus, using the dominated convergence theorem,

$$D\psi(t,u)|_{u=0} - \mathrm{Id} = \int_0^t DR\left(\psi(s,u)\right)D\psi(s,u)|_{u=0}\,ds, \quad t \geq 0,$$

where Id denotes the identity map on $\mathbb{S}_d^+$. From [11, Lemma 3.2 (iii)] we know that $\psi(t,u)$ is continuous in $\mathbb{R}_{\geq 0} \times \mathbb{S}_d^+$ and noting that $\psi(s,0) = 0$ (see [11, Remark 2.5]), we get

$$D\psi(t,0) - \mathrm{Id} = \int_0^t DR\left(0\right)D\psi(s,0)\mathrm{d}s, \quad t \geq 0.$$

From this and the precise formula for $\phi(t,h)$ we deduce that

$$D\psi(t,0) = \mathrm{e}^{tDR(0)} \quad \text{and} \quad D\phi(t,0) = \int_0^t DF(0)\mathrm{e}^{sDR(0)}\mathrm{d}s.$$

We use Lemma 3.2 to get that

$$DR(0)(u) = \widetilde{B}^\top(u) \quad \text{and} \quad DF(0)(u) = \langle b + \int_{\mathbb{S}_d^+\setminus\{0\}}\xi m(\mathrm{d}\xi), u\rangle.$$

Finally, combining this with (3.8) yields

$$\int_{\mathbb{S}_d^+} \langle u, \xi \rangle p_t(x, \mathrm{d}\xi) = \int_0^t (DF(0)) \, \mathrm{e}^{sDR(0)}(u)\mathrm{d}s + \langle x, \mathrm{e}^{tDR(0)}(u)\rangle$$

$$= \int_0^t \langle \mathrm{e}^{s\widetilde{B}} \left(b + \int_{\mathbb{S}_d^+ \setminus \{0\}} \xi m(\mathrm{d}\xi) \right), u\rangle \mathrm{d}s + \langle \mathrm{e}^{t\widetilde{B}}x, u\rangle.$$

Since the equality holds for each $u \in \mathbb{S}_d^+$, the assertion is proved. $\qquad\qquad\square$

4. Estimates on $\psi(t, u)$

We fix an admissible parameter set (α, b, B, m, μ) and let ψ be the unique solution to (2.3). In this section we study upper and lower bounds for ψ. Let us start with an upper bound for $\psi(t, u)$.

Proposition 4.1. *Let ψ be the unique solution to* (2.3). *Then*

$$\|\psi(t, u)\| \le M\|u\|\mathrm{e}^{-t\delta}, \quad t \ge 0, \tag{4.1}$$

where M and δ are given by (2.7).

Proof. The proof is divided into three steps.

Step 1: Denote by $q_t(x, \mathrm{d}\xi)$ the unique transition kernel of an affine process on $\mathbb{S}_d^+$ with admissible parameters $(\alpha, b, B, m = 0, \mu)$, that is, for each $u, x \in \mathbb{S}_d^+$, we have

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} q_t(x, \mathrm{d}\xi) = \exp\left(-\int_0^t \langle b, \psi(s, u)\rangle \mathrm{d}s - \langle x, \psi(t, u)\rangle \right), \quad t \ge 0. \tag{4.2}$$

Applying Jensen's inequality to the convex function $t \mapsto \exp(-t)$ yields

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} q_t(x, \mathrm{d}\xi) \ge \exp\left(-\int_{\mathbb{S}_d^+} \langle u, \xi\rangle q_t(x, \mathrm{d}\xi) \right)$$

$$= \exp\left(-\int_0^t \langle \mathrm{e}^{s\widetilde{B}}b, u\rangle \mathrm{d}s - \langle \mathrm{e}^{t\widetilde{B}}x, u\rangle \right),$$

where the last identity is a special case of Theorem 2.2. Using (4.2) we obtain

$$\langle x, \psi(t, u)\rangle + \int_0^t \langle b, \psi(t, u)\rangle \mathrm{d}s \le \langle \mathrm{e}^{t\widetilde{B}}x, u\rangle + \int_0^t \langle \mathrm{e}^{s\widetilde{B}}b, u\rangle \mathrm{d}s, \quad \text{for all } u, x \in \mathbb{S}_d^+, \ t \ge 0. \tag{4.3}$$

Step 2: Let $\alpha \in \mathbb{S}_d^+$ be fixed. We claim that (4.3) holds not only for $b \succeq (d-1)\alpha$ but also for any $b \in \mathbb{S}_d^+$. Aiming for a contradiction, suppose that there exist $t_0 > 0$ and ξ, x_0, $u_0 \in \mathbb{S}_d^+$ such that

$$I := \langle x_0, \psi(t_0, u_0)\rangle + \int_0^{t_0} \langle \xi, \psi(s, u_0)\rangle \mathrm{d}s - \langle x_0, \mathrm{e}^{t_0\widetilde{B}^\top}u_0\rangle - \int_0^{t_0} \langle \xi, \mathrm{e}^{s\widetilde{B}^\top}u_0\rangle \mathrm{d}s > 0.$$

We now take an arbitrary but fixed $b_0 \succeq (d-1)\alpha$. Noting that

$$\Delta := \int_0^{t_0} \langle b_0, \psi(s, u_0)\rangle \mathrm{d}s - \int_0^{t_0} \langle b_0, \mathrm{e}^{s\widetilde{B}^\top}u_0\rangle \mathrm{d}s$$

is finite, we find a constant $K > 0$ large enough so that $KI + \Delta > 0$, i.e.,

$$\langle Kx_0, \psi(t_0, u_0)\rangle + \int_0^{t_0} \langle b_0 + K\xi, \psi(s, u_0)\rangle \mathrm{d}s > \langle Kx_0, \mathrm{e}^{t_0\widetilde{B}^\top}u_0\rangle + \int_0^{t_0} \langle b_0 + K\xi, \mathrm{e}^{s\widetilde{B}^\top}u_0\rangle \mathrm{d}s. \tag{4.4}$$

Now, since $b_0 + K\xi \succeq (d-1)\alpha$, we see that (4.4) contradicts (4.3) if we chose $b = b_0 + K\xi$, $x = Kx_0$, $u = u_0$, and $t = t_0$. Hence (4.3) holds for all $b \in \mathbb{S}_d^+$.

Step 3: According to Step 2, we are allowed to choose $b = 0$ in (4.3), which implies

$$\langle x, \psi(t, u) \rangle \leq \langle x, e^{t\widetilde{B}^\top} u \rangle$$

for all $t \geq 0$ and $x, u \in \mathbb{S}_d^+$. This completes the proof.   □

We continue with a lower bound for $\psi(t, u)$.

Proposition 4.2. *Let ψ be the unique solution to (2.3) and suppose that $\alpha = 0$ and (2.10) is satisfied. Then, for each $u, \xi \in \mathbb{S}_d^+$,*

$$\langle \xi, \psi(t, u) \rangle \geq e^{-Kt} \langle \xi, u \rangle, \quad t \geq 0. \tag{4.5}$$

Proof. Fix $u \in \mathbb{S}_d^+$ and define $W_t(u) := \psi(t, u) - \exp(-Kt)u$. Using that $\exp(-Kt)u = \psi(t, u) - W_t(u)$ we obtain

$$\frac{\partial W_t(u)}{\partial t} = R(\psi(t, u)) + K\psi(t, u) - KW_t(u).$$

Since $W_0(u) = 0$, the latter implies

$$W_t(u) = \int_0^t e^{-K(t-s)} \left(K\psi(s, u) + R(\psi(s, u)) \right) \mathrm{d}s.$$

Fix $\xi \in \mathbb{S}_d^+$, then

$$\langle \xi, W_t(u) \rangle = \int_0^t e^{-K(t-s)} \left(K\langle \xi, \psi(s, u) \rangle + \langle \xi, R(\psi(s, u)) \rangle \right) \mathrm{d}s. \tag{4.6}$$

In the following we estimate the integrand. For this, we write $\langle \xi, R(\psi(s, u)) \rangle = I_1 + I_2$, where

$$I_1 = \langle \xi, B^\top (\psi(s, u)) \rangle \quad \text{and} \quad I_2 = -\int_{\mathbb{S}_d^+ \setminus \{0\}} \left(e^{-\langle \psi(s, u), \zeta \rangle} - 1 \right) \langle \xi, \mu(\mathrm{d}\zeta) \rangle,$$

and estimate I_1 and I_2 separately. For I_1, by (2.10) we get

$$I_1 = \langle B(\xi), \psi(s, u) \rangle \geq -K\langle \xi, \psi(s, u) \rangle,$$

where we used the self-duality of the cone $\mathbb{S}_d^+$ (see [25, Theorem 7.5.4]). Turning to I_2, we simply have

$$I_2 = \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - e^{-\langle \psi(s, u), \zeta \rangle} \right) \langle \xi, \mu(\mathrm{d}\zeta) \rangle \geq 0.$$

Collecting now the estimates for I_1 and I_2, we see that

$$(K\langle \xi, \psi(s, u) \rangle + \langle \xi, R(\psi(s, u)) \rangle) \geq 0$$

and, thus, $\langle \xi, W_t(u) \rangle \geq 0$ by (4.6) . This proves the assertion.   □

5. PROOF OF THE MAIN RESULTS

In this section we will prove Theorem 2.5, Proposition 2.6, and Corollary 2.7. Let $p_t(x, \mathrm{d}\xi)$ be the transition kernel of a subcritical affine process on $\mathbb{S}_d^+$ with admissible parameters (α, b, B, m, μ) and $\delta > 0$ be given by (2.7).

We note that $F(u) \geq 0$ for all $u \in \mathbb{S}_d^+$. Based on the estimates on $\psi(t, u)$ that we derived in the previous section, we easily obtain the following lemma.

Lemma 5.1. *Suppose that (2.8) holds. Then there exists a constant $C > 0$ such that*

$$F(\psi(s, u)) \leq C\|u\|e^{-s\delta}, \quad s \geq 0, \quad u \in \mathbb{S}_d^+. \tag{5.1}$$

Consequently,

$$\int_0^\infty F(\psi(s, u))\mathrm{d}s \leq \frac{C}{\delta}\|u\|, \quad u \in \mathbb{S}_d^+. \tag{5.2}$$

Proof. We know that

$$F(\psi(s,u)) = \langle b, \psi(s,u) \rangle + \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle} \right) m(\mathrm{d}\xi)$$
$$=: \langle b, \psi(s,u) \rangle + I(u).$$

Now, first note that, by (4.1),

$$\langle b, \psi(s,u) \rangle \leq \|b\| \|\psi(s,u)\| \leq \|b\| \|u\| \mathrm{e}^{-s\delta}. \tag{5.3}$$

We turn to estimate $I(u)$. Using once again (4.1), we obtain

$$I(u) = \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle} \right) m(\mathrm{d}\xi)$$
$$\leq \int_{\mathbb{S}_d^+ \setminus \{0\}} \min \left\{ 1, \langle \psi(s,u), \xi \rangle \right\} m(\mathrm{d}\xi)$$
$$\leq \int_{\mathbb{S}_d^+ \setminus \{0\}} \min \left\{ 1, \|\xi\| \|u\| \mathrm{e}^{-s\delta} \right\} m(\mathrm{d}\xi).$$

For all $a \geq 0$ it holds $1 \wedge a \leq \log(2)^{-1} \log(1+a)$, hence

$$I(u) \leq \frac{1}{\log(2)} \int_{\{\|\xi\| \leq 1\}} \|\xi\| \|u\| \mathrm{e}^{-s\delta} m(\mathrm{d}\xi) + \frac{1}{\log(2)} \int_{\{\|\xi\| > 1\}} \log \left(1 + \|\xi\| \|u\| \mathrm{e}^{-s\delta} \right) m(\mathrm{d}\xi)$$
$$=: J_1(u) + J_2(u).$$

Let $C > 0$ be a generic constant which may vary from line to line. Since $m(\mathrm{d}\xi)$ integrates $\|\xi\| \mathbb{1}_{\{\|\xi\| \leq 1\}}$ by definition, we have

$$J_1(u) \leq C \|u\| \mathrm{e}^{-s\delta}.$$

Moreover, noting that $m(\mathrm{d}\xi)$ integrates $\log \|\xi\| \mathbb{1}_{\{\|\xi\| > 1\}}$ by assumption, for $J_2(u)$ we use the elementary inequality (see [17, Lemma 8.5])

$$\log(1 + a \cdot c) \leq C \min \left\{ \log(1+a), \log(1+c) \right\} + C \log(1+a) \log(1+c)$$
$$\leq C \log(1+a) + Ca \log(1+c)$$
$$\leq Ca \left(1 + \log(1+c) \right)$$

for $a = \|u\| \exp(-s\delta)$ and $c = \|\xi\|$ to get

$$J_2(u) \leq C \|u\| \mathrm{e}^{-s\delta} \int_{\{\|\xi\| > 1\}} \left(1 + \log \left(1 + \|\xi\| \right) \right) m(\mathrm{d}\xi) \leq C \|u\| \mathrm{e}^{-s\delta}.$$

Combining the estimates for $J_1(u)$ and $J_2(u)$ yields

$$I(u) = J_1(u) + J_2(u) \leq C \|u\| \mathrm{e}^{-s\delta}. \tag{5.4}$$

So, by (5.3) and (5.4), we have (5.1) which proves the assertion. $\qquad \square$

We are now able to prove Theorem 2.5.

Proof of Theorem 2.5. Fix $x \in \mathbb{S}_d^+$. By means of Proposition 4.1, we see that

$$\lim_{t \to \infty} \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) = \lim_{t \to \infty} \exp \left(-\int_0^t F(\psi(s,u)) \mathrm{d}s - \langle x, \psi(t,u) \rangle \right)$$
$$= \exp \left(-\int_0^\infty F(\psi(s,u)) \mathrm{d}s \right),$$

and the limit on the right-hand side is finite according to Lemma 5.1. Clearly, by (5.2), we also have that $u \mapsto \int_0^\infty F(\psi(s,u)) \mathrm{d}s$ is continuous at $u = 0$. Now, Lévy's continuity theorem, cf.

[11, Lemma 4.5], implies that $p_t(x, \cdot) \to \pi$ weakly as $t \to \infty$. Moreover, π has Laplace transform (2.9). It remains to verify that π is the unique invariant distribution.

Invariance. Fix $u \in \mathbb{S}_d^+$ and let $t \geq 0$ be arbitrary. Then

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} \left(\int_{\mathbb{S}_d^+} p_t(x, \mathrm{d}\xi) \pi(\mathrm{d}x) \right) = \int_{\mathbb{S}_d^+} \left(\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) \right) \pi(\mathrm{d}x)$$

$$= \mathrm{e}^{-\phi(t,u)} \int_{\mathbb{S}_d^+} \exp\left(-\langle x, \psi(t, u) \rangle \right) \pi(\mathrm{d}x).$$

Noting that ψ satisfies the semi-flow equation[2] due to [11, Lemma 3.2] and using that the Laplace transform of π is given by (2.9), for each $u \in \mathbb{S}_d^+$, we obtain

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} \left(\int_{\mathbb{S}_d^+} p_t(x, \mathrm{d}\xi) \pi(\mathrm{d}x) \right) = \mathrm{e}^{-\phi(t,u)} \exp\left(-\int_0^\infty F\left(\psi\left(s, \psi(t, u) \right) \right) \mathrm{d}s \right)$$

$$= \mathrm{e}^{-\phi(t,u)} \exp\left(-\int_0^\infty F\left(\psi(t + s, u) \right) \mathrm{d}s \right)$$

$$= \mathrm{e}^{-\phi(t,u)} \exp\left(-\int_t^\infty F\left(\psi(s, u) \right) \mathrm{d}s \right)$$

$$= \exp\left(-\int_0^\infty F\left(\psi(s, u) \right) \mathrm{d}s \right)$$

$$= \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle x, u \rangle} \pi(\mathrm{d}x).$$

Consequently, π is invariant.

Uniqueness. Let π' be another invariant distribution. For fixed $u \in \mathbb{S}_d^+$ and $t \geq 0$ we have

$$\int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle x, u \rangle} \pi'(\mathrm{d}x) = \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} \left(\int_{\mathbb{S}_d^+} p_t(x, \mathrm{d}\xi) \pi'(\mathrm{d}x) \right)$$

$$= \int_{\mathbb{S}_d^+} \exp\left(-\phi(t, u) - \langle x, \psi(t, u) \rangle \right) \pi'(\mathrm{d}x).$$

Letting $t \to \infty$ shows that π' also satisfies (2.9). By uniqueness of the Laplace transforms, it holds that $\pi' = \pi$. $\qquad\square$

Proof of Proposition 2.6. Let $x \in \mathbb{S}_d^+$ and $\pi \in \mathcal{P}(\mathbb{S}_d^+)$ be such that $p_t(x, \cdot) \to \pi$ weakly as $t \to \infty$. It follows that

$$\lim_{t \to \infty} \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) = \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, y \rangle} \pi(\mathrm{d}y), \quad u \in \mathbb{S}_d^+,$$

and we obtain from (2.1)

$$\lim_{t \to \infty} \exp\left(-\int_0^t F(\psi(s, u)) \mathrm{d}s \right) = \lim_{t \to \infty} \mathrm{e}^{\langle x, \psi(t,u) \rangle} \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) = \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, y \rangle} \pi(\mathrm{d}y).$$

In particular, this implies

$$\int_0^\infty F(\psi(s, u)) \mathrm{d}s < \infty, \quad u \in \mathbb{S}_d^+.$$

---

[2]I.e., it holds that $\psi(t + s, u) = \psi\left(s, \psi(t, u) \right)$ for all $t, s \geq 0$.

Fix $u \in \mathbb{S}_d^{++}$. Assume that $\alpha = 0$ and (2.10) holds. By definition of F we have $F(u) \geq \int_{\mathbb{S}_d^+}(1 - \exp(-\langle u, \xi \rangle))m(\mathrm{d}\xi)$ and thereby

$$F(\psi(s, u)) \geq \int_{\{\langle \xi, u \rangle > 1\}} \left(1 - \mathrm{e}^{-\mathrm{e}^{-Ks}\langle \xi, u \rangle}\right) m(\mathrm{d}\xi),$$

where we used (4.5). Integrating over $[0, \infty)$ and using a change of variable $r := \exp(-Ks)\langle \xi, u \rangle$ with $\mathrm{d}s = -1/K \cdot \mathrm{d}r/r$ yields

$$
\begin{aligned}
\int_0^\infty F(\psi(s, u))\mathrm{d}s &\geq \frac{1}{K} \int_{\{\langle \xi, u \rangle > 1\}} \int_0^{\langle \xi, u \rangle} \frac{1 - \mathrm{e}^{-r}}{r} \mathrm{d}r m(\mathrm{d}\xi) \\
&\geq \frac{1}{K} \int_{\{\langle \xi, u \rangle > 1\}} \int_1^{\langle \xi, u \rangle} \frac{1 - \mathrm{e}^{-r}}{r} \mathrm{d}r m(\mathrm{d}\xi) \\
&\geq \frac{1 - \mathrm{e}^{-1}}{K} \int_{\{\langle \xi, u \rangle > 1\}} \log(\langle \xi, u \rangle)m(\mathrm{d}\xi),
\end{aligned}
$$

where we used in the last inequality that $1 - \exp(-r) \geq 1 - \exp(-1) > 0$ for $r \geq 1$. This leads to the estimate

$$\int_{\{\langle \xi, u \rangle > 1\}} \log(\langle \xi, u \rangle)m(\mathrm{d}\xi) \leq \frac{K}{1 - \mathrm{e}^{-1}} \int_0^\infty F(\psi(s, u))\mathrm{d}s < \infty.$$

Letting $u = \mathbb{1} \in \mathbb{S}_d^{++}$ gives $\langle \xi, \mathbb{1} \rangle = \mathrm{tr}(\xi) \geq \|\xi\|$ so that

$$\int_{\{\|\xi\| > 1\}} \log(\|\xi\|) m(\mathrm{d}\xi) \leq \int_{\{\langle \xi, \mathbb{1} \rangle > 1\}} \log(\langle \xi, \mathbb{1} \rangle) m(\mathrm{d}\xi) < \infty.$$

This completes the proof. $\qquad\qquad\square$

Proof of Corollary 2.7. Using that $\|\exp(t\widetilde{B})\| \leq M \exp(-\delta t)$, where δ is given by (2.7), we have

$$\lim_{t \to \infty} \int_{\mathbb{S}_d^+} y p_t(x, \mathrm{d}y) = \int_0^\infty \mathrm{e}^{s\widetilde{B}} \left(b + \int_{\mathbb{S}_d^+} \xi m(\mathrm{d}\xi)\right) \mathrm{d}s \in \mathbb{S}_d^+.$$

It remains to verify that $\lim_{t \to \infty} \int_{\mathbb{S}_d^+} y p_t(x, \mathrm{d}y) = \int_{\mathbb{S}_d^+} y \pi(\mathrm{d}y)$. To do so, we can proceed similar to the proof of Theorem 2.2. Indeed, by Lemma A.1, we estimate

$$\sup_{t \geq 0} \int_{\mathbb{S}_d^+} \|y\| p_t(x, \mathrm{d}y) \leq \sup_{t \geq 0} \mathrm{tr}\left(\int_{\mathbb{S}_d^+} y p_t(x, \mathrm{d}y)\right) \leq \sqrt{d} \sup_{t \geq 0} \left\|\int_{\mathbb{S}_d^+} y p_t(x, \mathrm{d}y)\right\| < \infty.$$

Therefore, applying the Lemma of Fatou yields

$$\int_{\mathbb{S}_d^+} \|y\| \pi(\mathrm{d}y) \leq \sup_{t \geq 0} \int_{\mathbb{S}_d^+} \|y\| p_t(x, \mathrm{d}y) < \infty.$$

So $\pi \in \mathcal{P}_1(\mathbb{S}_d^+)$. Now, let $\varepsilon > 0$. By dominated convergence theorem, we see that

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{S}_d^+} \frac{1 - \mathrm{e}^{-\langle \varepsilon u, y \rangle}}{\varepsilon} \pi(\mathrm{d}y) = \int_{\mathbb{S}_d^+} \langle u, y \rangle \pi(\mathrm{d}y).$$

Moreover, Noting that, by Proposition 4.1,

$$1 - \mathrm{e}^{-\langle \psi(s, \varepsilon u), \xi \rangle} \leq \langle \psi(s, \varepsilon u), \xi \rangle \leq \|\xi\| \|\varepsilon u\| \mathrm{e}^{-\delta t},$$

we can use once again the dominated convergence theorem to obtain

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{S}_d^+} \frac{1 - \mathrm{e}^{-\langle \varepsilon u, y \rangle}}{\varepsilon} \pi(\mathrm{d}y) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^\infty F(\psi(s, \varepsilon u))\mathrm{d}s$$

$$= \lim_{\varepsilon \searrow 0} \int_0^\infty \left(\langle b, \frac{\psi(s, \varepsilon u)}{\varepsilon} \rangle \mathrm{d}s + \int_{\mathbb{S}_d^+ \setminus \{0\}} \frac{1 - \mathrm{e}^{-\langle \psi(s, \varepsilon u), \xi \rangle}}{\varepsilon} m(\mathrm{d}\xi) \right) \mathrm{d}s$$

$$= \int_0^\infty \langle b, D\psi(s, 0)(u) \rangle \mathrm{d}s + \int_0^\infty \int_{\mathbb{S}_d^+ \setminus \{0\}} \langle D\psi(s, 0)(u), \xi \rangle m(\mathrm{d}\xi) \mathrm{d}s$$

$$= \int_0^\infty \langle \mathrm{e}^{s\widetilde{B}} \left(b + \int_{\mathbb{S}_d^+ \setminus \{0\}} \xi m(\mathrm{d}\xi) \right), u \rangle \mathrm{d}s,$$

where we used that $D\psi(s,0)(u) = \exp(s\widetilde{B}^\top)u$ (see the proof of Theorem 2.2). Since the latter identity holds for all $u \in \mathbb{S}_d^+$, we conclude with our proof. □

6. Proof of Theorem 2.8

Proof of Theorem 2.8. Suppose that (2.8) holds. By definition of d_L, we have

$$d_L \left(p_t(x, \mathrm{d}\xi), \pi(\mathrm{d}\xi) \right)$$

$$= \sup_{u \in \mathbb{S}_d^+ \setminus \{0\}} \frac{1}{\|u\|} \left| \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) - \int_{\mathbb{S}_d^+} \mathrm{e}^{-\langle u, \xi \rangle} \pi(\mathrm{d}\xi) \right|$$

$$= \sup_{u \in \mathbb{S}_d^+ \setminus \{0\}} \frac{1}{\|u\|} \left| \exp\left(-\int_0^t F(\psi(s, u)) \, \mathrm{d}s - \langle x, \psi(t, u) \rangle \right) - \exp\left(-\int_0^\infty F(\psi(s, u)) \, \mathrm{d}s \right) \right|.$$

$$\tag{6.1}$$

Let $C > 0$ be a generic constant that may vary from line to line. Using then (5.1), we have, for each $t \geq 0$,

$$\left| \exp\left(-\int_0^t F(\psi(s, u)) \, \mathrm{d}s - \langle x, \psi(t, u) \rangle \right) - \exp\left(-\int_0^\infty F(\psi(s, u)) \, \mathrm{d}s \right) \right|$$

$$\leq \left| \exp\left(-\langle x \psi(t, u) \rangle \right) - 1 \right| \cdot \left| \exp\left(-\int_0^t F(\psi(s, u)) \, \mathrm{d}s \right) \right|$$

$$+ \left| \exp\left(-\int_0^t F(\psi(s, u)) \, \mathrm{d}s \right) - \exp\left(-\int_0^\infty F(\psi(s, u)) \, \mathrm{d}s \right) \right|$$

$$\leq \left| \langle x, \psi(s, u) \rangle \right| + \left| \int_t^\infty F(\psi(s, u)) \, \mathrm{d}s \right|$$

$$\leq M \|x\| \|u\| \mathrm{e}^{-t\delta} + C \|u\| \int_t^\infty \mathrm{e}^{-s\delta} \mathrm{d}s$$

$$\leq C \left(1 + \|x\| \right) \|u\| \mathrm{e}^{-t\delta},$$

which when plugged back into (6.1) implies (2.11). □

7. Proof of Theorem 2.9

Proof of Theorem 2.9. Note that $\pi \in \mathcal{P}_1(\mathbb{S}_d^+)$ by Corollary 2.7. Let $q_t(x, \mathrm{d}\xi)$ be transition kernel for the conservative, subcritical affine processes with admissible parameters $(\alpha = 0, b = 0, B, m = 0, \mu)$. Using the particular form of the Laplace transform for $p_t(x, \cdot)$ (see (2.1)) it is not difficult to see that $p_t(x, \cdot) = q_t(x, \cdot) * p_t(0, \cdot)$, where '$*$' denotes the convolution of measures. Let H be any coupling with marginals δ_x and π, i.e., $H \in \mathcal{H}(\delta_x, \pi)$. Using the invariance of π, together with the convexity of W_1 (see [42, Theorem 4.8]) and [16, Lemma 2.3], we find

$$W_1 \left(p_t(x, \cdot), \pi \right) = W_1 \left(\int_{\mathbb{S}_d^+} p_t(y, \cdot) \delta_x(\mathrm{d}y), \int_{\mathbb{S}_d^+} p_t(y', \cdot) \pi(\mathrm{d}y') \right)$$

$$\leq \int_{\mathbb{S}_d^+ \times \mathbb{S}_d^+} W_1\left(p_t(y,\cdot), p_t(y',\cdot)\right) H(\mathrm{d}y, \mathrm{d}y')$$

$$\leq \int_{\mathbb{S}_d^+ \times \mathbb{S}_d^+} W_1\left(q_t(y,\cdot), q_t(y',\cdot)\right) H(\mathrm{d}y, \mathrm{d}y').$$

The integrand can now be estimated as follows

$$W_1\left(q_t(y,\cdot), q_t(y',\cdot)\right) \leq \int_{\mathbb{S}_d^+ \times \mathbb{S}_d^+} \|z - z'\| G(\mathrm{d}z, \mathrm{d}z')$$

$$\leq \int_{\mathbb{S}_d^+} \|z\| q_t(y, \mathrm{d}z) + \int_{\mathbb{S}_d^+} \|z'\| q_t(y', \mathrm{d}z')$$

$$\leq M\sqrt{d}\mathrm{e}^{-t\delta}\left(\|y\| + \|y'\|\right),$$

where G is any coupling of $(q_t(y,\cdot), q_t(y',\cdot))$ and we have used Lemma A.1 to obtain

$$\int_{\mathbb{S}_d^+} \|z\| q_t(y, \mathrm{d}z) \leq \mathrm{tr}\left(\int_{\mathbb{S}_d^+} z q_t(y, \mathrm{d}z)\right)$$

$$= \mathrm{tr}\left(\mathrm{e}^{t\widetilde{B}}y\right)$$

$$\leq \sqrt{d}\left\|\mathrm{e}^{t\widetilde{B}}y\right\|$$

$$\leq M\sqrt{d}\mathrm{e}^{-t\delta}\|y\|.$$

Combining these estimates, we obtain

$$W_1(p_t(x,\cdot), \pi) \leq M\sqrt{d}\mathrm{e}^{-t\delta} \int_{\mathbb{S}_d^+ \times \mathbb{S}_d^+} \left(\|y\| + \|y'\|\right) H(\mathrm{d}y, \mathrm{d}y')$$

$$\leq M\sqrt{d}\mathrm{e}^{-t\delta}\left(\|x\| + \int_{\mathbb{S}_d^+} \|y\| \pi(\mathrm{d}y)\right),$$

which yields (2.12). $\qquad\square$

8. APPLICATIONS

Let $(W_t)_{t\geq 0}$ be a $d \times d$-matrix of independent standard Brownian motions. Denote by $(J_t)_{t\geq 0}$ an $\mathbb{S}_d^+$-valued Lévy subordinator with Lévy measure m. Suppose that these two processes are independent of each other. Following [37], the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \left(b + \beta X_t + X_t\beta^\top\right)) \mathrm{d}t + \sqrt{X_t}\mathrm{d}W_t\Sigma + \Sigma^\top\mathrm{d}W_t^\top\sqrt{X_t} + \mathrm{d}J_t & t \geq 0, \\ X_0 = x \in \mathbb{S}_d^+, \end{cases} \tag{8.1}$$

has a unique weak solution if $b \succeq (d-1)\Sigma^\top\Sigma$ and Σ, β are real-valued $d \times d$-matrices. Moreover, according to [37, Corollary 3.2], if $b \succ (d+1)\Sigma^\top\Sigma$, then a unique strong solution also exists. The corresponding Markov process $X = (X_t)_{t\geq 0}$ is a conservative affine process with admissible parameters $(\alpha, b, B, m, 0)$ with diffusion $\alpha = \Sigma^\top\Sigma$ and linear drift $B(x) = \beta x + x\beta^\top$. The functions F and R are given by

$$F(u) = \langle b, u \rangle + \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle u, \xi \rangle}\right) m(\mathrm{d}\xi)$$

and

$$R(u) = -2u\alpha u + u\beta + \beta^\top u.$$

The generalized Riccati equations are now given by

$$\partial_t \phi(t,u) = \langle b, \psi(t,u) \rangle + \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(t,u), \xi \rangle}\right) m(\mathrm{d}\xi),$$

$$\partial_t \psi(t,u) = -2\psi(t,u)\alpha\psi(t,u) + \psi(t,u)\beta + \beta^\top \psi(t,u),$$

with initial conditions $\phi(0,u) = 0$ and $\psi(0,u) = u$. Let $\sigma_t^\beta : \mathbb{S}_d^+ \to \mathbb{S}_d^+$ be given by

$$\sigma_t^\beta(x) := 2 \int_0^t \mathrm{e}^{\beta s} x \mathrm{e}^{\beta^\top s} \mathrm{d}s, \quad t \geq 0.$$

According to [36, Section 4.3], we have

$$\phi(t,u) = \langle b, \int_0^t \psi(s,u)\mathrm{d}s \rangle + \int_0^t \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle}\right) m(\mathrm{d}\xi)\mathrm{d}s,$$

$$\psi(t,u) = \mathrm{e}^{\beta^\top t} \left(u^{-1} + \sigma_t^\beta(\alpha)\right)^{-1} \mathrm{e}^{\beta t}.$$

Since $\widetilde{B}(x) = B(x)$, Remark 2.4 implies that X is subcritical, provided β has only eigenvalues with negative real parts. If the Lévy measure m satisfies (2.8), then Theorem 2.5 implies existence, uniqueness, and convergence to the invariant distribution π whose Laplace transform satisfies

$$\int_0^\infty \mathrm{e}^{-\langle u,x \rangle} \pi(\mathrm{d}x) = \langle b, \int_0^\infty \psi(s,u)\mathrm{d}s \rangle \exp\left(-\int_0^\infty \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle}\right) m(\mathrm{d}\xi)\mathrm{d}s\right).$$

Moreover, if in addition $\int_{\{\|\xi\| \geq 1\}} \|\xi\| m(\mathrm{d}\xi) < \infty$, then we infer from Corollary 2.7 that

$$\lim_{t \to \infty} \mathbb{E}_x[X_t] = \int_0^\infty \mathrm{e}^{s\beta^\top} \left(b + \int_{\mathbb{S}_d^+ \setminus \{0\}} \xi m(\mathrm{d}\xi)\right) \mathrm{e}^{s\beta}\mathrm{d}s = \int_{\mathbb{S}_d^+} y\pi(\mathrm{d}y).$$

We end this section by considering the following examples.

Example 8.1 (The matrix-variate basic affine jump-diffusion and Wishart process). *Take $b = 2k\Sigma^\top\Sigma$ with $k \geq d-1$ in (8.1). This process is called matrix-variate basic affine jump-diffusion on $\mathbb{S}_d^+$ (MBAJD for short), see [36, Section 4]. Following [36, Section 4.3], $\phi(t,u)$ is precisely given by*

$$\phi(t,u) = k \log \det\left(\mathbb{1} + u\sigma_t^\beta(\alpha)\right) \int_0^t \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle}\right) m(\mathrm{d}\xi)\mathrm{d}s,$$

and Theorem 2.5 implies that the unique invariant distribution is given by

$$\int_0^\infty \mathrm{e}^{-\langle u,x \rangle} \pi(\mathrm{d}x) = \left(\det\left(\mathbb{1} + \sigma_\infty^\beta(\alpha)u\right)\right)^{-k} \exp\left(-\int_0^\infty \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{-\langle \psi(s,u), \xi \rangle}\right) m(\mathrm{d}\xi)\mathrm{d}s\right),$$

where $\sigma_\infty^\beta(\alpha) = \int_0^\infty \exp(s\beta)\alpha\exp(s\beta^\top)\mathrm{d}s$.

The well-known Wishart process, introduced by Bru [9], is a special case of the MBAJD with $m = 0$. Existence of a unique distribution was then obtained in [1, Lemma C.1]. In this case π is a Wishart distribution with shape parameter k and scale parameter $\sigma_\infty^\beta(\alpha)$.

Example 8.2 (Matrix-variate Ornstein-Uhlenbeck type processes). *For $b = 0$ and $\Sigma = 0$, we call the solutions to the stochastic differential equation (8.1) matrix-variate Ornstein-Uhlenbeck (shorted OU) type processes, see [7]. Properties of the stationary matrix-variate OU type processes were investigated in [39]. Provided $\int_{\{\|\xi\| \geq 1\}} \|\xi\| m(\mathrm{d}\xi) < \infty$, Theorem 2.9 implies that the matrix-variate OU type process is also exponentially ergodic in the Wasserstein-1-distance.*

Appendix A. Matrix calculus

For a $d \times d$ square matrix x, recall that $\mathrm{tr}(x) = \sum_{i=1}^{d} x_{ii}$. The Frobenius norm of x is given by $\|x\| = \mathrm{tr}(xx)^{1/2} = (\sum_{i,j=1}^{d} |x_{ij}|^2)^{1/2}$. Let us collect one property of this norm.

Lemma A.1. *Let $x \in \mathbb{S}_d^+$, then*

$$\|x\| \leq \mathrm{tr}(x) \leq \sqrt{d}\|x\|.$$

Proof. Write $x = u^\top \kappa u$, where u is orthogonal and κ is diagonal with its entries being given by $\lambda_i(x)$, $i, \ldots, d$, the eigenvalues of x. We have

$$\|x\|^2 = \mathrm{tr}\left(u^\top \kappa^2 u\right) = \sum_{i=1}^{d} \lambda_i(x)^2.$$

Since $x \in \mathbb{S}_d^+$, it holds that $\lambda_i(x) \geq 0$, $i = 1, \ldots, d$. Then

$$\|x\| = \left(\sum_{i=1}^{d} \lambda_i^2(x)\right)^{1/2} \leq \sum_{i=1}^{d} \lambda_i(x) \leq \sqrt{d}\left(\sum_{i=1}^{d} \lambda_i^2(x)\right)^{1/2} = \sqrt{d}\|x\|.$$

$\square$

References

1. Aurélien Alfonsi, Ahmed Kebaier, and Clément Rey, *Maximum likelihood estimation for Wishart processes*, Stochastic Process. Appl. **126** (2016), no. 11, 3243–3282. MR 3549707
2. Leif B. G. Andersen and Vladimir V. Piterbarg, *Moment explosions in stochastic volatility models*, Finance Stoch. **11** (2007), no. 1, 29–50. MR 2284011
3. Jan Baldeaux and Eckhard Platen, *Functionals of multidimensional diffusions with applications to finance*, Bocconi & Springer Series, vol. 5, Springer, Cham; Bocconi University Press, Milan, 2013. MR 3113191
4. Mátyás Barczy, Leif Döring, Zenghu Li, and Gyula Pap, *Stationarity and ergodicity for an affine two-factor model*, Adv. in Appl. Probab. **46** (2014), no. 3, 878–898. MR 3254346
5. Mátyás Barczy, Zenghu Li, and Gyula Pap, *Stochastic differential equation with jumps for multi-type continuous state and continuous time branching processes with immigration*, ALEA Lat. Am. J. Probab. Math. Stat. **12** (2015), no. 1, 129–169. MR 3340375
6. _____, *Moment formulas for multitype continuous state and continuous time branching process with immigration*, J. Theoret. Probab. **29** (2016), no. 3, 958–995. MR 3540486
7. Ole Eiler Barndorff-Nielsen and Robert Stelzer, *Positive-definite matrix processes of finite variation*, Probab. Math. Statist. **27** (2007), no. 1, 3–43. MR 2353270
8. Francesca Biagini, Alessandro Gnoatto, and Maximilian Härtel, *Long-term yield in an affine HJM framework on S_d^+*, Appl. Math. Optim. **77** (2018), no. 3, 405–441. MR 3787835
9. Marie-France Bru, *Wishart processes*, J. Theoret. Probab. **4** (1991), no. 4, 725–751. MR 1132135
10. Carl Chiarella, José Da Fonseca, and Martino Grasselli, *Pricing range notes within Wishart affine models*, Insurance Math. Econom. **58** (2014), 193–203. MR 3257351
11. Christa Cuchiero, Damir Filipović, Eberhard Mayerhofer, and Josef Teichmann, *Affine processes on positive semidefinite matrices*, Ann. Appl. Probab. **21** (2011), no. 2, 397–463. MR 2807963
12. D. A. Dawson and Zenghu Li, *Skew convolution semigroups and affine Markov processes*, Ann. Probab. **34** (2006), no. 3, 1103–1142. MR 2243880
13. D. Duffie, D. Filipović, and W. Schachermayer, *Affine processes and applications in finance*, Ann. Appl. Probab. **13** (2003), no. 3, 984–1053.
14. Damir Filipović and Eberhard Mayerhofer, *Affine diffusion processes: theory and applications*, Advanced financial modelling, Radon Ser. Comput. Appl. Math., vol. 8, Walter de Gruyter, Berlin, 2009, pp. 125–164. MR 2648460
15. José Da Fonseca, Martino Grasselli, and Claudio Tebaldi, *Option pricing when correlations are stochastic: an analytical framework*, Review of Derivatives Research **10** (2007), no. 2, 151–180.
16. Martin Friesen, Peng Jin, Jonas Kremer, and Barbara Rüdiger, *Exponential ergodicity for stochastic equations of nonnegative processes with jumps*, arXiv e-prints (2019), arXiv:1902.02833.
17. Martin Friesen, Peng Jin, and Barbara Rüdiger, *Stochastic equation and exponential ergodicity in Wasserstein distances for affine processes*, arXiv e-prints (2019), arXiv:1901.05815.

18. Paul Glasserman and Kyoung-Kuk Kim, *Moment explosions and stationary distributions in affine diffusion models*, Math. Finance **20** (2010), no. 1, 1–33. MR 2599675

19. Alessandro Gnoatto, *The Wishart short rate model*, Int. J. Theor. Appl. Finance **15** (2012), no. 8, 1250056, 24. MR 3011743

20. Alessandro Gnoatto and Martino Grasselli, *An affine multicurrency model with stochastic volatility and stochastic interest rates*, SIAM J. Financial Math. **5** (2014), no. 1, 493–531. MR 3252805

21. Christian Gourieroux and Razvan Sufana, *Derivative pricing with Wishart multivariate stochastic volatility*, J. Bus. Econom. Statist. **28** (2010), no. 3, 438–451. MR 2723611

22. _____, *Discrete time Wishart term structure models*, J. Econom. Dynam. Control **35** (2011), no. 6, 815–824. MR 2785429

23. Martino Grasselli and Giulio Miglietta, *A flexible spot multiple-curve model*, Quant. Finance **16** (2016), no. 10, 1465–1477. MR 3564921

24. Steven L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies (1993), 6:327?343.

25. Roger A. Horn and Charles R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013. MR 2978290

26. Rudra P. Jena, Kyoung-Kuk Kim, and Hao Xing, *Long-term and blow-up behaviors of exponential moments in multi-dimensional affine diffusions*, Stochastic Process. Appl. **122** (2012), no. 8, 2961–2993. MR 2931348

27. Peng Jin, Jonas Kremer, and Barbara Rüdiger, *Existence of limiting distribution for affine processes*, arXiv e-prints (2018), arXiv:1812.05402.

28. Kiyoshi Kawazu and Shinzo Watanabe, *Branching processes with immigration and related limit theorems*, Teor. Verojatnost. i Primenen. **16** (1971), 34–51. MR 0290475

29. Martin Keller-Ressel, *Moment explosions and long-term behavior of affine stochastic volatility models*, Math. Finance **21** (2011), no. 1, 73–98. MR 2779872 (2012e:91126)

30. Martin Keller-Ressel and Eberhard Mayerhofer, *Exponential moments of affine processes*, Ann. Appl. Probab. **25** (2015), no. 2, 714–752. MR 3313754

31. Martin Keller-Ressel and Aleksandar Mijatović, *On the limit distributions of continuous-state branching processes with immigration*, Stochastic Process. Appl. **122** (2012), no. 6, 2329–2345. MR 2922631

32. Martin Keller-Ressel, Walter Schachermayer, and Josef Teichmann, *Affine processes are regular*, Probab. Theory Related Fields **151** (2011), no. 3-4, 591–611. MR 2851694 (2012k:60219)

33. Martin Keller-Ressel and Thomas Steiner, *Yield curve shapes and the asymptotic short rate distribution in affine one-factor models*, Finance Stoch. **12** (2008), no. 2, 149–172. MR 2390186 (2009c:60222)

34. Markus Leippold and Fabio Trojani, *Asset pricing with matrix jump diffusions*, (Working paper (2010)).

35. Zenghu Li, *Measure-valued branching Markov processes*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR 2760602

36. Eberhard Mayerhofer, *Affine processes on positive semidefinite $d \times d$ matrices have jumps of finite variation in dimension $d > 1$*, Stochastic Process. Appl. **122** (2012), no. 10, 3445–3459. MR 2956112

37. Eberhard Mayerhofer, Oliver Pfaffel, and Robert Stelzer, *On strong solutions for positive definite jump diffusions*, Stochastic Process. Appl. **121** (2011), no. 9, 2072–2086. MR 2819242

38. Eberhard Mayerhofer, Robert Stelzer, and Johanna Vestweber, *Geometric Ergodicity of Affine Processes on Cones*, arXiv e-prints (2018), arXiv:1811.10542.

39. Christian Pigorsch and Robert Stelzer, *On the definition, stationary distribution and second order structure of positive semidefinite Ornstein-Uhlenbeck type processes*, Bernoulli **15** (2009), no. 3, 754–773. MR 2555198

40. Ken-iti Sato and Makoto Yamazato, *Operator-self-decomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type*, Stochastic Process. Appl. **17** (1984), no. 1, 73–100. MR 738769

41. Hans Schneider, *Positive operators and an inertia theorem*, Numer. Math. **7** (1965), 11–17. MR 0173678

42. Cédric Villani, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009, Old and new. MR 2459454

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