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Abstract

This work deals with the numerical solution of the extended Heston model with stochastic correlation using the Alternating Direction Implicit (ADI) type schemes. By imposing stochastic correlation driven by a stochastic process between the underlying asset and the stochastic volatility in the Heston model, we thus obtain three-dimensional equation with the mixed derivative terms. This problem is solved using the ADI type schemes, whereas we investigate the effective implementation in matrix form. Our numerical results demonstrate the achieved acceleration using the proposed implementation in the ADI schemes, where several classical ADI schemes are considered and compared.

Keywords *Heston model, stochastic correlation, ADI schemes, acceleration*

1 Introduction

The Heston model [11] is a well-known and widely used stochastic volatility model for pricing equity. For example, the values of European options are given by the (two-dimensional) Heston partial differential equation (PDE) that is supplemented with the initial and boundary conditions [11]. It is well-known that only a semi-analytical solution in a closed form is available for the Heston model, many various methods have thus been proposed for its solution over the past decades. For example, the reader may refer to [1] for the Monte Carlo-based approaches and [3, 4, 9] for the Fourier-based methods. Furthermore, some numerical methods were proposed to numerically solve the Heston PDE. We refer to [22] for the finite element method (FEM) and [12] for the finite difference method (FDM).

It is well-known that in many cases the Heston model can not generate enough skews or smiles in the implied volatility as the market required. For this problem, several works have been proposed to extend the Heston model. For the extension by allowing time-dependent parameters we refer to [2, 8, 15, 16]. Furthermore, the pure Heston model has been also extended by additionally including a random factor driven by a stochastic process, e.g., volatility of the volatility process, interest rate and correlation, see [5, 10, 18, 19].

In this work we study the ADI scheme for solving the Heston model extended by including a stochastic correlation [18, 19], which is a three-dimensional problem. As mentioned above, the investigation on the ADI scheme for the pure Heston model has been provided in [12]. To the best of our knowledge, the ADI scheme is usually implemented in vector form. This kind of implementation is computationally satisfactory for the two-factor but not for the three-factor problem. Thus, in order to accelerate the computation we investigate the implementation of ADI scheme in matrix form. Our idea is to design the difference and solution matrix (instead of solution vector) properly so that ADI scheme can be performed only with matrix-based computation.

The reminder of paper is organized as follows. The next section specializes the problem, namely the extended Heston PDE by including a stochastic correlation. In Section 3, we reformulate firstly ADI scheme for the Heston model from vector to matrix form, and apply the reformulation for extended Heston model as well. The numerical results are shown in Section 4. Finally, we conclude this work in Section 5.

2 Stochastic correlation in the Heston model

Heston's stochastic volatility model [11] under the risk-neutral measure is specified as

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{\nu_t}S_t\,dW_t^S, & S_0 > 0, \\ d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)\,dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, & \nu_0 > 0, \end{cases} \tag{2.1}$$

where S_t is the spot price of the underlying asset, ν_t is the volatility and the Brownian motions W_t^S and W_t^ν are correlated with a constant $\rho \in [-1, 1]$. Note that under the risk-neutral measure, the market price of volatility risk is embedded in the parameters of $d\nu_t$. Based on (2.1) the corresponding Heston model PDE can be derived as [11]

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_\nu S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma_\nu\nu S\frac{\partial^2 V}{\partial S\partial\nu}+\frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 V}{\partial\nu^2}+rS\frac{\partial V}{\partial S}+\kappa_\nu(\mu_\nu-\nu_t)\frac{\partial V}{\partial\nu}-rV=0, \quad (2.2)$$

where $V(S,\nu,t)$ is the value of a derivative.

We extend (2.1) by imposing a stochastic correlation driven by the bounded Jacobi process [14, 17, 20] between the asset price and the volatility:

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{\nu_t}S_t\,dW_t^S, & S_0>0, \\ d\nu_t = \kappa_\nu(\mu_\nu-\nu_t)dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, & \nu_0>0, \\ d\rho_t = \kappa_\rho(\mu_\rho-\rho_t)dt + \sigma_\rho\sqrt{1-\rho_t^2}dW_t^\rho, & \rho_0\in(-1,1), \end{cases} \quad (2.3)$$

where

$$dW_t^S dW_t^\nu = \rho_t\,dt, \quad dW_t^S dW_t^\rho = \rho_1\,dt, \quad dW_t^\nu dW_t^\rho = \rho_2\,dt, \quad (2.4)$$

i.e. the asset price and the volatility process are set to be correlated randomly, driven by the correlation process ρ_t which is by itself correlated with the asset price process by ρ_1 and with the volatility by ρ_2, respectively. Note that $d\rho_t$ in (2.3) is assumed to be under the risk-neutral measure, i.e., the market price of correlation risk is embedded in the parameters of $d\rho_t$, see [16, 18]. For another type of the stochastic correlation process we refer to [17]. Note that the family of correlation matrices in (2.3) reads

$$\mathcal{C}_t = \begin{pmatrix} 1 & \rho_t & \rho_1 \\ \rho_t & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}, \quad t\geq 0, \quad (2.5)$$

which is symmetric and must be positive semi-definite as well. Obviously, the positive semi-definiteness is provided that

$$1-\rho_1^2-\rho_2^2+2\rho_1\rho_2\rho_t-\rho_t^2\geq 0 \quad (2.6)$$

which implies e.g.,

$$\rho_1\rho_t-\sqrt{(1-\rho_1^2)(1-\rho_t^2)}\leq\rho_2\leq\rho_1\rho_t+\sqrt{(1-\rho_1^2)(1-\rho_t^2)}. \quad (2.7)$$

We denote the conditional expectation under the risk-neutral measure $\mathbb{Q}$, and let $V(S,\nu,\rho,t) = e^{-r(T-t)}E^{\mathbb{Q}}[V(S,\nu,\rho,T)|\mathcal{F}_t]$ be the value of any financial derivative, whose Kolmogorov's backward equation is given by

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa_\nu(\mu_\nu - \nu)\frac{\partial V}{\partial \nu} + \kappa_\rho(\mu_\rho - \rho)\frac{\partial V}{\partial \rho} + \frac{1}{2}\nu S^2\frac{\partial^2 V}{\partial S^2} \\
&\quad + \frac{1}{2}\nu\sigma_\nu^2\frac{\partial^2 V}{\partial \nu^2} + \frac{1}{2}\sigma_\rho^2(1-\rho^2)\frac{\partial^2 V}{\partial \rho^2} + \rho_2\sigma_\rho\sqrt{1-\rho^2}\sigma_\nu\sqrt{\nu}\frac{\partial^2 V}{\partial \rho \partial \nu} \\
&\quad + \sigma_\nu\nu\rho S\frac{\partial^2 V}{\partial \nu \partial S} + \rho_1\sigma_\rho\sqrt{1-\rho^2}S\sqrt{\nu}\frac{\partial^2 V}{\partial \rho \partial S} - rV = 0
\end{aligned}
\tag{2.8}
$$

subject to the terminal condition known from the payoff. We note that (2.8) can also be derived by constructing a hedging portfolio.

3 Implementation of the ADI schemes in matrix form

In this section we investigate the implementation of ADI schemes in matrix form for fast computing. We firstly define the discrete grid from the defined domain of (2.8). Without loss of generality we split $[0,T]$, $[S_{min}, S_{max}]$, $[\nu_{min}, \nu_{max}]$ and $[\rho_{min}, \rho_{max}]$ into N_T, D_S, D_ν and D_ρ equidistant sub-intervals, respectively. We thus obtain the following mesh on $[S_{min}, S_{max}] \times [\nu_{min}, \nu_{max}] \times [\rho_{min}, \rho_{max}] \times [0,T]$:

$$
\begin{cases}
t_i = i\Delta t; \quad \Delta t = \frac{T}{N_T}; \quad i = 0, \cdots, N_T, \\
S_j = S_{min} + j\Delta S; \quad \Delta S = \frac{S_{max}-S_{min}}{D_S}; \quad j = 0, \cdots, D_S, \\
\nu_k = \nu_{min} + k\Delta \nu; \quad \Delta \nu = \frac{\nu_{max}-\nu_{min}}{D_\nu}; \quad k = 0, \cdots, D_\nu, \\
\rho_m = \rho_{min} + m\Delta \rho; \quad \Delta \rho = \frac{\rho_{max}-\rho_{min}}{D_\rho}; \quad m = 0, \cdots, D_\rho.
\end{cases}
\tag{3.1}
$$

3.1 The CS scheme for the Heston model

For a simpler explanation we start with the Craig-Sneyd (CS) scheme [6] to the pure Heston PDE (2.2), which can be rewritten as the ordinary differential equation (ODE) as follows:

$$
\begin{aligned}
V'(t) &= F(V,t) \\
&= F_0(V,t) + F_1(V,t) + F_2(V,t)
\end{aligned}
\tag{3.2}
$$

where

$$\begin{cases} F_0\left(V,t\right) = \rho\sigma_\nu\nu S\dfrac{\partial^2 V}{\partial S\partial\nu}, \\ F_1\left(V,t\right) = rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\nu S^2\dfrac{\partial^2 V}{\partial S^2} - \dfrac{1}{2}rV, \\ F_2\left(V,t\right) = \kappa_\nu\left(\mu_\nu - \nu\right)\dfrac{\partial V}{\partial \nu} + \dfrac{1}{2}\sigma_\nu^2\nu\dfrac{\partial^2 V}{\partial \nu^2} - \dfrac{1}{2}rV. \end{cases} \tag{3.3}$$

We denote the approximation of $V(S,\nu,t)$ at t_i and grid point (S_j,ν_k) by $U_i^{j,k}$. In order to apply the CS scheme to the Heston model, to the best of our knowledge, the most widely used implementation requires the formulation

$$\mathbf{U}_i = \begin{pmatrix} U_i^{0,0} \\ \vdots \\ U_i^{D_S,0} \\ - \\ \vdots \\ - \\ U_i^{0,D_\nu} \\ \vdots \\ U_i^{D_S,D_\nu} \end{pmatrix} \in \mathbb{R}^{(D_S+1)(D_\nu+1)}. \tag{3.4}$$

Furthermore, for all the derivatives in (3.3) we use finite difference methods of second order and obtains the corresponding matrices

$$\begin{aligned} F_0 &= A_0 && \in \mathbb{R}^{(D_S+1)(D_\nu+1)\times(D_S+1)(D_\nu+1)} \\ F_1 &= A_1 - \frac{1}{2}rI && \in \mathbb{R}^{(D_S+1)(D_\nu+1)\times(D_S+1)(D_\nu+1)} \\ F_2 &= A_2 - \frac{1}{2}rI && \in \mathbb{R}^{(D_S+1)(D_\nu+1)\times(D_S+1)(D_\nu+1)} \end{aligned} \tag{3.5}$$

where I is an identity matrix and A_n, $n = 0,1,2$ exhibits a form of

$$\begin{pmatrix} A^{0,0} & A^{0,1} & A^{0,2} & \dots & A^{0,D_\nu-2} & A^{0,D_\nu-1} & A^{0,D_\nu} \\ A^{1,0} & A^{1,1} & A^{1,2} & \dots & A^{1,D_\nu-2} & A^{1,D_\nu-1} & A^{1,D_\nu} \\ A^{2,0} & A^{2,1} & A^{2,2} & \dots & A^{2,D_\nu-2} & A^{2,D_\nu-1} & A^{2,D_\nu} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A^{D_\nu-2,0} & A^{D_\nu-2,1} & A^{D_\nu-2,2} & \dots & A^{D_\nu-2,D_\nu-2} & A^{D_\nu-2,D_\nu-1} & A^{D_\nu-2,D_\nu} \\ A^{D_\nu-1,0} & A^{D_\nu-1,1} & A^{D_\nu-1,2} & \dots & A^{D_\nu-1,D_\nu-2} & A^{D_\nu-1,D_\nu-1} & A^{D_\nu-1,D_\nu} \\ A^{D_\nu,0} & A^{D_\nu,1} & A^{D_\nu,2} & \dots & A^{D_\nu,D_\nu-2} & A^{D_\nu,D_\nu-1} & A^{D_\nu,D_\nu} \end{pmatrix} \tag{3.6}$$

with $A^{k_1,k_2} = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & 0 \\ a_{1,0} & a_{1,1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{D_s-1,D_s} \\ 0 & \cdots & a_{D_s,D_s-1} & a_{D_s,D_s} \end{pmatrix} \in \mathbb{R}^{(D_S+1)\times(D_S+1)}, k_1 = 0,\cdots,D_\nu, k_2 = 0,\cdots,D_\nu$. Note that the matrix A_1 and A_2 are tridiagonal and pentadiagonal, respectively. One needs then to apply the CS scheme for the above ODE and obtains loop for each time step:

$$\begin{cases} U_0 = U_i + \Delta t\left(F_0\left(\mathbf{U}_i\right) + F_1\left(\mathbf{U}_i\right) + F_2\left(\mathbf{U}_i\right)\right) \\ U_1 = \left(I - \theta\Delta t F_1\right)^{-1}\left(U_0 - \theta\Delta t F_1\left(\mathbf{U}_i\right)\right) \\ U_2 = \left(I - \theta\Delta t F_2\right)^{-1}\left(U_1 - \theta\Delta t F_2\left(\mathbf{U}_i\right)\right) \\ \hat{U}_0 = U_0 + \frac{1}{2}\Delta t\left(F_0\left(U_2\right) - F_0\left(\mathbf{U}_i\right)\right) \\ \hat{U}_1 = \left(I - \theta\Delta t F_1\right)^{-1}\left(\hat{U}_0 - \theta\Delta t F_1\left(\mathbf{U}_i\right)\right) \\ \hat{U}_2 = \left(I - \theta\Delta_t F_2\right)^{-1}\left(\hat{U}_1 - \theta\Delta_t F_2\left(\mathbf{U}_i\right)\right) \\ \mathbf{U}_{i+1} = \hat{U}_2 \end{cases} \tag{3.7}$$

3.2 Implementation of the CS scheme in matrix form

Instead of the vector form in (3.4) we write the solution in matrix form

$$\mathbf{U}_i^S = \begin{pmatrix} U^{0,0} & U^{0,1} & \cdots & U^{0,D_\nu-1} & U^{0,D_\nu} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U^{D_S,0} & U^{D_S,1} & \cdots & U^{D_S,D_\nu-1} & U^{D_S,D_\nu} \end{pmatrix} \in \mathbb{R}^{(D_S+1)\times(D_\nu+1)} \tag{3.8}$$

for derivative in the direction of S,

$$\mathbf{U}_i^\nu = \begin{pmatrix} U^{0,0} & U^{0,1} & \cdots & U^{0,D_S-1} & U^{0,D_S} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U^{D_\nu,0} & U^{D_\nu,1} & \cdots & U^{D_\nu,D_S-1} & U^{D_\nu,D_S} \end{pmatrix} \in \mathbb{R}^{(D_\nu+1)\times(D_S+1)} \tag{3.9}$$

for derivative in the direction of ν and

$$\mathbf{U}_i^{S,\nu} = \begin{pmatrix} 0 & U^{0,0} & \cdots & U^{0,D_\nu-2} & U^{0,D_\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & U^{D_S,0} & \cdots & U^{D_S,D_\nu-2} & U^{D_S,D_\nu-1} \\ U^{0,0} & U^{0,1} & \cdots & U^{0,D_\nu-1} & U^{0,D_\nu} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U^{D_S,0} & U^{D_S,1} & \cdots & U^{D_S,D_\nu-1} & U^{D_S,D_\nu} \\ U^{1,1} & U^{1,2} & \cdots & U^{1,D_\nu} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U^{D_S,1} & U^{D_S,2} & \cdots & U^{D_S,D_\nu} & 0 \end{pmatrix} \in \mathbb{R}^{3(D_S+1)\times(D_\nu+1)} \tag{3.10}$$

for the mixed derivative. Let $A_{S,1}$ and $A_{S,2}$ be difference matrices of $rS\frac{\partial V}{\partial S}$ and $\frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}$, $A_{\nu,1}$ and $A_{\nu,2}$ be such matrices for $\kappa_\nu(\mu_\nu - \rho)\frac{\partial V}{\partial \nu}$ and $\frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 V}{\partial \nu^2}$, and $A_{S,\nu}$ be difference matrix of $\rho\sigma_\nu S\frac{\partial^2 V}{\partial S \partial \nu}$. According to the matrix forms of solution we write (3.3) as

$$\begin{cases} F_0^k(\nu_k) & = \nu_k A_{S,\nu} \in \mathbb{R}^{(D_S+1)\times 3(D_S+1)}, \quad k = 0, 1, \ldots, D_\nu \\ F_1^k(\nu_k) & = A_{S,1} + \nu_k A_{S,2} - \frac{1}{2}rI_S \in \mathbb{R}^{(D_S+1)\times(D_S+1)}, \quad k = 0, 1, \ldots, D_\nu \\ F_2 & = A_{\nu,1} + A_{\nu,2} - \frac{1}{2}rI_\nu \in \mathbb{R}^{(D_\nu+1)\times(D_\nu+1)}, \end{cases} \tag{3.11}$$

where the identity matrices I_S and I_ν are of size $(D_S + 1) \times (D_S + 1)$ and $(D_\nu + 1) \times (D_\nu + 1)$, respectively. Note that in (3.11) we have considered the difference matrices in S-direction for each $\nu_k, k = 1, \cdots, D_\nu$. Since there is no derivative with respect to S in F_2, which is thus only one matrix. To demonstrate the reformulation from vector form to matrix form we highlight e.g., the difference matrix $F_1^k(\nu_k)$ in (3.6) with red color and underline, which corresponds the matrix in $A^{k,k}$

$$\begin{pmatrix} \underline{A^{0,0}} & A^{0,1} & A^{0,2} & \cdots & A^{0,D_\nu-2} & A^{0,D_\nu-1} & A^{0,D_\nu} \\ A^{1,0} & \underline{A^{1,1}} & A^{1,2} & \cdots & A^{1,D_\nu-2} & A^{1,D_\nu-1} & A^{1,D_\nu} \\ A^{2,0} & A^{2,1} & \underline{A^{2,2}} & \cdots & A^{2,D_\nu-2} & A^{2,D_\nu-1} & A^{2,D_\nu} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A^{D_\nu-2,0} & A^{D_\nu-2,1} & A^{D_\nu-2,2} & \cdots & \underline{A^{D_\nu-2,D_\nu-2}} & A^{D_\nu-2,D_\nu-1} & A^{D_\nu-2,D_\nu} \\ A^{D_\nu-1,0} & A^{D_\nu-1,1} & A^{D_\nu-1,2} & \cdots & A^{D_\nu-1,D_\nu-2} & \underline{A^{D_\nu-1,D_\nu-1}} & A^{D_\nu-1,D_\nu} \\ A^{D_\nu,0} & A^{D_\nu,1} & A^{D_\nu,2} & \cdots & A^{D_\nu,D_\nu-2} & A^{D_\nu,D_\nu-1} & \underline{A^{D_\nu,D_\nu}} \end{pmatrix}. \tag{3.12}$$

For each time step we obtain CS loop in matrix form as

$$
\begin{cases}
\bar{U}_0^S\left(:,k\right) & = F_0^k\left(\mathbf{U}_i^{S,\nu}\left(:,k\right)\right), \qquad k=0,\dots,D_\nu,\\
\bar{U}_1^S\left(:,k\right) & = F_1^k\left(\mathbf{U}_i^{S}\left(:,k\right)\right), \qquad k=0,\dots,D_\nu,\\
\bar{U}_2^\nu\left(:,j\right) & = F_2\left(\mathbf{U}_i^{\nu}\left(:,j\right)\right), \qquad j=0,\dots,D_S,\\
U_0^S & = U_i^S + \Delta t(\bar{U}_0^S + \bar{U}_1^S + \bar{U}_2^S),\\
U_1^S\left(:,k\right) & = \left(I-\theta\Delta t F_1^k\right)^{-1}\left(U_0^S\left(:,k\right) - \theta\Delta t F_1^k\left(\mathbf{U}_i^S\left(:,k\right)\right)\right), \qquad k=0,\dots,D_\nu,\\
U_2^\nu\left(:,j\right) & = \left(I-\theta\Delta_t F_2\right)^{-1}\left(U_1^\nu\left(:,j\right) - \theta\Delta t F_2\left(\mathbf{U}_i^\nu\left(:,j\right)\right)\right), \qquad j=0,\dots,D_S,\\
\breve{U}^S\left(:,k\right) & = F_0^k\left(U_2^{S,\nu}\left(:,k\right)\right), \qquad k=0,\dots,D_\nu,\\
\hat{U}_0^S & = U_0^S + \frac{1}{2}\Delta_t\left(\breve{U}^S - \bar{U}_0^S\right),\\
\hat{U}_1^S\left(:,k\right) & = \left(I-\theta\Delta t F_1^k\right)^{-1}\left(\hat{U}_0^S\left(:,k\right) - \theta\Delta t F_1^k\left(\mathbf{U}_i^S\left(:,k\right)\right)\right), \qquad k=0,\dots,D_\nu,\\
\hat{U}_2^\nu\left(:,j\right) & = \left(I-\theta\Delta t F_2\right)^{-1}\left(\hat{U}_1^\nu\left(:,j\right) - \theta\Delta t F_2\left(\mathbf{U}_i^\nu\left(:,j\right)\right)\right), \qquad j=0,\dots,D_S,\\
\mathbf{U}_{i+1}^S & = \hat{U}_2^S.
\end{cases}
$$

It is worth to note the notation used above, e.g., $\bar{U}_2^S$ in the fourth equation can be directly obtained by reformulating $\bar{U}_2^\nu$ in the third equation.

3.3 The application to the extended Heston model

In this section we apply the Craig-Sneyd (CS) scheme to the Heston PDE extended by including a stochastic correlation (2.8), which can be rewritten as the ODE:

$$
\begin{aligned}
V'\left(t\right) &= \bar{F}(V,t)\\
&= \bar{F}_0\left(V,t\right) + \bar{F}_1\left(V,t\right) + \bar{F}_2\left(V,t\right) + \bar{F}_3\left(V,t\right)
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{cases}
\bar{F}_0\left(V,t\right) = \rho\sigma_\nu\nu S\dfrac{\partial^2 V}{\partial S\partial\nu} + \rho_1\sqrt{\nu}S\sigma_\rho\sqrt{1-\rho^2}\dfrac{\partial^2 V}{\partial S\partial\rho} + \rho_2\sigma_\rho\sqrt{1-\rho^2}\sigma_\nu\sqrt{\nu}\dfrac{\partial^2 V}{\partial\nu\partial\rho},\\
\bar{F}_1\left(V,t\right) = rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\nu S^2\dfrac{\partial^2 V}{\partial S^2} - \dfrac{1}{3}rV,\\
\bar{F}_2\left(V,t\right) = \kappa_\nu\left(\mu_\nu - \nu\right)\dfrac{\partial V}{\partial\nu} + \dfrac{1}{2}\sigma_\nu^2\nu\dfrac{\partial^2 V}{\partial\nu^2} - \dfrac{1}{3}rV,\\
\bar{F}_3\left(V,t\right) = \kappa_\rho\left(\mu_\rho - \rho\right)\dfrac{\partial V}{\partial\rho} + \dfrac{1}{2}\sigma_\rho^2(1-\rho^2)\dfrac{\partial^2 V}{\partial\rho^2} - \dfrac{1}{3}rV.
\end{cases}
\tag{3.14}
$$

We denote the approximation of $V(S,\nu,\rho,t)$ at t_i and grid point (S_j,ν_k,ρ_m) by $U_i^{j,k,m}$. Compare to (3.3) for the pure Heston model, we have the derivatives $\frac{\partial^2 V}{\partial S\partial\rho}$, $\frac{\partial^2 V}{\partial\nu\partial\rho}$, $\frac{\partial V}{\partial\rho}$ and $\frac{\partial V^2}{\partial\rho^2}$ more, which are approximated using the finite difference method with second order accuracy as well. For the same derivatives as in (3.3) we use the same notation as in (3.11) for the difference matrices. Besides, we use $A_{S,\nu}$, $A_{S,\rho}$, $A_{\nu,\rho}$, $A_{\rho,1}$ and $A_{\rho,2}$ for $\sigma_\nu S\frac{\partial^2 V}{\partial S\partial\nu}$, $\rho_1\sigma_\rho\sqrt{1-\rho^2}\frac{\partial^2 V}{\partial S\partial\rho}$, $\rho_2\sigma_\rho\sigma_\nu\sqrt{1-\rho^2}\frac{\partial^2 V}{\partial\nu\partial\rho}$, $\kappa_\rho(\mu_\rho-\rho)\frac{\partial V}{\partial\rho}$ and $\frac{1}{2}\sigma_\rho^2(1-\rho^2)\frac{\partial^2 V}{\partial\rho^2}$, respectively. Finally, (3.14) can be written in matrix form as

$$\begin{cases} \bar{F}_{01}^{j,k} = \sqrt{\nu\left(k\right)}S\left(j\right)A_{S,\rho} \in \mathbb{R}^{(D_\rho+1)\times 3(D_\rho+1)}, \quad j=0,\dots,D_S,\ k=0,\dots,D_\nu, \\ \bar{F}_{02}^{k} = \sqrt{\nu_t\left(k\right)}A_{\nu,\rho} \in \mathbb{R}^{(D_\rho+1)\times 3(D_\rho+1)}, \quad k=0,\dots,D_\nu, \\ \bar{F}_{03}^{k,m} = \rho(m)\nu\left(k\right)A_{S,\nu} \in \mathbb{R}^{(D_S+1)\times 3(D_S+1)}, \quad k=0,\dots,D_\nu,\ m=0,\dots,D_\rho, \\ \bar{F}_{1}^{k} = A_{S,1}+\nu\left(k\right)A_{S,2}-\dfrac{1}{3}rI_S \in \mathbb{R}^{(D_S+1)\times(D_S+1)}, \quad j=0,\dots,D_\nu, \\ \bar{F}_{2} = A_{\nu,1}+A_{\nu,2}-\dfrac{1}{3}rI_\nu \in \mathbb{R}^{(D_\nu+1)\times(D_\nu+1)}, \\ \bar{F}_{3} = A_{\rho,1}+A_{\rho,2}-\dfrac{1}{3}rI_\rho \in \mathbb{R}^{(D_\rho+1)\times(D_\rho+1)}, \end{cases} \tag{3.15}$$

where I_S, I_ν and I_ρ are the identity matrices of corresponding size given above.

Similar to (3.8), (3.9) and (3.10) we need also to formulate solution matrices

$$\begin{aligned} \bar{\mathbf{U}}_i^{S_{\nu\rho}} &\in \mathbb{R}^{(D_S+1)\times(D_\nu+1)\times(D_\rho+1)}, \\ \bar{\mathbf{U}}_i^{\nu_{S\rho}} &\in \mathbb{R}^{(D_\nu+1)\times(D_S+1)\times(D_\rho+1)}, \\ \bar{\mathbf{U}}_i^{\rho_{S\nu}} &\in \mathbb{R}^{(D_\rho+1)\times(D_S+1)\times(D_\nu+1)}, \\ \bar{\mathbf{U}}_i^{\rho,S} &\in \mathbb{R}^{3(D_\rho+1)\times(D_S+1)\times(D_\nu+1)}, \\ \bar{\mathbf{U}}_i^{\rho,\nu} &\in \mathbb{R}^{3(D_\rho+1)\times(D_\nu+1)\times(D_S+1)}, \\ \bar{\mathbf{U}}_i^{S,\nu} &\in \mathbb{R}^{3(D_S+1)\times(D_\nu+1)\times(D_\rho+1)}, \end{aligned}$$

for (3.15). For example, $\bar{\mathbf{U}}_i^{S_{\nu\rho}} \in \mathbb{R}^{(D_S+1)\times(D_\nu+1)\times(D_\rho+1)}$ reads

$$
\begin{array}{ccccc}
U^{0,0,D_\rho} & U^{0,1,D_\rho} & \cdots & U^{0,D_\nu-1,D_\rho} & U^{0,D_\nu,D_\rho} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U^{D_S,0,D_\rho} & U^{D_S,1,D_\rho} & \cdots & U^{D_S,D_\nu-1,D_\rho} & U^{D_S,D_\nu,D_\rho}
\end{array}
$$

$$
\begin{array}{ccccc}
U^{0,0,m} & U^{0,1,m} & \cdots & U^{0,D_\nu-1,m} & U^{0,D_\nu,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U^{D_S,0,m} & U^{D_S,1,m} & \cdots & U^{D_S,D_\nu-1,m} & U^{D_S,D_\nu,m}
\end{array}
$$

$$
\begin{array}{|ccccc|}
\hline
U^{0,0,0} & U^{0,1,0} & \cdots & U^{0,D_\nu-1,0} & U^{0,D_\nu,0} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U^{D_S,0,0} & U^{D_S,1,0} & \cdots & U^{D_S,D_\nu-1,0} & U^{D_S,D_\nu,0} \\
\hline
\end{array}
\tag{3.16}
$$

for $\bar{F}_1^k$, and $\bar{\mathbf{U}}_i^{S,\nu} \in \mathbb{R}^{3(D_S+1)\times(D_\nu+1)\times(D_\rho+1)}$ reads for a fix m

$$
\begin{pmatrix}
0 & U^{0,0,m} & \cdots & U^{0,D_\nu-2,m} & U^{0,D_\nu-1,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & U^{D_S,0,m} & \cdots & U^{D_S,D_\nu-2,m} & U^{D_S,D_\nu-1,m} \\
U^{0,0,m} & U^{0,1,m} & \cdots & U^{0,D_\nu-1,m} & U^{0,D_\nu,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U^{D_S,0,m} & U^{D_S,1,m} & \cdots & U^{D_S,D_\nu-1,m} & U^{D_S,D_\nu,m} \\
U^{0,1,m} & U^{0,2,m} & \cdots & U^{0,D_\nu-1,m} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U^{D_S,1,m} & U^{D_S,2,m} & \cdots & U^{D_S,D_\nu-1,m} & 0
\end{pmatrix}
\tag{3.17}
$$

for $\bar{F}_{03}^{k,m}$. Finally, for each time step we can obtain CS loop in matrix form for the extended Heston model as

$$
\left\{
\begin{aligned}
\bar{U}_1^{\rho,S}\left(:,j,k\right) &= \bar{F}_{01}^{j,k}\left(\bar{\mathbf{U}}_i^{\rho,S}\left(:,j,k\right)\right), \quad j=0,\dots,D_S,\, k=0,\dots,D_\nu,\\
\bar{U}_2^{\rho,\nu}\left(:,k,j\right) &= \bar{F}_{02}^{k}\left(\bar{\mathbf{U}}_i^{\rho,\nu}\left(:,k,j\right)\right), \quad k=0,\dots,D_\nu,\, j=0,\dots,D_S,\\
\bar{U}_3^{S,\nu}\left(:,k,m\right) &= \bar{F}_{03}^{k,m}\left(\bar{\mathbf{U}}_i^{S,\nu}\left(:,k,m\right)\right), \quad k=0,\dots,D_\nu,\, m=0,\dots,D_\rho,\\
\bar{U}_4^{S_{\nu\rho}}\left(:,k,m\right) &= \bar{F}_{1}^{k}\left(\bar{\mathbf{U}}_i^{S_{\nu\rho}}\left(:,k,m\right)\right), \quad k=0,\dots,D_\nu,\, m=0,\dots,D_\rho,\\
\bar{U}_5^{\nu_{S\rho}}\left(:,j,m\right) &= \bar{F}_{2}\left(\bar{\mathbf{U}}_i^{\nu_{S\rho}}\left(:,j,m\right)\right), \quad j=0,\dots,D_S,\, m=0,\dots,D_\rho,\\
\bar{U}_6^{\rho_{S\nu}}\left(:,j,k\right) &= \bar{F}_{3}\left(\bar{\mathbf{U}}_i^{\rho_{S\nu}}\left(:,j,k\right)\right), \quad j=0,\dots,D_S,\, k=0,\dots,D_\nu,\\
\bar{U}_0^{S_{\nu\rho}} &= \bar{\mathbf{U}}_i^{S_{\nu\rho}} + \Delta t\sum_{l=1}^{6}\bar{U}_l^{S_{\nu\rho}},\\
U_1^{S_{\nu\rho}}\left(:,k,m\right) &= \left(I-\theta\Delta t\bar{F}_1^k\right)^{-1}\left(\bar{U}_0^{S_{\nu\rho}}\left(:,k,m\right)-\theta\Delta t\bar{F}_1^k\left(\bar{\mathbf{U}}_i^{S_{\nu\rho}}\left(:,k,m\right)\right)\right),\\
&\quad k=0,\dots,D_\nu,\, m=0,\dots,D_\rho,\\
U_2^{\nu_{S\rho}}\left(:,j,m\right) &= \left(I-\theta\Delta t\bar{F}_2\right)^{-1}\left(U_1^{\nu_{S\rho}}\left(:,j,m\right)-\theta\Delta t\bar{F}_2\left(\bar{\mathbf{U}}_i^{\nu_{S\rho}}\left(:,j,m\right)\right)\right),\\
&\quad j=0,\dots,D_S,\, m=0,\dots,D_\rho,\\
U_3^{\rho_{S\nu}}\left(:,k,j\right) &= \left(I-\theta\Delta t\bar{F}_3\right)^{-1}\left(U_2^{\rho_{S\nu}}\left(:,k,j\right)-\theta\Delta t\bar{F}_3\left(\bar{\mathbf{U}}_i^{\rho_{S\nu}}\left(:,k,j\right)\right)\right),\\
&\quad k=0,\dots,D_\nu,\, j=0,\dots,D_S,\\
\breve{U}_1^{\rho,S}\left(:,j,k\right) &= \bar{F}_{01}^{j,k}\left(U_3^{\rho,S}\left(:,j,k\right)\right), j=0,\dots,D_S,\, k=0,\dots,D_\nu,\\
\breve{U}_2^{\rho,\nu}\left(:,k,j\right) &= \bar{F}_{02}^{k}\left(U_3^{\rho,\nu}\left(:,k,j\right)\right), k=0,\dots,D_\nu,\, j=0,\dots,D_S,\\
\breve{U}_3^{S,\nu}\left(:,k,m\right) &= \bar{F}_{03}^{k,m}\left(U_3^{S,\nu}\left(:,k,m\right)\right), k=0,\dots,D_\nu,\, m=0,\dots,D_\rho,\\
\hat{U}_0^{S_{\nu\rho}} &= \bar{U}_0^{S_{\nu\rho}} + \frac{1}{2}\Delta t\sum_{l=1}^{3}\left(\breve{U}_l^{S_{\nu\rho}}-\bar{U}_l^{S_{\nu\rho}}\right),\\
\hat{U}_1^{S_{\nu\rho}}\left(:,k,m\right) &= \left(I-\theta\Delta t\bar{F}_1^k\right)^{-1}\left(\hat{U}_0^{S_{\nu\rho}}\left(:,k,m\right)-\theta\Delta t\bar{F}_1^k\left(\bar{\mathbf{U}}_i^{S_{\nu\rho}}\left(:,k,m\right)\right)\right),\\
&\quad k=0,\dots,D_\nu,\, m=0,\dots,D_\rho,\\
\hat{U}_2^{\nu_{S\rho}}\left(:,j,m\right) &= \left(I-\theta\Delta t\bar{F}_2\right)^{-1}\left(\hat{U}_1^{\nu_{S\rho}}\left(:,j,m\right)-\theta\Delta t\bar{F}_2\left(\bar{\mathbf{U}}_i^{\nu_{S\rho}}\left(:,j,m\right)\right)\right),\\
&\quad j=0,\dots,D_S,\, m=0,\dots,D_\rho,\\
\hat{U}_3^{\rho_{S\nu}}\left(:,j,k\right) &= \left(I-\theta\Delta t\bar{F}_3\right)^{-1}\left(\hat{U}_2^{\rho_{S\nu}}\left(:,j,k\right)-\theta\Delta t\bar{F}_3\left(\bar{\mathbf{U}}_i^{\rho_{S\nu}}\left(:,j,k\right)\right)\right),\\
&\quad j=0,\dots,D_S,\, k=0,\dots,D_\nu,\\
\bar{\mathbf{U}}_{i+1}^{S_{\nu\rho}} &= \hat{U}_3^{S_{\nu\rho}}.
\end{aligned}
\right.
$$

Similarly, we can also implement e.g., Douglas (DO) [7], modified CS (MCS) [13] and Hundsdorfer-Verwer (HV) [21] scheme in matrix form.

4 Numerical results

As an example we calculate the value of a European call option, i.e., the PDE in (2.8) is solved numerically subject to the following boundary conditions:

$$\begin{cases} V(S,\nu,\rho,0) = (S-K)^+, \\ V(0,\nu,\rho,\tau) = 0, \\ V(S_{max},\nu,\rho,\tau) = S,\ \frac{\partial V}{\partial S}(S_{max},\nu,\rho,\tau) = 1,\ \frac{\partial^2 V}{\partial S^2}(S_{max},\nu,\rho,\tau) = 0, \\ -\dfrac{\partial V}{\partial \tau}(S,0,\rho,\tau) + rS\dfrac{\partial V}{\partial S}(S,0,\rho,\tau) + \kappa_\nu\mu_\nu\dfrac{\partial V}{\partial \nu}(S,0,\rho,\tau) \\ +\kappa_\rho(\mu_\rho-\rho)\dfrac{\partial V}{\partial \rho}(S,0,\rho,\tau) + \dfrac{1}{2}\sigma_\rho^2(1-\rho^2)\dfrac{\partial^2 V}{\partial \rho^2}(S,0,\rho,\tau) - rV(S,0,\rho,\tau) = 0, \\ V(S,\nu_{max},\rho,\tau) = S,\ \frac{\partial V}{\partial S}(S,\nu_{max},\rho,\tau) = 1,\ \frac{\partial^2 V}{\partial S^2}(S,\nu_{max},\rho,\tau) = 0, \\ -\dfrac{\partial V}{\partial \tau}(S,\nu,-1,\tau) + rS\dfrac{\partial V}{\partial S}(S,\nu,-1,\tau) + \kappa_\nu(\mu_\nu-\nu)\dfrac{\partial V}{\partial \nu}(S,\nu,-1,\tau) \\ +\kappa_\rho(\mu_\rho+1)\dfrac{\partial V}{\partial \rho}(S,\nu,-1,\tau) + \dfrac{1}{2}\nu S^2\dfrac{\partial^2 V}{\partial S^2}(S,\nu,-1,\tau) \\ +\dfrac{1}{2}\nu\sigma_\nu^2\dfrac{\partial^2 V}{\partial \nu^2}(S,\nu,-1,\tau) - \sigma_\nu\nu S\dfrac{\partial^2 V}{\partial\nu\partial S} - rV = 0, \\ -\dfrac{\partial V}{\partial \tau}(S,\nu,1,\tau) + rS\dfrac{\partial V}{\partial S}(S,\nu,1,\tau) + \kappa_\nu(\mu_\nu-\nu)\dfrac{\partial V}{\partial \nu}(S,\nu,1,\tau) \\ +\kappa_\rho(\mu_\rho+1)\dfrac{\partial V}{\partial \rho}(S,\nu,1,\tau) + \dfrac{1}{2}\nu S^2\dfrac{\partial^2 V}{\partial S^2}(S,\nu,1,\tau) \\ +\dfrac{1}{2}\nu\sigma_\nu^2\dfrac{\partial^2 V}{\partial \nu^2}(S,\nu,1,\tau) - \sigma_\nu\nu S\dfrac{\partial^2 V}{\partial\nu\partial S} - rV = 0. \end{cases} \tag{4.1}$$

The boundary conditions for the Heston PDE (2.1) can be directly obtained from (4.1) by considering ρ as constant. We use CS, MCS, DO and HV scheme to verify and validate our accelerated implementation. Numerical experiments were performed with Intel(R) Core(TM) i7-4470 CPU @ 3.4 GHz and 8GB RAM. For our numerical results we use uniform grid.

4.1 The results for the Heston model

For a premium's comparison between using implementation in vector and matrix form we firstly consider the pure Heston model. For our numerical results we use the sets of parameters: spot price $S_0 = 1200$, strike price $K = 1250$, risk-free rate $r = 0.25\%$, a zero dividend rate, stochastic volatility $\nu_0 = 0.15, \kappa_\nu = 1, \mu_\nu = 0.15, \sigma_\nu = 0.4$, constant correlation $\rho = -0.8$, Maturity $T = 0.125$, $S_{max} = 3000, S_{min} = 400, \nu_{max} = 5, \nu_{min} = 0$. The analytical solution is 43.01 being used as the reference price, absolute error is computed. In Table 1, we compare premiums between using implementation in vector and matrix form, where several ADI schemes are considered. Note that the

Full matrix	Mesh grid			Vector form			Matrix form		
ADI scheme	N_T	D_S	D_ν	Approx.	(CPU time)	Abs. Error	Approx.	(CPU time)	Abs. Error
DO	10	20	15	40.21	(0.11)	2.80	40.21	(0.01)	2.80
	20	40	35	42.44	(5.81)	0.57	42.44	(0.04)	0.57
	30	60	55	42.69	(45.63)	0.32	42.69	(0.22)	0.32
	40	80	75	42.85	(251.99)	0.16	42.85	(1.92)	0.16
	50	100	95	42.97	(1012.40)	0.04	42.97	(9.56)	0.04
CS	10	20	15	40.16	(0.20)	2.85	40.16	(0.01)	2.85
	20	40	35	42.39	(10.61)	0.62	42.39	(0.08)	0.62
	30	60	55	42.66	(82.91)	0.35	42.66	(0.46)	0.35
	40	80	75	42.83	(470.55)	0.18	42.83	(3.67)	0.18
	50	100	95	42.96	(1915.30)	0.05	42.96	(17.04)	0.05
MCS	10	20	15	40.16	(0.18)	2.85	40.16	(0.01)	2.85
	20	40	35	42.39	(11.28)	0.62	42.39	(0.08)	0.62
	30	60	55	42.66	(96.02)	0.35	42.66	(0.52)	0.35
	40	80	75	42.83	(517.94)	0.18	42.83	(4.08)	0.18
	50	100	95	42.96	(2038.10)	0.05	42.96	(19.18)	0.05
HV	10	20	15	40.16	(0.17)	2.84	40.16	(0.01)	2.84
	20	40	35	42.39	(11.34)	0.62	42.39	(0.08)	0.62
	30	60	55	42.66	(93.53)	0.35	42.66	(0.45)	0.35
	40	80	75	42.83	(508.40)	0.18	42.83	(3.81)	0.18
	50	100	95	42.96	(2020.30)	0.05	42.96	(17.88)	0.05

Table 1: Comparison of implementation in vector and matrix form using DO, CS, MCS and HV scheme, where $\theta_{DO} = \theta_{CS} = \theta_{HV} = 0.5$, $\theta_{MCS} = 0.75$, CPU time in seconds are given in brackets.

computation for results in Table 1 is performed using a full matrix structure. Due to the tridiagonal structure in the ADI scheme we can firstly convert full matrices into sparse form. Therefore, we run our algorithm again with the same parameter setting but based on the sparse matrices, the results are shown in Table 2. From our numerical results in Table 1 and Table 2 we observe that the proposed implementation in matrix form provide a significant acceleration, especially when full matrices structure is used. That allows for using ADI schemes to compute 3D problem with a reasonable computational time.

Sparse matrix	Mesh grid			Vector form			Matrix form		
ADI scheme	N_T	D_S	D_ν	Approx.	(CPU time)	Abs. error	Approx.	(CPU time)	Abs. error
DO	10	20	15	40.21	(0.04)	2.80	40.21	(0.00)	2.80
	20	40	35	42.44	(0.78)	0.57	42.44	(0.03)	0.57
	30	60	55	42.69	(4.06)	0.32	42.69	(0.18)	0.32
	40	80	75	42.85	(15.57)	0.16	42.85	(0.74)	0.16
	50	100	95	42.97	(46.77)	0.04	42.97	(7.32)	0.04
CS	10	20	15	40.16	(0.06)	2.85	40.16	(0.01)	2.85
	20	40	35	42.39	(0.84)	0.62	42.39	(0.05)	0.62
	30	60	55	42.66	(4.20)	0.35	42.66	(0.34)	0.35
	40	80	75	42.83	(17.47)	0.18	42.83	(1.49)	0.18
	50	100	95	42.96	(47.46)	0.05	42.96	(9.50)	0.05
MCS	10	20	15	40.16	(0.05)	2.85	40.16	(0.01)	2.85
	20	40	35	42.39	(1.46)	0.62	42.39	(0.06)	0.62
	30	60	55	42.66	(7.69)	0.35	42.66	(0.43)	0.35
	40	80	75	42.83	(29.73)	0.18	42.83	(1.79)	0.18
	50	100	95	42.96	(89.72)	0.05	42.96	(11.02)	0.05
HV	10	20	15	40.16	(0.05)	2.84	40.16	(0.01)	2.84
	20	40	35	42.39	(1.49)	0.62	42.39	(0.06)	0.62
	30	60	55	42.66	(7.56)	0.35	42.66	(0.36)	0.35
	40	80	75	42.83	(31.18)	0.18	42.83	(1.46)	0.18
	50	100	95	42.96	(89.82)	0.05	42.96	(10.43)	0.05

Table 2: Reproduction of Table 1 using the sparse matrix form.

4.2 The results for the extended Heston model

We apply the accelerated ADI schemes to the Heston model extended by a stochastic correlation, namely (2.8). For numerical results we use the sets of parameters: spot price $S_0 = 100$, risk-free rate $r = 0$, a zero dividend rate, stochastic volatility $\nu_0 = 0.02, \kappa_\nu = 2.1, \mu_\nu = 0.03, \sigma_\nu = 0.06$, stochastic correlation $\rho_0 = -0.3, \kappa_\rho = 3.7, \mu_\rho = -0.5, \sigma_\rho = 0.2$, $S_{max} = 4K^*, S_{min} = 0, \nu_{max} = 0.5, \nu_{min} = 0, \rho_{max} = 1, \rho_{min} = -1$. Furthermore, we set $\rho_1 = 0$ and choose $\rho_2 = -0.1, 0$ or 0.1 such that (2.7) is satisfied when $\rho_t \in (-1, 1)$. For this parameter setting we use $N_T = 80, D_S = 120, D_\nu = 100$ and $D_\rho = 80$. As the reference, we take the implied volatility for the proposed approach in [18]. In Table 3, we compare the approximated implied volatility (IV) for all the different ADI scheme in matrix form to the reference implied volatility (Ref. IV.), by varying strike K, maturity T and ρ_2. Note that the absolute error values in Table 3 are given in percent. The given CPU time in Table 3 evaluated without converting the matrices into sparse form. For a faster computation one can

*Strike price.

ADI scheme			DO		CS		MCS		HV	
ρ_2	Strike	Ref. IV.	Approx. IV.	Abs. Err.	Approx. IV.	Abs. Err.	Approx. IV.	Abs. Err.	Approx. IV.	Abs. Err.
Maturity $T = 0.5$										
-0.1	90	15.78	15.80 (124.54)	0.02	15.80 (205.19)	0.02	15.80 (278.54)	0.02	15.80 (124.54)	0.02
	100	15.45	15.51 (123.20)	0.06	15.51 (207.80)	0.06	15.51 (278.71)	0.06	15.51 (123.20)	0.06
	110	14.91	15.23 (123.20)	0.32	15.23 (207.64)	0.32	15.23 (277.40)	0.32	15.23 (123.20)	0.32
0	90	15.78	15.80 (124.96)	0.02	15.80 (207.44)	0.02	15.80 (276.51)	0.02	15.80 (124.96)	0.02
	100	15.46	15.51 (125.20)	0.05	15.51 (207.32)	0.05	15.51 (274.83)	0.05	15.51 (125.20)	0.05
	110	14.92	15.23 (121.90)	0.31	15.23 (209.90)	0.31	15.23 (277.39)	0.31	15.23 (121.90)	0.31
0.1	90	15.78	15.80 (125.34)	0.02	15.80 (206.25)	0.02	15.80 (279.95)	0.02	15.80 (125.34)	0.02
	100	15.46	15.51 (125.43)	0.05	15.51 (206.01)	0.05	15.51 (277.63)	0.05	15.51 (125.43)	0.05
	110	14.92	15.23 (124.29)	0.31	15.23 (207.38)	0.31	15.23 (279.30)	0.31	15.23 (124.29)	0.31
Maturity $T = 1$										
-0.1	90	16.26	16.30 (126.55)	0.04	16.30 (207.36)	0.04	16.30 (276.94)	0.04	16.30 (126.55)	0.04
	100	16.03	16.07 (124.43)	0.04	16.07 (207.00)	0.04	16.07 (275.92)	0.04	16.07 (124.43)	0.04
	110	15.82	15.84 (125.06)	0.02	15.84 (205.69)	0.02	15.84 (279.45)	0.02	15.84 (125.06)	0.02
0	90	16.26	16.30 (122.95)	0.04	16.30 (205.41)	0.04	16.30 (278.63)	0.04	16.30 (122.95)	0.04
	100	16.03	16.07 (123.85)	0.04	16.07 (207.19)	0.04	16.07 (276.79)	0.04	16.07 (123.85)	0.04
	110	15.82	15.84 (124.48)	0.02	15.84 (206.19)	0.02	15.84 (279.53)	0.02	15.84 (124.48)	0.02
0.1	90	16.26	16.30 (123.64)	0.04	16.30 (204.45)	0.04	16.30 (278.29)	0.04	16.30 (123.64)	0.04
	100	16.03	16.07 (123.67)	0.04	16.07 (209.02)	0.04	16.07 (276.67)	0.04	16.07 (123.67)	0.04
	110	15.82	15.84 (123.76)	0.02	15.84 (209.95)	0.02	15.84 (278.68)	0.02	15.84 (123.76)	0.02
Maturity $T = 5$										
-0.1	90	17.07	17.07 (123.85)	0.00	17.07 (207.47)	0.00	17.07 (277.94)	0.00	17.07 (123.85)	0.00
	100	16.99	16.99 (121.96)	0.00	16.99 (204.12)	0.00	16.99 (276.83)	0.00	16.99 (121.96)	0.00
	110	16.92	16.91 (121.42)	0.01	16.91 (207.90)	0.01	16.91 (277.18)	0.01	16.91 (121.42)	0.01
0	90	17.06	17.07 (123.03)	0.01	17.06 (205.06)	0.00	17.06 (277.56)	0.00	17.07 (123.03)	0.01
	100	16.99	16.99 (122.85)	0.00	16.99 (208.31)	0.00	16.99 (277.11)	0.00	16.99 (122.85)	0.00
	110	16.92	16.91 (123.40)	0.01	16.91 (207.82)	0.01	16.91 (277.48)	0.01	16.91 (123.40)	0.01
0.1	90	17.06	17.06 (124.79)	0.00	17.06 (207.05)	0.00	17.06 (278.89)	0.00	17.06 (124.79)	0.00
	100	16.99	16.99 (125.55)	0.00	16.99 (208.19)	0.00	16.99 (277.72)	0.00	16.99 (125.55)	0.00
	110	16.91	16.91 (125.31)	0.00	16.91 (206.95)	0.00	16.91 (279.46)	0.00	16.91 (125.31)	0.00

Table 3: Comparison of implied volatility for different ADI schemes by varying strike K, maturity T and ρ_2, CPU time in seconds are given in round brackets.

use sparse form as well. Our numerical results are quite promising for the three-dimensional problem.

5 Conclusion

In this work we have investigated an effective implementation of ADI scheme for the three-dimensional PDE problem with the mixed derivative term, namely the Heston model extended by including a stochastic correlation. Instead of the standard implementation in vector form, we have shown how to implement the ADI scheme in matrix form. The presented numerical results show that a significant acceleration can be achieved by designing the ADI scheme in matrix form.

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