Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 19/17

Nneka Umeorah, Matthias Ehrhardt and Phillip Mashele

Valuation of basket credit default swaps under stochastic default intensity models

May 2019

http://www.math.uni-wuppertal.de

Valuation of basket credit default swaps under stochastic default intensity models

Nneka Umeorah*[1], Matthias Ehrhardt†[2], and Phillip Mashele‡[1]

[1]Business Mathematics and Informatics, North-West University, Potchefstroom, South Africa

[2]Applied Mathematics and Numerical Analysis, Bergische Universität Wuppertal, Germany

Abstract

Portfolio credit derivatives, including the basket credit default swaps, are designed to facilitate the transfer of credit risk amongst market participants. Investors consider them as cheap tools to hedge a portfolio of credits, instead of individual hedging of the credits. The prime aim of this work is to model the hazard rate process using stochastic default intensity models, as well as extend the results to the pricing of basket default swaps.

We focused on the nth-to-default swaps whereby the spreads are dependent on the nth default time, and we estimated the joint survival probability distribution functions of the intensity models under the risk-neutral pricing measure, for both the homogeneous and the heterogeneous portfolio.

This work further employed the Monte-Carlo method, under the one-factor Gaussian copula model to numerically approximate the distribution function of the default time, and thus, the numerical experiments for pricing the nth default swaps were made viable. Finally, we compared the effects of different swap parameters to various nth-to-default swaps.

Keywords: Portfolio credit derivatives, basket default swaps, Gaussian copula, Monte-Carlo simulations, stochastic intensity modelling, hazard rate, joint survival probability distribution.

1 Introduction

Basket default swaps (BDS) are financial contracts that payoff whenever there is a default or multiple defaults among a portfolio of entities or obligors. From the investor's point of view, BDS are still preferable because they limit the credit portfolio that an investor can easily monitor compared to the large portfolio obtainable in a typical synthetic collateralised debt obligation (CDO)[1][24]. BDS are generally classified into first-to-default, nth-to-default, n-out-of-m-to-default and all-to-default. The protection buyer finds the BDS attractive because the

*umeorahnnekaozioma@gmail.com

†ehrhardt@uni-wuppertal.de

‡phillip.mashele@nwu.ac.za

[1]Small scale portfolio size includes 5-10 credits, and large portfolio scale ranges from 100-150 credits.

cost of purchasing protection on a portfolio of entities is less expensive compared to the purchase of an individual protection. Considering the nth-to-default (n2D), the protection seller enjoys a limited downside risk, and this stems from the fact that at most one default is expected for the seller to cover [2]. After the payments pending default and contract finalised, the loss incurred resulting from further default is borne only by the protection buyer. The recovery rate, number of entities in the portfolio, the entity's credit ratings, as well as the default correlations are some of the major factors that affect the BDS spread.

Several research works have been carried out in the area of basket default swaps. Laurent and Gregory (2005) considered the valuation of BDS and the CDO. They obtained semi-analytic values for the above contingent claims under the assumption of independent default times conditioned on a low dimensional factor, and based their valuation on mean-variance mixture models and frailty models. Kijima and Muromachi (2000) priced BDS using a stochastic intensity-based model under the assumption of conditional independence. Using the joint survival probability, they further provided closed form values when the intensity process is defined within the extended Vasicek model. Liang *et al.* (2011) provided the limitation of using the Vasicek model in modelling the intensity process. They explained using some numerical examples that the Vasicek model can be efficient when the portfolio has relatively few correlated risky assets and their valuations are extended to the prices of BDS, credit default swap (CDS) index and CDOs. Fathi and Nader (2007a) provided Monte-Carlo methods and semi-explicit expressions which improves the cost-effectiveness of the Monte-Carlo approach to price multi-named credit derivatives like the basket default swaps and CDO tranches. They further calibrated the prices of BDS to the Japanese financial markets [8]. Iscoe and Kreinin (2006) proposed a recursive algorithm to value BDS based on a continuous-time model in the conditional independence framework. They employed the concept of the order statistics of the default times of the entities in a portfolio, and then applied it in the estimation of first-to-default and the second-to-default contracts.

Regardless of whether the reduced form method or the structural approach in modelling the joint default events in portfolio credit derivatives, there is always the problem of the joint probability distribution of the default times, and many researchers have employed Copula models. Frey and McNeil (2003) explored the role of copulas in latent variable models and used the modified Gaussian copula method to model the dependent defaults evident in a portfolio credit risk. Mashal and Naldi (2002) used the student t-copulas which posses non-trivial tail dependency structure, as well as the ability for more joint extreme events. They applied it to the price estimations of the multi-name instruments, like the n2D baskets and CDOs. Schönbucher and Schubert (2001) used Archimedean copulas in general, and in particular, employed the Gumbel and Clayton copulae in the modelling of the default dependency structure in the intensity models. Li *et al.* (2015) focused on the use of the single-factor Gaussian-NIG-copula model in the simulation of the distribution functions, as well as to obtain the correlation structure between the assets and further applied the concepts to the valuation of BDS.

Choe and Jang (2011) included the one factor Gaussian copula model to derive the probability distribution function of the nth default time explicitly and thus, valued the n2D and n-out-of-m-to-default. Bluhm *et al.* (2002) defined the nth default time as an order statistic for $n \leq N$, where N is the number of reference entities and thus, they obtained the distribution of the time $\tau^{n\text{th}}$ of the nth default. The Joshi-Kainth algorithm [16] is an innovative importance sampling technique also employed in modelling the default time of the process. Chen and Glasserman (2008) proposed a modification of the technique and ensured that there exist variance reduction even when the defaults does not seem to occur. Usually, in the pricing of n2D swaps, as the size of the entities increase, the pricing becomes computationally intense. Schröter and Heider, (2013) derived the default time distribution and simplified the pricing problem from an n-dimensional quadrature to a one-dimensional quadrature, thereby breaking its curse of dimensionality. Jouanin, *et al.* (2002) incorporated the modelling of correlated default events in the intensity-based framework. They estimated the marginal probability distribution functions for each default and then, modelled the joint distribution with the aid of a copula function.

In this work, however, we modelled the intensity process using the Vasicek model and Cox-Ingersoll-Ross (CIR) model, owing to their analytical tractability. The joint survival probability distributions (JSPD) for the homogeneous portfolio under both models and the JSPD for the heterogeneous portfolio under the Vasicek model are obtained. For a heterogeneous portfolio, we consider one which consists of five entities from the corporate sector (with different credit ratings), obtained the default intensity, estimated the parameters of the default intensity framework and solved for their JSPD. We further considered a homogenised portfolio and employed the Monte-Carlo method to simulate their default time distribution function and thus, obtained the prices for different categories of n2D.

The organisation of the work is as follows: Section 1 introduces the topic and highlights some of the recent work done on the pricing of BDS. Section 2 discusses the model structure, outlines the density functions of both the Vasicek and the CIR model. It further introduces the model for intensity default and highlights the model for bond valuation under both Vasicek and the CIR processes. Section 3 focuses on the pricing of BDS and explains how the swap spread can be obtained. It further introduces the concept of default time modelling. Section 4 outputs results on the parameter estimation of the heterogeneous portfolios, as well as their JSPD. It also gives some numerical experiments on the BDS valuations, as well as some sensitivity analysis on the swap spreads. Section 5 concludes our research study.

2 Model Structure

In modelling the intensity process, we consider the 'economically viable' property of mean-reverting and Ornstein–Uhlenbeck process. This process ensures that the hazard rate does not explode and hence tends to infinity. Consider the given stochastic differential equation (SDE)

$$\mathrm{d}\lambda(t) = \alpha\big(\beta - \lambda(t)\big)\,\mathrm{d}t + \sigma\big(\lambda(t)\big)^{\gamma}\,\mathrm{d}W(t)\,, \tag{1}$$

where α denotes the speed of reversion, β is the long term mean, σ is volatility and $W(t)$ denotes the standard Brownian motion. The parameters α, β, σ are all positive constants. The drift term which is of the form $\alpha(\beta-\lambda(t))$ is linear and captures the full rate of mean reversion. The volatility term defined by $\sigma(\lambda(t))^{\gamma}$ is non-linear and the term γ parametrizes the extent to which the process $\lambda(t)$ depends on its level [7]. The above SDE (1) follows the model proposed by Chan, Karolyi, Longstaff and Sanders (CKLS) in which many one-factor and multi-factor stochastic interest rate models are nested together [4]. They compared the models to obtain which best fits the short-term interest rate process. From their findings, the best fit models possess the characteristics that their conditional volatility of interest rate changes fully depend on the interest rate level.

For $\gamma = 0$, we have the Vasicek model, $\gamma = 1/2$ gives the CIR model, $\gamma = 1$ gives the Brennan-Schwartz model, etc. Let the given process $\lambda(t)$, been considered to be the hazard rate function or the default process. This function refers to the probability of default for a given time interval which is conditioned on no prior default. These defaults, also known as credit events could be as a result of failure to pay, bankruptcy, downgrade, restructuring, etc. [24]. In this work, our focus shall be on the Vasicek and CIR models.

Under the risk-neutral measure Q, the process λ defined by the Vasicek model follows a normal distribution with mean and variance given respectively as:

$$\mathbb{E}^Q\big[\lambda(T)|\mathcal{F}(t)\big] = \lambda(t)\,\mathrm{e}^{-\alpha(T-t)} + \beta\big(1-\mathrm{e}^{-\alpha(T-t)}\big)\,,$$
$$\mathrm{Var}^Q\big[\lambda(T)\big] = \frac{\sigma^2}{2\alpha}\big(1-\mathrm{e}^{-2\alpha(T-t)}\big)\,.$$

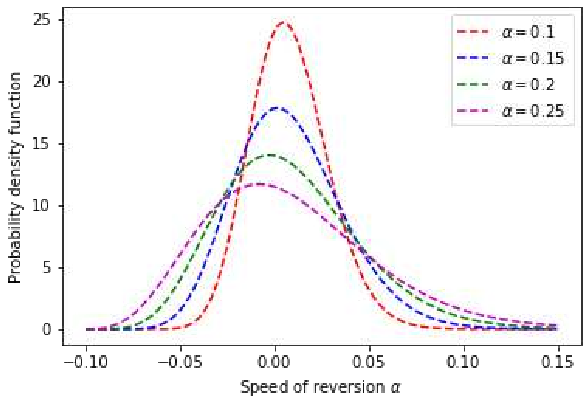
For the CIR model, the process $\lambda(t)$ follows a non-central chi-squared distribution $\chi^2_{v,\eta}$ with v and η denoting the degree of freedom and the non-centrality parameter, respectively. Under the risk-neutral pricing measure, the probability density function is defined as [3]:

$$f\big[\lambda(T)|\lambda(t)\big] := c\cdot\chi^2_{v,\eta}\big[c\lambda(t)\big]\,,$$

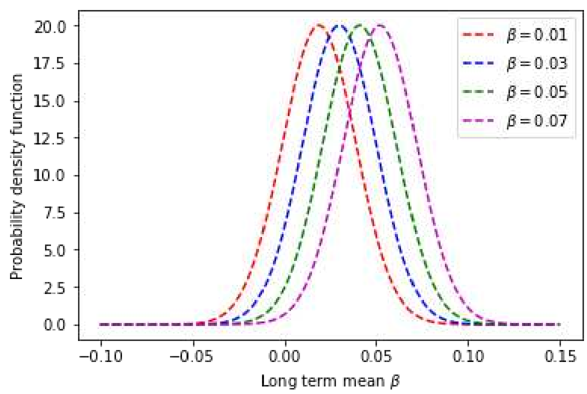
where

$$c = \frac{4\alpha}{\sigma^2(1-\mathrm{e}^{-\alpha(T-t)})}, \qquad v = \frac{4\alpha\beta}{\sigma^2}, \qquad \eta = c\lambda(T)\mathrm{e}^{-\alpha(T-t)}\,.$$

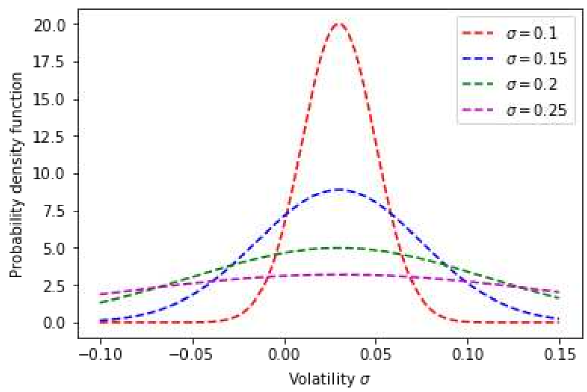
The following plots depict the probability density functions (PDF) of both the Vasicek and the CIR models[2]. We plot the PDF with respect to various parameters of each of the model.



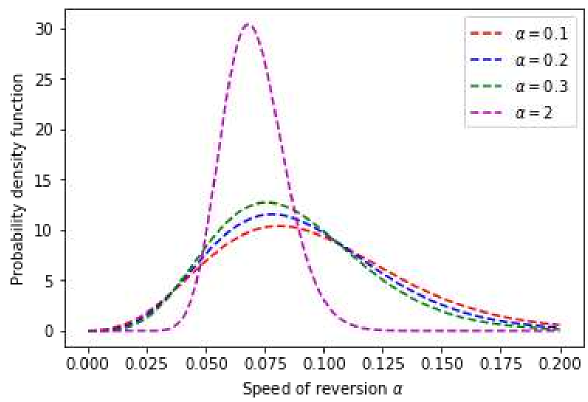
(a) PDF of Vasicek vs α



(b) PDF of Vasicek vs β

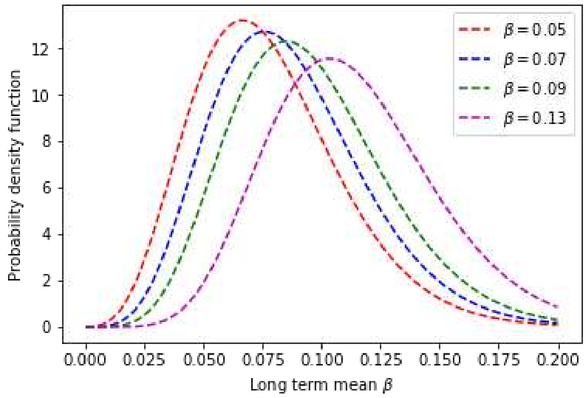


(c) PDF of Vasicek vs σ

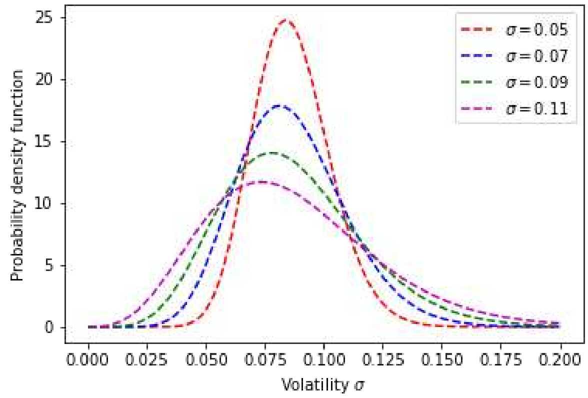


(d) PDF of CIR vs α

[2]See the link `https://github.com/NnekaU/Codes/blob/master/BDS%20codes.ipynb` for the `ipython notebook` codes of all the plots and tables obtained in this work.



(e) PDF of CIR vs β



(f) PDF of CIR vs σ

Figure 1: PDF of the Vasicek and the CIR model versus the parameters α, β and σ.

For both models, the speed reversion parameter α identifies the velocity at which the process evolves around the long term mean β. Increasing α reduces the variance, and the process attains to β faster. The size of the distribution is not affected when β is increased, though the mean of the distribution is affected. Larger volatility σ results to a more flatter surface and thus greater randomness of the process is observed. Additional observations from the CIR model are the presence of the non-negative values of the hazard rate function, resulting from the standard deviation factor $\sigma\sqrt{\lambda(t)}$, as well as an adjustment in the shape of the density function owing to the β increment.

2.1 A model for Intensity Default

CIR Model: Let the default process $\lambda_i(t)$ be modelled by the following SDE:

$$\mathrm{d}\lambda_i(t) = \alpha_i\big(\beta_i - \lambda_i(t)\big)\,\mathrm{d}t + \sigma_i\sqrt{\lambda_i(t)}\,\mathrm{d}W_i(t)\,, \tag{2}$$

where $t \geq 0$, $\alpha_i, \beta_i, \sigma_i$ are all positive constants, with the condition $2\alpha_i\beta_i > \sigma_i^2$ and $i = 0, 1, \dots, n$. The correlation of the standard Brownian motion $W_i(t)$ and $W_j(t)$, where $i, j = 0, 1, \dots, n$, under the risk-neutral probability measure $\mathbb{Q}$ reads:

$$\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \begin{cases} \rho_{ij}\,\mathrm{d}t & \text{for } i \neq j \\ \mathrm{d}t & \text{for } i = j\,. \end{cases}$$

The model posses analytical tractability and thus the SDE (2) has a solution of the form:

$$\lambda_i(t) = \lambda_i(0)\,\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i\mathrm{e}^{-\alpha_i(t-s)}\,\mathrm{d}s + \sigma_i\int_0^t \mathrm{e}^{-\alpha_i(t-s)}\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\,. \tag{3}$$

According to Brigo and Mercurio (2007), the first two moments can be obtained as:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[\lambda_i(t)\big] &= \lambda_i(0)\,\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i\,\mathrm{e}^{-\alpha_i(t-s)}\,\mathrm{d}s\,, \\
\mathrm{Var}^{\mathbb{Q}}\big[\lambda_i(t)\big] &= \frac{\sigma_i^2\lambda_i(0)}{\alpha_i}\big(\mathrm{e}^{-\alpha_i t} - \mathrm{e}^{-2a_i t}\big) + \frac{\sigma_i^2\beta_i}{2\alpha_i}\big(1 - \mathrm{e}^{-\alpha_i t}\big)^2\,.
\end{aligned}$$

Consider a credit event of n values whose default times are denoted by τ_i, for $i = 1, 2, \dots, n$. We first aim at obtaining the joint probability of default times to price basket derivatives and under the intensity process, it is defined as [18]:

$$P(\tau_0 > t_0, \tau_1 > t_1, \ldots, \tau_n > t_n) = \prod_{i=0}^{n} P(\tau_i > t_i) := \mathbb{E}\left[\exp\left(-\sum_{i=0}^{n} H_i(t_i)\right)\right], \tag{4}$$

where $H_i(t) = \int_0^t \lambda_i(s)\,\mathrm{d}s$. Thus, we have

$$\prod_{i=0}^{n} P(\tau_i > t_i) =$$

$$\mathbb{E}\left[\exp\left(-\sum_{i=0}^{n}\left[\int_0^{t_i}\left(\lambda_i(0)\,\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i \mathrm{e}^{-\alpha_i(t-s)}\,\mathrm{d}s + \sigma_i \int_0^t \mathrm{e}^{-a_i(t-s)}\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\right)\,\mathrm{d}t\right]\right)\right]$$

$$= \mathbb{E}\left[\exp\left(-\sum_{i=0}^{n} B_i(t_i) - \sum_{i=0}^{n}\sigma_i I_i(t_i)\right)\right],$$

where

$$B_i(t_i) = \int_0^{t_i}\left(\lambda_i(0)\,\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i\,\mathrm{e}^{-\alpha_i(t-s)}\,\mathrm{d}s\right)\mathrm{d}t = \frac{\lambda_i(0)}{\alpha_i}\left(1 - \mathrm{e}^{-\alpha_i t_i}\right) + \int_0^{t_i}\beta_i\left(1 - \mathrm{e}^{-\alpha_i(t_i-s)}\right)\mathrm{d}s \tag{5}$$

and

$$I_i(t_i) = \int_0^{t_i}\int_0^t \mathrm{e}^{-\alpha_i(t-s)}\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\mathrm{d}t = \frac{1}{\alpha_i}\int_0^{t_i}\left(1 - \mathrm{e}^{-\alpha_i(t-s)}\right)\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s).$$

We seek for the covariance

$$\mathrm{Cov}(I_i(t_i), I_j(t_j)) = \mathbb{E}\big[I_i(t_i)\cdot I_j(t_j)\big] - \mathbb{E}\big[I_i(t_i)\big]\cdot\mathbb{E}\big[I_j(t_j)\big]$$

and denote the covariance for the ith and jth counterparties as $c_{ij}(t_i, t_j)$. Then

$$\begin{aligned}
c_{ij}(t_i, t_j) =& \mathbb{E}\left[\int_0^{t_i}\int_0^{t_j}\frac{1}{\alpha_i\alpha_j}\left(1 - \mathrm{e}^{-\alpha_i(t_i-s)}\right)\left(1 - \mathrm{e}^{-\alpha_j(t_j-s)}\right)\cdot\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\sqrt{\lambda_j(s)}\,\mathrm{d}W_j(s)\right] \\
=& \frac{1}{\alpha_i\alpha_j}\int_0^{t_i}\int_0^{t_j}\left(1 - \mathrm{e}^{-\alpha_i(t_i-s)}\right)\left(1 - \mathrm{e}^{-\alpha_j(t_j-s)}\right)\cdot\mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\,\mathrm{d}W_i(s)\,\mathrm{d}W_j(s)\right] \\
=& \frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j}\left(1 - \mathrm{e}^{-\alpha_i(t_i-s)}\right)\left(1 - \mathrm{e}^{-\alpha_j(t_j-s)}\right)\cdot\mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\right]\mathrm{d}s
\end{aligned} \tag{6}$$

from the correlation properties of standard Brownian motion. Also, let us define $t_i \wedge t_j$ as the minimum of t_i and t_j.

Next, we denote the quantity $I = \sum_{i=0}^{n}\sigma_i I_i(t_i)$. We note that I follows a normal distribution

$$I \sim \mathcal{N}\left(0, \sum_{j=0}^{n}\sum_{i=0}^{n}\sigma_i\sigma_j c_{ij}(t_i, t_j)\right).$$

That is,

$$\mathrm{Var}(I) = \mathrm{Var}\left[\sum_{i=0}^{n}\sigma_i I_i(t_i)\right] = \mathrm{Var}\left[\sum_{i=0}^{n}\sigma_i\frac{1}{\alpha_i}\int_0^{t_i}\left(1 - \mathrm{e}^{-\alpha_i(t-s)}\right)\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\right]$$

$$
\begin{aligned}
\mathrm{Var}(I) &= \mathbb{E}\left[\sum_{i=0}^{n}\sigma_i\frac{1}{\alpha_i}\int_0^{t_i}\big(1-\mathrm{e}^{-\alpha_i(t-s)}\big)\sqrt{\lambda_i(s)}\,\mathrm{d}W_i(s)\cdot\sum_{j=0}^{n}\sigma_j\frac{1}{\alpha_j}\int_0^{t_j}\big(1-\mathrm{e}^{-\alpha_j(t-s)}\big)\sqrt{\lambda_j(s)}\,\mathrm{d}W_j(s)\right] \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n}\sigma_i\sigma_j.\frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j}\big(1-\mathrm{e}^{-\alpha_i(t_i-s)}\big)\big(1-\mathrm{e}^{-\alpha_j(t_j-s)}\big)\cdot\mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\right]\mathrm{d}s \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\,.
\end{aligned}
$$

From the moment generating function of I, we have that

$$
\mathbb{E}\big[\mathrm{e}^{-I}\big] = \mathrm{e}^{\frac{1}{2}\mathrm{Var}(I)}\,. \tag{7}
$$

Summing up equations (4) and (6), the JSPD function of the default times τ_i is given as

$$
P(\tau_0>t_0,\tau_1>t_1,\cdots,\tau_n>t_n) = \prod_{i=0}^{n}P(\tau_i>t_i) = \exp\left(-\sum_{i=0}^{n}B_i(t_i)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\right)\,. \tag{8}
$$

For the **Vasicek Model**, the same process is obtainable with the CIR model above, and the difference is the nature of the diffusion process which is of the form $\sigma_i\,\mathrm{d}W_i$. The model is also analytical tractable with the same expectation value as that of the CIR model. The JSPD function under the Vasicek model is given by [17]:

$$
P(\tau_0>t_0,\tau_1>t_1,\ldots,\tau_n>t_n) = \exp\left(-\sum_{i=0}^{n}B_i(t_i)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\right)\,, \tag{9}
$$

where $B_i(t_i)$ is the same as in equation (**??**) and c_{ij} is defined as

$$
\begin{aligned}
c_{ij}(t_i,t_j) &= \mathbb{E}\left[\int_0^{t_i}\int_0^{t_j}\frac{1}{\alpha_i a_j}\big(1-\mathrm{e}^{-\alpha_i(t_i-s)}\big)\big(1-\mathrm{e}^{-\alpha_j(t_j-s)}\big)\cdot\,\mathrm{d}W_i(s)\,\mathrm{d}W_j(s)\right] \\
&= \frac{1}{\alpha_i\alpha_j}\int_0^{t_i}\int_0^{t_j}\big(1-\mathrm{e}^{-\alpha_i(t_i-s)}\big)\big(1-\mathrm{e}^{-\alpha_j(t_j-s)}\big)\cdot\mathbb{E}\left[\mathrm{d}W_i(s)\,\mathrm{d}W_j(s)\right] \\
&= \frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j}\big(1-\mathrm{e}^{-\alpha_i(t_i-s)}\big)\big(1-\mathrm{e}^{-\alpha_j(t_j-s)}\big)\,\mathrm{d}s \qquad (10)\\
&= \frac{\rho_{ij}}{\alpha_i\alpha_j}\left[s-\frac{e^{-\alpha_i(t_i-s)}}{\alpha_i}-\frac{e^{-\alpha_j(t_j-s)}}{\alpha_j}+\frac{e^{-\alpha_i(t_i-s)-\alpha_j(t_j-s)}}{\alpha_i+\alpha_j}\right]_{s=0}^{s=t_i\wedge t_j}\,. \qquad (11)
\end{aligned}
$$

Consider homogenized risky assets in the basket portfolio, where $\alpha_i=\alpha_j=\alpha$, $\beta_i=\beta_j=\beta$, $\sigma_i=\sigma_j=\sigma$, $\rho_{ij}=\rho$, $\lambda_i=\lambda_j=\lambda$, for $1\le i,\ j,\le N$. Then the JSPD function is given by

$$
P(\tau_0>t_0,\tau_1>t_1,\ldots,\tau_n>t_n) = \exp\Big(-B(t)+\frac{1}{2}\sigma^2c(t,t)\Big)\,,
$$

where $B(t)$ for both the Vasicek and CIR is

$$
B(t) = \frac{\lambda(0)(1-e^{-at})}{a}+b\Big(t-\frac{1}{a}(1-e^{-at})\Big)\,,
$$

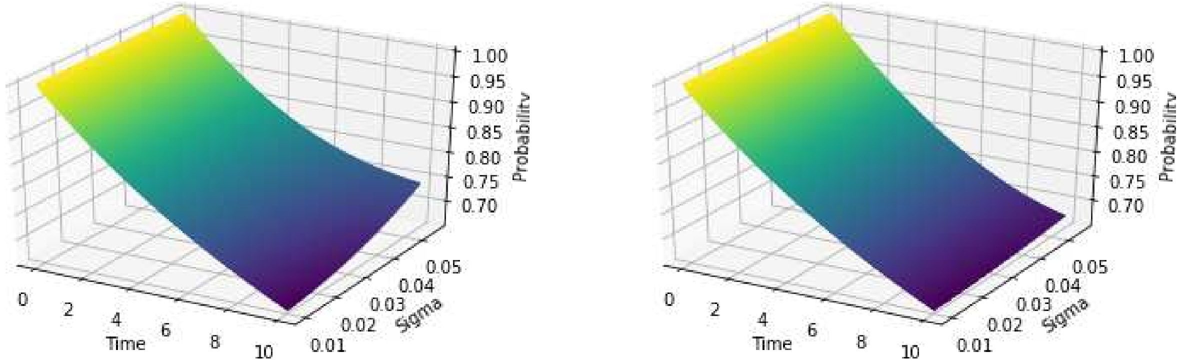
and $c(t,t)$ for the Vasicek reads

$$
c(t,t) = \frac{\rho(2at-3+4e^{-at}-e^{-2at})}{2a^3}\,,
$$

and for the CIR model, we have

$$c(t,t) = \rho \left[\frac{e^{-2at}\big(b - 2\lambda(0)\big) + 4e^{at}\big(b + a(b - \lambda(0))t\big) + e^{2at}\big(2\lambda(0) + b(2at - 5)\big)}{2a^3} \right] .$$

The plots of the above homogenised process for both the Vasicek and the CIR are given below. The parameters considered are $\sigma = 0.05$, $\alpha = 0.1$, $\beta = 0.03$, $\lambda(0) = 0.05$, $T = 10$, $t = 0$ and $\rho = 0.5$. From the plots, we observe that the survival probabilities are declining with increase in time, which in turn, increases the probability of default.



(a) Vasicek Model

(b) CIR model

Figure 2: Joint survival probability distribution for homogenized portfolio.

2.2 Bond Valuation

Let $C(\bar{\lambda}, t, T)$ be the price of a defaultable zero coupon bond with no recovery rate and a face value of 1. Let $\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, where λ_i is the default intensity or default hazard rate defined in (1). Define the pre-first-default bond value as $\hat{C}(\bar{\lambda}, t, T)$ and according to Liang *et al.* (2011), the price at time t of the bond can be written as:

$$C(\bar{\lambda}, t, T) := e^{-r(T-t)} P\{\tau_1 > T, \ldots, \tau_N > T | \mathcal{F}_t\} ,$$

where P is the survival probability and $\mathcal{F}_t$ is the filtration available at the current time t. Under the risk-neutral measure, $\mathbb{E}[\mathrm{d}\hat{C}] = (r + \bar{\lambda})\hat{C}\mathrm{d}t$ and applying Itô's formula, the following PDE for $\hat{C}(\bar{\lambda}, t, T)$ is satisfied [20]:

$$\frac{\partial \hat{C}}{\partial t} + \sum_{i=1}^{N} \alpha_i(\beta_i - \lambda_i)\frac{\partial \hat{C}}{\partial \lambda_i} + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 \hat{C}}{\partial \lambda_i \lambda_j}\left[\lambda_i^{2\gamma}\sigma_i\sigma_j\rho_{ij}\right] = \left(r + \sum_{i=1}^{N}\lambda_i\right)\hat{C} . \tag{12}$$

The affine term structure of the bond is given below:

$$\hat{C} = \exp\left\{ A(t, r; T) - \sum_{i=1}^{N} B_i(t; T)\lambda_i \right\} , \tag{13}$$

The parameters for the **Vasicek model** are obtained from the solution of the ODE

$$\begin{cases} \dfrac{\mathrm{d}A(t,r;T)}{\mathrm{d}t} - \displaystyle\sum_{i=1}^{N} \alpha_i\beta_i B_i(t,T) + \frac{1}{2}\sum_{i,j=1}^{N} B_i(t;T)B_j(t;T)\sigma_i\sigma_j\rho_{ij} - r = 0, \\ \dfrac{\mathrm{d}B_i(t;T)}{\mathrm{d}t} - \alpha_i B_i(t;T) + 1 = 0, \end{cases} \tag{14}$$

subject to the conditions, $A(T,r;T)=0$ and $B_i(T;T)=0$. The solution to equation (13) has the following parameters

$$B_i(t;T)=\frac{1}{\alpha_i}(1-\mathrm{e}^{-\alpha_i(T-t)})$$

and

$$\begin{aligned}A(t,r;T)=&\frac{1}{2}\sum_{i,j=1}^{N}\frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i\alpha_j}\left[(T-t)-B_i(t;T)-B_j(t;T)+\frac{1-\mathrm{e}^{-(\alpha_i+\alpha_j)(T-t)}}{\alpha_i+\alpha_j}\right]\\&-\sum_{i=1}^{N}\beta_i[(T-t)-B_i(t;T)]-r(T-t).\end{aligned}$$

Thus, the bond price is

$$\hat{C}(\bar{\lambda},t,T)=\exp\Big\{A(t;T)-\sum_{i=1}^{N}B_i(t;T)\lambda_i\Big\}. \tag{15}$$

The parameters for the **CIR model** are obtained from the solution of the ODE:

$$\begin{cases}\dfrac{\partial A(t,r;T)}{\partial t}-\displaystyle\sum_{i=1}^{N}\alpha_i\beta_iB_i(t,T)-r=0\\ \dfrac{\partial B_i(t;T)}{\partial t}-\alpha_iB_i(t;T)-\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{N}B_i(t;T)B_j(t;T)\sigma_i\sigma_j\rho_{ij}+1=0,\end{cases} \tag{16}$$

subject to the conditions, $A(T,r;T)=0$ and $B_i(T;T)=0$. For homogenous portfolio basket derivatives, the ODE reduces to

$$\begin{cases}\dfrac{\mathrm{d}A(t,r;T)}{\mathrm{d}t}-\alpha\beta B(t;T)-r=0,\\ \dfrac{\mathrm{d}B(t;T)}{\mathrm{d}t}-\alpha B(t;T)-\dfrac{1}{2}B^2(t;T)\sigma^2\rho+1=0,\end{cases} \tag{17}$$

Remark 1 *The second part of* (16) *takes a Ricatti form, with solution*

$$B(t;T)=\frac{2(\mathrm{e}^{h(T-t)}-1)}{2h+(\alpha\beta+h)(\mathrm{e}^{h(T-t)}-1)}, \tag{18}$$

where $h=\sqrt{(\alpha\beta)^2+2\sigma^2\rho}$.

Furthermore,

$$A(t,r;T)=A(T,r;T)-\int_t^T\partial_sA(s,r,T)\,\mathrm{d}s+r(T-t)-\alpha\beta\int_t^TB(s;T)\,\mathrm{d}s-r(T-t)$$

$$=\frac{-2(\alpha\beta)^2\big[\zeta(T-t)+\log 4-\alpha\log\mathrm{e}^{h(T-t)}\zeta+2\log\alpha\beta\big]}{\zeta(\alpha\beta-h)}-r(T-t),$$

with $\zeta=\alpha\beta+h$. Thus, we have the bond price $\hat{C}$ under the CIR model for the homogeneous basket portfolio with parameters $A(t,r;T)$ and $B(t;T)$.

2.2.1 Demerits on the use of Vasicek model for intensity process

The JSPD of default time τ_i is given by

$$P = P(\tau_1 > T, \tau_2 > T, \ldots, \tau_N > T|\mathcal{F}_t) = \mathrm{e}^{r(T-t)}\hat{C}(\bar{\lambda}, t, T)\,, \tag{19}$$

where $\hat{C}(\bar{\lambda}, t, T)$ is already defined in equation (14). Liang *et al.* (2011) defined the JSPD function, in connection with the number of reference entities N as

$$P(N) = \exp\Big\{\frac{\sigma^2\rho K N^2}{2\alpha^2} - \frac{\sigma^2(\rho-1)KN}{2\alpha^2} + ZN\Big\}\,, \tag{20}$$

where

$$K = (T-t) - \frac{3}{2\alpha} + \frac{2\mathrm{e}^{-\alpha(T-t)}}{\alpha} - \frac{\mathrm{e}^{-2\alpha(T-t)}}{2\alpha} \qquad \text{and}$$

$$Z = \frac{\big(\beta - \lambda(t)\big)\big(1 - e^{-\alpha(T-t)}\big)}{\alpha} - \beta(T-t)\,.$$

The Vasicek model produces a negative default intensity, and as a result, the corresponding JSPD function would exceed 1. Kijima and Muromachi (2000), and Kijima (2000) explained that a restriction has to be imposed on the use of the model. For example, consider a homogenized basket portfolio of $N = 40$ reference obligors, with other parameters: $\alpha = 0.3, \beta = 0.03, \rho = 0.5, \lambda(0) = 0.05, \sigma = 0.035, t = 0, T = 5$. The JSPD for the portfolio under the Vasicek is given below:

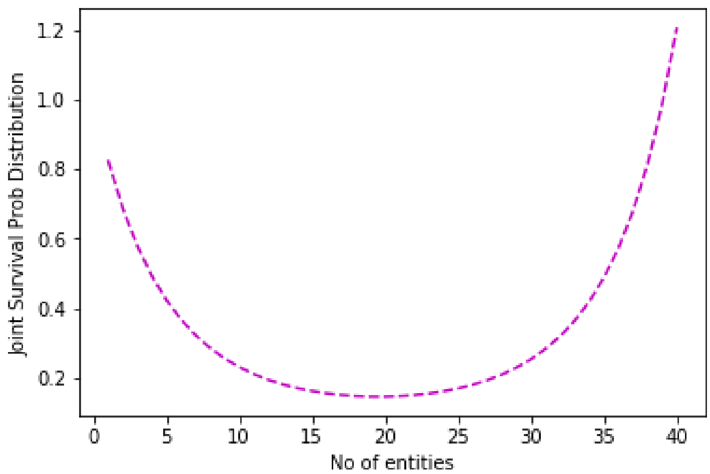


Figure 3: Survival distribution function with respect to reference entities.

From Figure 3, we observe that the survival curve decreases from $P(1) = 0.8255$ with increase in the number of reference entities, attains minimum at $P(20) = 0.14652$, and then goes up to $P(40) = 1.2068$. It is seen that the probability $P(N) > 1$ after $N > 39$. Liang *et al.* (2011) further analysed this restriction by focusing on the number of risky assets in the given portfolio.

3 Pricing basket credit default swaps

Consider an n2D swap which is a bilateral swap contract whose payoff is dependent on the specified cumulative credit default event of a given portfolio of reference entities. The reference entities mentioned here could be bonds, loans, corporations, etc. The pricing of the above contingent claim needs the knowledge of the joint probability distribution functions of the default times. The default leg (DL) which is the payment by the protection seller in case of a credit event and premium leg (PL) which is paid by the protection buyer prior to default must be calculated. Under the risk-neutral probability measure, the fair swap spread is obtained as the ratio between the default payments and the premium payments.

Let the price of a bond with t maturity be

$$B(0,t) = \exp\Big(-\int_0^t f(0,x)\mathrm{d}x\Big)\,,$$

where $f(0,x)$ is the instantaneous forward rate at initial time $t=0$. Let the discrete premium dates be $0 = t_0 < t_1 < \cdots < t_N = T$ and the frequency payment dates be $\delta_i = t_i - t_{i-1}$ (in units of years). Furthermore, denote S_n = swap spread; τ^n = default time for asset n; $M = \sum_{i=1}^N = M_i$ = total face value of the portfolio; $\mathbb{I}_{\{\tau^n>t\}}$ = indicator function of the credit event; R = recovery rate; and T = maturity. If the nth asset defaults on or before maturity, the protection seller has to pay $M_i(1-R_i)$ to the protection buyer at time τ^n or at maturity. On the other hand, suppose no default occurs, the buyer continues to pay $S_n\delta$ at time t_i. The present value for the **default leg** is given by

$$\mathrm{DL} = M(1-R^n)B(0,\tau^n)\mathbb{I}_{\{\tau^n\le T\}}\,. \tag{21}$$

Taking its expectation value, we have

$$\begin{aligned}
\mathbb{E}[DL] &= \mathbb{E}[M(1-R^n)B(0,\tau^n)\,\mathbb{I}_{\{\tau^n\le T\}}] \\
&= M(1-R^n)\int_0^T B(0,\tau^n)\,\mathrm{d}F^n(t) \\
&= M(1-R^n)\int_0^T \exp\left(-\int_0^t f(0,x)\mathrm{d}x\right)\,\mathrm{d}F^n(t) \\
&= M(1-R^n)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\,\mathrm{d}t\right]\,.
\end{aligned} \tag{22}$$

The present value of the **premium leg** is given by

$$PL = S_n M \sum_{i=1}^N \delta B(0,t_i)\mathbb{I}_{\{\tau^n>t\}}\,. \tag{23}$$

Taking its expectation, we have

$$\mathbb{E}[PL] = \sum_{i=1}^N S_n M\delta B(0,t_i)[1-F^n(t_i)]\,. \tag{24}$$

Under the risk-neutral measure, the value of the fair swap spread is obtained by equating the expected discounted value for the premium leg (PL) with that of the default leg (DL) and solving for S_n. Theorem 1 and Theorem 2 give the values of the n2D swaps in the absence and the presence of accrued premium respectively.

Theorem 1 *[9, 6] Assume that there is no accrued premium, the risk neutral pricing measure for the annualized n2D swap is given below:*

$$S_n = \frac{(1-R^n)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\,\mathrm{d}t\right]}{\sum_{i=1}^N \delta B(0,t_i)[1-F^n(t_i)]}\,,$$

where $F^n(t) = \mathbb{P}(\tau^n \le t)$ is the distribution function of the default time τ^n.

In the presence of accrued premium, suppose a credit event occurs at the commencement of a premium period, then no premium will have accrued. If the credit event happens at the end of the period, then a full premium payment will have to be sorted out. But if however, any of the entities default within the two premium time period (t_{i-1}, t_i), then the protection buyer is under obligation to pay an extra amount of the accrued premium of $S_n \delta(t_{i-1}, \tau_i)$. The calculation follows the same basic convention used to pay other premium.

Thus, the present value of the **accrued leg** due to a default in the nth premium period is given by

$$AP = S_n M \sum_{i=1}^{N} \delta \left(\frac{\tau^n - t_{i-1}}{t_i - t_{i-1}} \right) B(0, \tau^n) \mathbb{I}_{\{t_{i-1} < \tau^n \leq t_i\}} \,. \tag{25}$$

Taking its expectation, we have

$$\mathbb{E}[AP] = S_n M \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (q - t_{i-1}) \delta B(0, q) F_n \mathrm{d}q \,. \tag{26}$$

Theorem 2 *[9] In the presence of accrued premium, the risk neutral pricing measure for the annualized n2D swap is given as*

$$S_n = \frac{(1 - R^n) \left[B(0,T) F^n(T) + \int_0^T f(0,t) B(0,t) F^n(t) \mathrm{d}t \right]}{\sum_{i=1}^{N} \delta B(0, t_i)[1 - F^n(t_i)] + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (q - t_{i-1}) B(0,q) F_n \mathrm{d}q} \,,$$

where $F^n(t) = \mathbb{P}(\tau^n \leq t)$ is the distribution function of the default time τ^n.

3.1 Modelling the default time

The expectation values in the pricing of an n2D basket swap are over the joint distribution of default times, and thus, it is vital to know the joint probability distribution of the default times of the entities in the given portfolio. This probability can be modelled via a copula function, which connects the marginal default probability to their joint probability. Market information relating to that entity can provide the corresponding marginal probability distribution of each of the entities [5]. This information is essential in the approximation of the instantaneous intensity rate λ_i for each entity i, and then we can define the probability distribution [17]:

$$F_{\tau_i}(t_i) = \mathbb{P}(\tau_i \leq t_i) := 1 - \exp\Big(- \int_0^{t_i} \lambda_i(u) \mathrm{d}u \Big) \,. \tag{27}$$

Using the bond market data and CDS spread date, one can calibrate the distribution function $F_{\tau_i}(t)$ to the market data, thus we have that $F_{\tau_i}(t)$ and its probability density function $f_{\tau_i}(t)$ are given functions, with

$$f_{\tau_i}(t_i) := \frac{d}{dt} \mathbb{P}(\tau_i \leq t_i) := \lambda_i(t) \exp\Big(- \int_0^{t_i} \lambda_i(u) \mathrm{d}u \Big) \,. \tag{28}$$

In the analysis of the default time modelling, the joint distribution function is decomposed into univariate marginals and the dependency structure of default time (as described by the suitable copula), and they are made evident in Theorem 3 (Sklar's theorem).

Theorem 3 *[25] Denote $\tau_1, \ldots, \tau_N$ as random variables with marginal distribution functions F_{τ_i} and joint distribution function $F_{\tau_1,\ldots,\tau_N}$. Then, there exists an N-dimensional copula function $C : [0,1]^N \to [0,1]$, such that $\forall\ \tau_{\mathbf{i}} \in \mathbb{R}^N$, we have*

$$F_{\tau_1,\ldots,\tau_N}(\mathbf{t}) = \mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_N \leq t_N) = C(F_{\tau_1}(t_1), \ldots, F_{\tau_N}(t_N)) \,.$$

If $F_{\tau_1}, \ldots, F_{\tau_N}$ are continuous, then C is unique. Otherwise C is uniquely determined on $RanF_{\tau_1} \times, \ldots, \times RanF_{\tau_N}$, where $RanF_{\tau_i}$ denotes the range of F_{τ_i}, for $i = 1, \ldots, N$.

Thus, the copula function C is significant in the modelling of joint default time distribution, and it reflects the dependency amongst the default times in the pricing of BDS and CDO tranches.

In this work, we focus on implementing the **one-factor Gaussian copula** given by Laurent and Gregory (2005) and Choe and Jang (2009). The Gaussian copula function is defined by $C(t_1, \cdots, t_n) = \Phi_\Sigma(\Phi^{-1}(t_1), \cdots, \Phi^{-1}(t_1))$, where Φ is the cumulative standard normal distribution function and Φ_Σ is the joint distribution function for a multivariate random normal vector, having a covariance matrix Σ and zero mean. Define the correlated standard Gaussian random variables Y_i as:

$$Y_i = \rho_i Z + \sqrt{1-\rho_i^2}\epsilon_i = \Phi^{-1}\Big(F_i(\tau_i)\Big),$$

where Z is a common factor (systematic risk) for all the entities and ϵ_i is the error term (or entity-specific risk). Both the random variables Z and ϵ_i are independent standard normally distributed, so that for $i \neq j$, $\mathrm{Cov}(Y_i, Y_j) = \rho$ and $\mathrm{Var}(Y_i) = 1$. We seek to obtain the joint probability distribution function for the correlated default times

$$\prod_{i=1}^{N} \mathbb{P}(\tau_i \leq t_i) = \mathbb{P}(\tau_1 \leq t_1, \cdots, \tau_N \leq t_N) = \mathbb{P}\Big(F_1^{-1}\big(\Phi(Y_1)\big) \leq t_1, \cdots, F_N^{-1}\big(\Phi(Y_N) \leq t_N\big)\Big).$$

Taking expectations of both sides with respect to Z, we have

$$\begin{aligned}\mathbb{E}_Z\left[\prod_{i=1}^{N} \mathbb{P}(\tau_i \leq t_i)|Z\right] &= \mathbb{E}_Z[\mathbb{P}(F_1^{-1}(\Phi(Y_1) \leq t_1, \cdots, F_N^{-1}(\Phi(Y_N) \leq t_N)|Z] \\ &= \mathbb{E}_Z[\mathbb{P}(F_1^{-1}(\Phi(Y_1) \leq t_1|Z) \times \cdots \times \mathbb{P}(F_N^{-1}(\Phi(Y_N) \leq t_N|Z)]\,.\end{aligned}$$

Since

$$\mathbb{P}(F_i^{-1}(\Phi(Y_i) \leq t_i|Z) = \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right),$$

and since the default times conditional on Z are independent, the joint probability of default times reads

$$\prod_{i=1}^{N} \mathbb{P}(\tau_i \leq t_i) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i z}{\sqrt{1-\rho_i^2}}\right)\psi(z)\mathrm{d}z\,.$$

On the other hand, the joint probability of the survival time is

$$\prod_{i=1}^{N} \mathbb{P}(\tau_i > t_i) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Phi\left(\frac{\rho_i z - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\psi(z)\mathrm{d}z\,,$$

where $\psi(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-z^2}{2}}$ denotes the standard normal density function of Z.

For the simulation of the default time, we employ the Monte-Carlo method. In the numerical experiment for Section 4, we used the default times to calculate the expectation of each leg by averaging over the discounted payoff for each the protection legs and premium. When the default boundary is a deterministic function, then each of the default time τ_i, for random variables Y_i can be defined as [19]:

$$\tau_i := \inf\left\{t \geq 0 : \int_0^t \lambda_i(u)\mathrm{d}u \geq -\ln(Y_i)\right\}.$$

Thus, by mapping the cumulative normal distribution between the Gaussian variable Y_i and the default time τ_i, the default time of the reference entity i can be simulated using

$$\tau_i = \frac{-\ln(1-\Phi(Y_i))}{\lambda_i}\,.$$

4 Parameter Analysis and Numerical Experiments

In this section, we consider a portfolio of five corporate entities with different credit ratings given by Standard & Poor's as of 28-02-2017. The entities have the grades[3] AAA, AA, A, BBB and BB. For the data, we used the OAS spread bid of the entities from the periods of 30-10-2015 till 28-02-2017, monitored monthly with a dollar ($) denomination.

Table 1: Entities with their credit ratings

Entities	Microsoft	Apple	Pepsi	General Motors	Free Port Inc.
Ratings	AAA	AA	A	BBB	BB

For the parameters of the default intensities for each of the entities, we employ the Maximum Likelihood Estimation under the Vasicek model, using the historical bond spread data. We set the initial parameters for all the entities at $\alpha_i = 0.2$, $\beta_i = 0.02$, $\sigma_i = 0.002$, $\lambda_i(0) = 0.02$ and $t = 5$ years. We further assume that since the portfolio consists of 5 entities in the same sector (corporate), then their correlation coefficients are the same, that is, $\rho_{ij} = \rho$, for $i \neq j$. Next, we compute the first and second moments for each of the entities using both Vasicek and the CIR model, as well as the default intensity for each entity. Table 2 shows the values obtained:

Table 2: Parameter estimation of entities, with their default intensities (DI)

Entities	i	DI	Parameters (α, β, σ)	Vasicek & CIR $\mathbb{E}[\lambda_i(t)]$	Vasicek $\mathrm{Var}[\lambda_i(t)]$	CIR $\mathrm{Var}[\lambda_i(t)]$
AAA	1	42	(0.66196, 0.00263, 0.00064)	32.64413	0.00309	0.00001
AA	2	69	(0.95874, 0.00421, 0.00059)	43.40769	0.00182	0.00001
A	3	128	(0.38473, 0.00834, 0.00088)	100.43209	0.00985	0.00011
BBB	4	266	(0.26734, 0.01816, 0.00248)	186.43389	0.10709	0.00203
BB	5	1019	(0.25302, 0.06547, 0.01482)	526.37863	3.99455	0.18157

From Table 2 we observe that firms with higher credit ratings have lesser default intensity (i.e., their conditional probabilities of no earlier default per year) compared to lower rated entities. These highly rated firms have a strong capacity of meeting up to their financial obligations, and their corresponding low bond yield serves as a security and high repayment probability. The lower rated firms, on the other hand, have higher bond spreads and this follows from their vulnerability and a higher risk of default. We further observe that as the entity gradually transits to a lower credit rating, the expected values of their hazard rates increase, together with their probabilities of default[4]. Using a constant recovery rate $R_i = R = 40\%$, we obtain the average default intensity of each firm from the given historical data, using the formula $S_i^{-1}(1 - R)$, where S_i is the spread of each entity. This recovery rate is a common assumption of market participants [12]. The expectation values of the hazard rates remain the same for both the Vasicek and the CIR model, whereas their variances differ. The result is evident because in the CIR model, the volatility parameter contains an extra $\sqrt{\lambda_i(t)}$ term and this reduces the effect of the volatility change. We observed that as a fixed change in the volatility affect the intensity rate of the CIR more than the Vasicek, thus, leading to a smaller variance, as shown in Table 2. Moreover, the expected value of the intensity rate remains unchanged, as the value for σ changes, but only the variances are affected.

[3]Here, AA^+ and AA^- are viewed as AA, A^+ and A^- are viewed as A, BBB^+ and BBB^- are viewed as BBB, BB^+ and BB^- are viewed as BB

[4]This probability can be obtained from CDS spreads, Merton's structural model, bond prices, or historical data.

Furthermore, when the portfolio of the 5 entities are observed, the marginal survival distribution is smooth and strictly decreasing. These survival functions give the probability that the portfolio will attain at a specific time, and their JSPD values are shown in Table 3:

Table 3: JSPD of the entities in Table 1 with different maturity time

T	1	2	3	4	5	6	7	8	9
JSPD	0.91022	0.83490	0.76703	0.70416	0.64547	0.59071	0.53978	0.49261	0.44909

Consider the valuation of the basket default swap for a homogeneous portfolio of $N = 10$ entities. They have the same notional value M, recovery rate $R = 0.5$. The maturity of the swap is $T = 5$ years, the payment frequency is $\delta = 0.25$ (quarterly payment), $r = 0.04$, $\lambda = 0.07$ and $\rho = 0.45$. Using 10000 number of simulations, we estimate the values for the default leg, premium leg and accrued premium leg for different nth basket or rank of the default protection, as displayed in Table 4:

Table 4: Default leg, premium leg, and accrued premium values for increasing rank of the default protection

Rank	Default leg	Premium leg	Accrued premium leg
1	0.36126	2.14996	0.08695
2	0.27968	2.92183	0.07028
3	0.21665	3.40114	0.05442
4	0.16368	3.73715	0.04011
5	0.12217	3.96857	0.03058
6	0.08657	4.14992	0.02157
7	0.05925	4.27591	0.01476
8	0.03720	4.37210	0.00953
9	0.01951	4.44181	0.00496
10	0.00721	4.44864	0.00188

Table 4 shows that as the rank level ($n = 1$ for F2D, $n = 2$ for S2D, etc.) increases, the default leg and the accrued premium legs decrease, whereas, the premium leg increases. This result, in turn, leads to a decrease in the value of the n2D basket spread and the lower the rank level, the less risky. The accrued premium is a fraction of the premium which has accrued starting from the date the previous payment was made, till the time when a default occurs. It is the smallest in comparison with the other legs, and its effect can be ignored since it is very insignificant. Considering the same 10 entities-portfolio, the probability of having a fewer number of entities to default is more than having many entities to default. The protection seller experiences loss when there is more than one default. Thus, the payment from the default leg in case of a default reduces as the number of entities to default increases. Furthermore, since the probability of default as the rank level increases and since there is less likelihood for many entities to default in the portfolio, the protection buyer has to pay more premium at regular intervals.

Table 5 considers the same parameters as in Table 4, but we vary and increase the hazard rate (λ) and the default correlation (ρ) for the valuations of the first to fourth-default swap. The following values are obtained:

Table 5: Effects of default intensities and default correlations on basket default swap prices

F2D

λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	487	453	416	367	324	289	258	216	172	129	66
0.02	1008	879	770	675	588	512	427	354	295	220	133
0.03	1517	1306	1115	973	831	729	620	530	409	310	193
0.04	2003	1738	1474	1273	1100	944	780	663	540	423	260
0.05	2519	2119	1781	1564	1312	1126	956	794	636	498	318

S2D

λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	86	100	111	119	124	120	110	107	97	84	61
0.02	275	281	263	262	259	254	226	219	192	169	120
0.03	494	467	461	418	407	376	342	316	273	245	183
0.04	743	684	636	595	551	505	458	419	376	310	236
0.05	1002	897	822	762	683	615	564	521	467	387	298

T2D

λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	11	22	36	45	54	61	70	71	75	64	62
0.02	59	92	108	127	139	143	148	145	148	139	117
0.03	154	194	217	224	226	241	233	224	216	200	174
0.04	277	310	326	321	325	333	325	310	295	267	229
0.05	421	433	439	440	434	425	401	396	378	368	264

Fo2D

λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	1	5	10	20	28	33	42	45	52	55	51
0.02	9	27	42	57	79	84	96	107	110	117	106
0.03	39	66	95	115	132	148	153	169	175	178	156
0.04	88	127	163	181	199	216	225	231	237	228	208
0.05	164	204	235	259	281	285	289	292	294	288	266

Table 5 gives the valuations of the premium prices for the first-to-default (F2D), second-to-default (S2D), third-to-default (T3D), and fourth-to-default (FoD) values, together with the effects of the default correlations and hazard rate. The prices for the S2D and T3D swaps are generally lesser compared to the F2D swaps, and Fo2D swaps are the cheapest among the others because the payoff is paid when the fourth entity defaults, notwithstanding prior defaults. The protection buyer is entitled to a net loss on the entity which has defaulted, and not on the cumulative loss. The hazard rate is a significant factor in the pricing of n2D swaps. There is a positive correlation between the premium and the hazard rate, with every other parameter kept constant, and this evident in the following relation $S = \lambda(1-R)$, for a constant recovery rate R. For example, when $\lambda = 0.02$ for an F2D swap, the swap spread becomes 1008 basis point(bp), but the spread increases to 2519 bp when the hazard rate increases to 0.05. Furthermore, we observe that the intensity for the probability of default is entirely dependent on the hazard rate value. Thus, from the perspective of the protection seller, lesser spread has lower default probability and more likelihood to offset its debt.

A negative correlation exists between the BDS spread values (in this case F2D) and the default correlation, as the spread decreases with an increase in the default correlation. Lower default correlation implies that fewer losses are more likely to occur and this strengthens the effects of risk diversification, thereby lowering the BDS spreads[5]. The more entities stay correlated, the higher the chances of the portfolio to exhibit characteristics of a single-named entity. Furthermore, zero correlation implies no relationship among the portfolio entities and selling protection on non-correlated entities (for F2D swaps) is the same as selling individual credit protections thus; this accounts for their large spread. Hence, investors seeking to achieve the highest swap premium should sell protection on a portfolio of entities with low correlation or zero correlation, like over a portfolio of many unrelated industries.

As we gradually move away from the valuation of the F2D swaps, we noticed that monotonicity is no longer applicable. The result is evident from Table 5 because for a fixed hazard rate and increasing default correlation, the spread values started to fluctuate; that is, increasing and decreasing at some points. For example: consider the S2D swap, where $\lambda \geq 0.03$ in Table 5, we notice a consistent decrease in the spread value as the default correlation increases. But the reverse is the case for lower hazard rate $\lambda < 0.03$. Also, we observe the non-linearity of the BDS spread for a fixed correlation in the valuation of the T3D and Fo2D swaps. Jabbour, *et al.* (2008) gave the reason for this phenomena [14].

5 Conclusion

Basket default swaps are financial credit derivatives which are linked to an underlying basket of entities, bonds, loans or assets. The pricing is dependent on the joint distribution of the corresponding default times of the entities in question. We modelled the dependency structure of the default time, defined under the one-factor Gaussian copula method, with the Monte-Carlo method. Furthermore, this work focused on the valuation of an n2D swaps which are based on a homogeneous portfolio of entities. Under the stochastic intensity models, we modelled the hazard rate and estimated the model parameters of five corporate entities, having different credit ratings. We further obtained the JSPD of the same entities, and we observed that an increase in the maturity time T leads to a decrease in the survival probability. We also considered the survival probability of a homogeneous and a heterogeneous portfolio under both Vasicek and the CIR model.

Hedging portfolio credit derivatives involve an appropriate calculation of the sensitivities of the swap value with respect to the associated parameters. Thus, this work conducted a sensitivity analysis for the prices of the n2D basket, in connection to the rank of the default protection, the intensity rate and the default correlation. We observed that the value of the n2D swaps behaves differently with regards to changes in the intensity rate and the default correlation, as the rank of default protection increases. Finally, we observed that investors that want to trade F2D swaps with highest swap premium should sell protections on entities with low correlations.

Acknowledgement

This work commenced while the first author was a visiting research student at the Bergische Universität Wuppertal, Germany, under the sponsorship of Deutscher Akademischer Austauschdienst (DAAD). The authors would like to acknowledge the financial support from DAAD, in collaboration with the African Institute for Mathematical Sciences (AIMS), South Africa and

[5] This might be different for a portfolio which consists of sufficiently large entities, or when higher order swaps are considered.

the Centre for Business Mathematics and Informatics, North-West University, Potchefstroom, South Africa.

References

[1] Bluhm, C., Overbeck, L., and Wagner, C., 2002. *An introduction to credit risk modeling.* Chapman and Hall/CRC.

[2] Bomfim, A.N., 2015. *Understanding credit derivatives and related instruments.* Academic Press.

[3] Brigo, D., and Mercurio, F., 2007. *Interest rate models-theory and practice: with smile, inflation and credit.* Springer Science & Business Media.

[4] Chan, K.C., Karolyi, G.A., Longstaff, F.A., and Sanders, A.B., 1992. *An empirical comparison of alternative models of the short-term interest rate.* The Journal of Finance, 47(3), pp.1209-1227.

[5] Chen, Z., and Glasserman, P., 2008. *Fast pricing of basket default swaps.* Operations Research, 56(2), pp.286-303.

[6] Choe, G.H., and Jang, H.J., 2011. *The kth default time distribution and basket default swap pricing.* Quantitative Finance, 11(12), pp.1793-1801.

[7] Choi, Y., Wirjanto, T.S., 2007. *An analytic approximation formula for pricing zero-coupon bonds.* Finance Research Letters, 4(2), pp.116-126.

[8] Fathi, A., and Nader, N., 2007a. *Price Calibration of basket default swap: Evidence from Japanese market.* Retrieved from MPRA-Munich Personal RePEc Archive, at `https://mpra.ub.uni-muenchen.de/6013/1/MPRA_paper_6013.pdf`

[9] Fathi, A., and Nader, N., 2007b. *Copula based simulation procedures for pricing basket Credit Derivatives.* Retrieved from MPRA-Munich Personal RePEc Archive, at `https://mpra.ub.uni-muenchen.de/6014/1/MPRA_paper_6014.pdf`.

[10] Frey, R., and McNeil, A.J., 2003. *Dependent Defaults in Models of Portfolio Credit Risk. Journal of Risk* 6(1), pp.59-92.

[11] Jouanin, J.F., Riboulet, G., and Roncalli, T., 2002. *Beyond conditionally independent defaults.* Available at SSRN 1032564.

[12] Hull, J. C., Predescu, M., and White, A., 2005. *Bond prices, default probabilities and risk premiums.* Default Probabilities and Risk Premiums (March 9, 2005). Accessed at `http://www-2.rotman.utoronto.ca/~hull/downloadablepublications/CreditSpreads.pdf`

[13] Iscoe, I., and Kreinin, A., 2006. *Recursive valuation of basket default swaps.* Journal of Computational Finance, 9(3), pp.95-116.

[14] Jabbour, G.M., Kramin, M.V., and Young, S.D., 2008. *Nth-to-default swaps: valuation and analysis.* Managerial Finance, 35(1), pp.25-47.

[15] Jiang, G.J., 1998. *Nonparametric modeling of US interest rate term structure dynamics and implications on the prices of derivative securities.* Journal of Financial and Quantitative Analysis, 33(4), pp.465-497.

[16] Joshi, M.S., and Kainth, D., 2004. *Rapid and accurate development of prices and Greeks for nth to default credit swaps in the Li model.* Quantitative Finance, 4(3), pp.266-275.

[17] Kijima, M., 2000. *Valuation of a credit swap of the basket type.* Review of Derivatives Research, 4(1), pp.81-97.

[18] Kijima, M., and Muromachi, Y., 2000. *Credit events and the valuation of credit derivatives of basket type.* Review of Derivatives Research, 4(1), pp.55-79.

[19] Laurent, J.P., and Gregory, J., 2005. *Basket default swaps, CDOs and factor copulas.* Journal of Risk, 7(4), pp.103-122.

[20] Liang, J., Ma, J.M., Wang, T. and Ji, Q., 2011. *Valuation of portfolio credit derivatives with default intensities using the Vasicek model.* Asia-Pacific Financial Markets, 18(1), pp.33-54.

[21] Li, D.X., 2000. *On Default Correlation: A Copula Function Approach.* The Journal of Fixed Income, 9(4), pp.43-54.

[22] Li, P., Liu, J., Zhang, X., and Huang, G., 2015. *Pricing of Basket Default Swaps Based on Factor Copulas and NIG.* Procedia Computer Science, 55, pp.566-574.

[23] Mashal, R., and Naldi, M., 2002. *Pricing multiname credit derivatives: heavy tailed hybrid approach.* Available at SSRN 296402.

[24] O'Kane, D., 2011. *Modelling single-name and multi-name credit derivatives* (Vol. 573). John Wiley & Sons.

[25] Schönbucher, P.J., 2003. *Credit derivatives pricing models: models, pricing and implementation.* John Wiley & Sons.

[26] Schönbucher, P.J., and Schubert, D., 2001. *Copula-dependent default risk in intensity models.* Working paper, Department of Statistics, Bonn University.

[27] Schröter, A., and Heider, P., 2013. *An analytical formula for pricing m-th to default swaps.* Journal of Applied Mathematics and Computing, 41(1-2), pp.229-255.