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ABSTRACT. We study transport processes on infinite metric graphs with non-constant velocities and matrix boundary conditions in the L^∞-setting. We apply the theory of bi-continuous operator semigroups to obtain well-posedness of the problem under different assumptions on the velocities and for general stochastic matrices appearing in the boundary conditions.

1. INTRODUCTION

Consider a very large network, whose actual size may not be known but some structural properties are understood well. One way to model this situation is to consider an infinite graph with combinatorially reasonable assumptions based on the a priori knowledge about the structure of the network. Along the edges of the network some transport processes take place that are coupled in the vertices in which the edges meet. This means that we consider each edge as an interval and describe functions on it, that is, we consider a *metric graph*. Systems of partial differential equations on a metric graph are also known as *quantum graphs*. The transport processes (or *flows*) on the edges are given by partial differential equations of the form $\frac{\partial}{\partial t}u_j(t,x) = c_j\frac{\partial}{\partial x}u_j(t,x)$ and are interlinked in the common nodes via some prescribed transmission conditions.

Such a problem was considered by Dorn et al. [12, 13, 14] on the state space $L^1\left([0,1],\ell^1\right)$ applying the theory of strongly continuous operator semigroups. A semigroup approach to flows in finite metric graphs was first presented by Kramar-Sikolya [21] and further used in [15, 14, 9, 4, 8] while transport processes in infinite networks were also studied in [5, 7]. However, all these results were obtained in the L^1-setting. By considering problems in infinite graphs, the flow problem in the L^∞-setting should be interesting for applications as well. Von Below and Lubary [27, 28], for example, study eigenvalues of the Laplacian on infinite networks in an L^∞-setting. To the best of our knowledge, transport equations on infinite metric graphs with an L^∞-state space have not yet been studied. We consider this problem on the state space $L^\infty\left([0,1],\ell^1\right)$ where the obtained operator semigroup is not strongly continuous. To tackle this we apply the theory of *bi-continuous semigroups* that was introduced by Kühnemund [23, 24] and further developed by Farkas [17, 18, 19] and Albanese-Lorenzi-Manco [2].

This paper is organised as follows. Section 2 is a preliminary section which we start by a short introduction to the theory of bi-continuous semigroups. We also recall certain duality concepts of Bochner L^p-spaces that are needed later to obtain the generation theorem, and introduce some notions for networks and metric graphs. In Section 3 we present our flow problem for an infinite metric graph. We first prove the well posedness in the case when all flow velocities c_j equal 1. Next, we generalise this result to the case with rationally dependent velocities satisfying a finiteness condition. Finally, we show that the general problem on a finite metric graph is well-posed.

2. PRELIMINARIES

2.1. **Bi-Continuous Semigroups.** A family of linear bounded operators $(T(t))_{t \geq 0}$ on a Banach space X is called an (one-parameter) *strongly continuous semigroup* if it satisfies the following properties,

(1) $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and

(2) the function $t \mapsto T(t)f$ is continuous for all $f \in X$.

Strongly continuous semigroups and their applications to evolution equations have been studied intensively in the last decades, we refer to monographs [16, 8] and references therein. It is well-known, however, that there are important examples of semigroups which fail to satisfy property (2) above, i.e., that are not strongly continuous with respect to the Banach space norm. The standard example is the left translation semigroup $(T_l(t))_{t \geq 0}$ on $C_b(\mathbb{R})$ defined by $T_l(t)f(x) = f(x+t)$, $t \geq 0$. It is also known that this semigroup is strongly continuous with respect to the so-called compact open topology τ_{co}. This is a locally convex topology induced by the family of seminorms $\mathscr{P} = \{p_K \mid K \subseteq \mathbb{R} \text{ compact}\}$ where

$$p_K(f) = \sup_{x \in K} |f(x)|, \quad f \in C_b(\mathbb{R}).$$

Hence, it might be useful to equip the given Banach space $(X, \|\cdot\|)$ with an additional locally convex topology τ. This is the general idea of the so-called bi-continuous semigroups. Before giving the proper definition we state the main assumptions on the interplay between the norm and the locally convex topology τ.

Assumption 2.1. (i) τ is a Hausdorff topology and is coarser then the norm-topology on X, i.e., the identity map $(X, \|\cdot\|) \to (X, \tau)$ is continuous.

(ii) τ is sequentially complete on norm-bounded sets, i.e., every $\|\cdot\|$-bounded τ-Cauchy sequence in τ-convergent.

(iii) The dual space of (X, τ) is norming for X, i.e.,

$$\|f\| = \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} |\varphi(f)|.$$

Remark 2.2. (a) One can re-formulate the third assumption by the following equivalent statement: There is a set $\mathscr{P}$ of τ-continuous seminorms defining the topology τ, such that

$$\|f\| = \sup_{p \in \mathscr{P}} p(f).$$

(b) The above mentioned compact-open topology τ_{co} on $C_b(\mathbb{R})$ satisfies all these assumptions.

Now we are in the state to formulate the definition of a bi-continuous semigroup.

Definition 2.3 (Kühnemund [24]). Let X be a Banach space with norm $\|\cdot\|$ together with a locally convex topology τ, such that conditions in Assumption 2.1 are satisfied. We call $(T(t))_{t \geq 0}$ a *bi-continuous semigroup* on X if the following holds.

(i) $(T(t))_{t \geq 0}$ satisfies the *semigroup property*, i.e., $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $s, t \geq 0$.

(ii) $(T(t))_{t \geq 0}$ is *strongly τ-continuous*, i.e., the map $\varphi_f : [0, \infty) \to (X, \tau)$ defined by $\varphi_f(t) = T(t)f$ is continuous for every $f \in X$.

(iii) $(T(t))_{t \geq 0}$ is *exponentially bounded*, i.e., there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for each $t \geq 0$.

(iv) $(T(t))_{t\geq 0}$ is *locally-bi-equicontinuous*, i.e., if $(f_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in X which is τ-convergent to 0, then also $(T(s)f_n)_{n\in\mathbb{N}}$ is τ-convergent to 0 uniformly for $s \in [0, t_0]$ for each fixed $t_0 \geq 0$.

The *growth bound* of $(T(t))_{t\geq 0}$ is defined as

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \text{ there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}.$$

As in the case of strongly continuous semigroups one can define a generator and relate it to the well-posedness property of abstract initial value problems. The *generator A* of bi-continuous semigroup $(T(t))_{t\geq 0}$ is defined as

$$Af := \tau - \lim_{t\to 0} \frac{T(t)f - f}{t}$$

with the domain

$$D(A) := \left\{ f \in X \,\middle|\, \tau - \lim_{t\to 0} \frac{T(t)f - f}{t} \text{ exists and } \sup_{t\in(0,1]} \frac{\|T(t)f - f\|}{t} < \infty \right\}.$$

Let us recall some more notions from the bi-continuous setting. A subset $M \subseteq X$ is called *bi-dense* if for every $f \in X$ there exists a $\|\cdot\|$-bounded sequence $(f_n)_{n\in\mathbb{N}}$ in M which is τ-convergent to f. An operator A is called *bi-closed*, whenever for $f_n \xrightarrow{\tau} f$ and $Af_n \xrightarrow{\tau} g$, where both sequences are norm-bounded, it holds $f \in D(A)$ and $Af = g$.

We have collected here some basic properties of generators of bi-continuous semigroups. For the proofs we refer to [23, 24, 18].

Proposition 2.4. *The following assertions hold for the generator $(A, D(A))$ of a bi-continuous semigroup $(T(t))_{t\geq 0}$.*

(a) *Operator A is bi-closed and its domain $D(A)$ is bi-dense in X.*

(b) *For $f \in D(A)$ one has $T(t)f \in D(A)$ and $T(t)Af = AT(t)f$ for all $t \geq 0$.*

(c) *For $t > 0$ and $f \in X$ one has*

$$\int_0^t T(s)f \, ds \in D(A) \quad and \quad A \int_0^t T(s)f \, ds = T(t)f - f.$$

(d) *For $\lambda > \omega_0(T)$ one has $\lambda \in \rho(A)$ and for every $f \in X$*

$$(2.1) \qquad\qquad R(\lambda, A)f = \int_0^\infty e^{-\lambda s}T(s)f \, ds,$$

where the integral is a τ-improper integral.

(e) *The semigroup $(T(t))_{t\geq 0}$ is uniquely determined by its generator $(A, D(A))$.*

The following generalisation of the classical notion of well-posedness is due to Farkas [17, Def. 4.1.1]. By $B_{loc}(\mathbb{R}_+, X)$ we denote the space of functions that are bounded on each compact subset of $\mathbb{R}_+$ and the differentiation is understood in the vector valued sense with respect to τ.

Definition 2.5. The abstract Cauchy problem

$$(\text{ACP}) \qquad\qquad \begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = f \in D(A), \end{cases}$$

is *well-posed* in X if

(i) for every $f \in D(A)$ there exists a unique solution $u(t) := u(t, f)$ of (ACP) with $u \in B_{loc}(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, (X, \tau))$ and $\dot{u} \in B_{loc}(\mathbb{R}_+, X)$,

(ii) the solution is unique, and

(iii) the solution u of (ACP) depends continuously on the initial data f, i.e., if the sequence $(f_n)_{n \in \mathbb{N}}$ is norm-bounded and τ-convergent to 0 then the solutions $u_n(t) := u_n(t, f_n)$ converge to 0 in τ and uniformly on compact intervals $[0, t_0]$.

Theorem 2.6. [17, Thm. 4.1.2] *If $(A, \mathrm{D}(A))$ generates a bi-continuous semigroup then the abstract Cauchy problem* (ACP) *is well-posed.*

This result is very useful for applications in combination with some version of the generation theorem for bi-continuous semigroups. We only recall here a variant of Trotter–Kato approximation theorem, see [23, 3, 17]. For that we also evoke the notion of uniformly bi-continuous semigroups [23, Def. 2.1].

Definition 2.7. Let $(T_n(t))_{t \geq 0}$, $n \in \mathbb{N}$, be bi-continuous semigroups on X. They are called *uniformly bi-continuous* (of type ω) if the following conditions hold.

(i) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_n(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}$.

(ii) $(T_n(t))_{t \geq 0}$ are locally bi-equicontinuous uniformly for $n \in \mathbb{N}$, i.e., for every $t_0 \geq 0$ and for every $\|\cdot\|$-bounded sequence $(f_k)_{k \in \mathbb{N}}$ in X which is τ-convergent to 0 we have

$$\tau - \lim_{k \to \infty} T_n(t)f_k = 0,$$

uniformly for $0 \leq t \leq t_0$ and $n \in \mathbb{N}$.

Let us also recall the notion of a bi-core, as defined in [23, Def. 1.20]. A subspace D of the domain of a linear operator $(A, \mathrm{D}(A))$ on a Banach space X is a *bi-core* for A if for all $f \in \mathrm{D}(A)$ there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in D such that $(f_n)_{n \in \mathbb{N}}$ and $(Af_n)_{n \in \mathbb{N}}$ are $\|\cdot\|$-bounded and $\lim_{n \to \infty} f_n = f$ with respect to the locally convex (graph) topology τ_A generated by the family of seminorms

$$\mathscr{P}_A := \{p(\cdot) + q(A\cdot) \mid p, q \in \mathscr{P}\}.$$

Having these definitions in mind, we can formulate the Trotter–Kato approximation theorem for bi-continuous semigroups which was first proven by Kühnemund [23, Thm. 2.6] and later on in a more general version by Albanese and Mangino [3, Thm. 3.6]. For locally equicontinuous semigroups on locally convex space this was done by Albanese and Kühnemund [1, Thm. 16].

Theorem 2.8. [23, Thm. 2.6] *Let $(T_n(t))_{t \geq 0}$, $n \in \mathbb{N}$, be bi-continuous semigroups with generators $(A_n, \mathrm{D}(A_n))$ such that they are uniformly bi-continuous of type ω and let $\lambda_0 > \omega$. Consider the following assertions.*

(a) *There exists a bi-densely defined operator $(A, \mathrm{D}(A))$ such that $A_n f \xrightarrow{\|\cdot\|} Af$ for all f in a bi-core of A and such that $\mathrm{Ran}(\lambda_0 - A)$ is bi-dense in X.*

(b) *There exists an operator $R \in \mathscr{L}(X)$ such that $R(\lambda_0, A_n)f \xrightarrow{\|\cdot\|} Rf$ for all f in a subset of $\mathrm{Ran}(R)$ which is bi-dense in X.*

(c) *There exists a bi-continuous semigroup $(T(t))_{t \geq 0}$ with generator $(B, \mathrm{D}(B))$ such that $T_n(t)f \xrightarrow{\tau} T(t)f$ for all $f \in X$ uniformly for t in compact intervals*

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold. In this case, $B = \overline{A}^\tau$ (the bi-closure of A).*

Remark 2.9. In the proof of [23, Thm. 2.6] one observes that operator R in assertion (b) gives rise to a pseudo-resolvent that is used to define operator $(B, \mathrm{D}(B))$ in assertion (c).

We refer to [23, 24, 17, 10, 18, 3] for further properties of bi-continuous semigroups and their generators.

2.2. **Some observations on Bochner L^p-spaces and duality.** We consider spaces of the form $L^p([0,1], X)$ where X is a Banach space. From [11, Chapter IV, Section 1] and [22, Theorem 2.22] we obtain that for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$(2.2) \qquad L^p([0,1], X)' \cong L^q([0,1], X') \quad \text{and} \quad L^1([0,1], X)' \cong L^\infty([0,1], X')$$

whenever X' has the Radon–Nikodym property. If this is not the case, one only has an isometric inclusion $L^q([0,1], X') \hookrightarrow L^p([0,1], X)'$ for $1 \leq p < \infty$.

Example 1. It is known that the space ℓ^1 has the Radon-Nikodym property while the spaces c_0, c, and ℓ^∞ do not. Recall that $(\ell^1)' = \ell^\infty$ while $(c_0)' \cong \ell^1$ as well as $(c)' \cong \ell^1$ (that is, space ℓ^1 does not have a unique predual space, see also [25]). By (2.2) we obtain that

$$L^\infty([0,1], \ell^1) \cong L^1([0,1], c_0)' \cong L^1([0,1], c)'$$

while $L^1([0,1], \ell^1)'$ is only isomorphic to a subset of $L^\infty([0,1], \ell^\infty)$.

The *weak*-topology* on the dual spaces is defined as follows: $f_n \to f$ with respect to the weak*-topology on $L^p([0,1], X)'$ for $1 \leq p < \infty$ if and only if $\langle f_n - f, g \rangle \to 0$ for all $g \in L^p([0,1], X)$, where the pairing is defined by

$$(2.3) \qquad \langle f, g \rangle := \int_0^1 \langle f(s), g(s) \rangle_X \, ds, \quad f \in L^p([0,1], X)' = L^q([0,1], X'), \; g \in L^p([0,1], X),$$

and $\langle \cdot, \cdot \rangle_X$ denotes the dual pairing between X and X'.

2.3. **Infinite Networks, Metric Graphs.** We use the notation introduced in [21] for finite and expanded in [12] to infinite networks. Network is modelled with an *infinite directed graph* $G = (V, E)$ with a set of *vertices* $V = \{v_i \mid i \in I\}$ and a set of *directed edges* $E = \{e_j \mid j \in J\} \subseteq V \times V$ for some countable sets $I, J \subseteq \mathbb{N}$. For a directed edge $e = (v_i, v_k)$ we call v_i the *tail* and v_k the *head* of e. Further, the edge e is an *outgoing edge* of the vertex v_i and an *incoming edge* for the vertex v_k. We assume that graph G is *simple*, i.e., there are no loops or multiple edges, and *locally finite*, i.e., each vertex only has finitely many outgoing edges.

Graph G is *weighted*, that is equipped with some weights $0 \leq w_{ij} \leq 1$ such that

$$(2.4) \qquad \sum_{i \in J} w_{ij} = 1 \text{ for all } j \in J.$$

The structure of a graph can be described by its incidence and/or adjacency matrices. We shall only use the so-called *weighted (transposed) adjacency matrix of the line graph* $\mathbb{B} = (\mathbb{B}_{ij})_{i,j \in J}$ defined as

$$(2.5) \qquad \mathbb{B}_{ij} := \begin{cases} w_{ij} & \text{if } \xrightarrow{e_j} v \xrightarrow{e_i}, \\ 0 & \text{otherwise.} \end{cases}$$

By (2.4), matrix $\mathbb{B}$ is column stochastic and defines a bounded positive operator on ℓ^1 with $r(\mathbb{B}) = \|\mathbb{B}\| = 1$. It reflects many properties of graph G. For example, $\mathbb{B}$ is irreducible iff graph G is strongly connected (see [12, Prop. 4.9]).

We identify every edge of our graph with the unit interval, $e_j \equiv [0,1]$ for each $j \in J$, and parametrise it contrary to its direction, so that it is assumed to have its tail at the endpoint 1 and its head at the endpoint 0. For simplicity we use the notation $e_j(1)$ and $e_j(0)$ for the tail and the head, respectively. In this way we obtain a *metric graph*.

For the unexplained terminology we refer to [8, Sect. 18] and [12].

3. Transport problems in (in)finite metric graphs

We now consider a transport process (or a flow) along the edges of an infinite network, modelled by a metric graph G. The distribution of material along edge e_j at time $t \geq 0$ is described by function $u_j(x,t)$ for $x \in [0,1]$. The material is transported along edge e_j with constant velocity $c_j > 0$, $j \in J$. We assume that

$$(3.1) \qquad 0 < c_{\min} \leq c_j \leq c_{\max} < \infty$$

for all $j \in J$. Let $C := \operatorname{diag}(c_j)_{j \in J}$ be a diagonal velocity matrix and define another weighted adjacency matrix of the line graph by

$$\mathbb{B}^C := C^{-1} \mathbb{B} C.$$

In the vertices the material gets redistributed according to some prescribed rules. This is modelled in the boundary conditions by using the adjacency matrix $\mathbb{B}_C$. The flow process on G is thus given by the following infinite system of equations

$$(3.2) \qquad \begin{cases} \frac{\partial}{\partial t} u_j(x,t) = c_j \frac{\partial}{\partial x} u_j(x,t), & x \in (0,1), \ t \geq 0, \\ u_j(1,t) = \sum_{k \in J} \mathbb{B}^C_{jk} u_k(0,t), & t \geq 0, \\ u_j(x,0) = f_j(x), & x \in (0,1), \end{cases}$$

for every $j \in J$, where $f_j(x)$ are the initial distributions along the edges.

One can give different interpretations to the weights w_{ij}, i.e, entries of the matrix $\mathbb{B}$, resulting in different transport problems. The two most obvious are the following.

(1) w_{ij} is the proportion of the material arriving from edge e_j leaving on edge e_i.

(2) w_{ij} is the proportion of the material arriving from vertex $e_j(0) = e_i(1)$ leaving on edge e_i.

Note, that in both situations (2.4) represents a conservation of mass and the assumption on local finiteness of the graph guarantees that all the sums are finite. While the latter situation is the most common one (see e.g. [12, 21, 8]) the first one was considered for finite networks in [9, Sect. 5]. Here, we will not give any particular interpretation and will treat all the cases simultaneously.

Remark 3.1. By replacing in (3.2) the graph matrix $\mathbb{B}^C$ with some other matrix, one obtains a more general initial-value problem that does not necessarily consider a process in a physical network. Such a problem from population dynamics was for example studied in [5]. Furthermore, a question when can such a general problem be identified with a corresponding problem on a metric graph was raised in [4].

3.1. **The simple case.** First we assume that all the velocities are the same: $c_j = 1$ for each $j \in J$. As the state space we set $X := \mathrm{L}^\infty\left([0,1], \ell^1\right)$ equipped with the norm

$$\|f\|_X := \operatorname*{ess\,sup}_{s \in [0,1]} \|f(s)\|_{\ell^1}.$$

On Banach space X we define operator $(A, \mathrm{D}(A))$ by

$$(3.3) \qquad A := \operatorname{diag}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

$$\mathrm{D}(A) := \left\{ v \in \mathrm{W}^{1,\infty}\left([0,1], \ell^1\right) \mid v(1) = \mathbb{B}v(0) \right\}$$

Observe that the corresponding abstract Cauchy problem

$$(3.4) \qquad \begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = (f_j)_{j \in J}, \end{cases}$$

on X is equivalent to the flow problem (3.2) in case when all the velocities equal 1.

This problem was considered by Dorn [12] on the state space $\mathrm{L}^1\left([0,1], \ell^1\right)$ where an explicit formula for the solution semigroup in terms of a shift and matrix $\mathbb{B}$ was derived. Here, the left

translation semigroup $(T_l(t))_{t \geq 0}$ was used, which is not strongly continuous on X. By using duality arguments we will show that it is, however, a bi-continuous semigroup on X. First note that the left translation semigroup on $X = \mathrm{L}^1([0,1], \mathrm{c}_0)'$ is the adjoint semigroup of the right translation semigroup on $\mathrm{L}^1([0,1], \mathrm{c}_0)$, see Subsection 2.2 and [16, II.5.14].

Lemma 3.2. *The right translation semigroup $(T_r(t))_{t \geq 0}$, defined by*

$$T_r(t)f(s) := \begin{cases} f(s-t), & s-t \geq 0, \\ 0, & s-t < 0, \end{cases}$$

for $t \geq 0$ and $s \in [0,1]$, is strongly continuous on $\mathrm{L}^1([0,1], \mathrm{c}_0)$.

Proof. Let $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \mathrm{c}_0$ and let $f := \mathbf{x} \cdot \chi_\Omega$ for a measurable subset $\Omega \subseteq [0,1]$. We first show that $T_r(t)f \to f$ with respect to the norm on $\mathrm{L}^1([0,1], \mathrm{c}_0)$ as $t \to 0$:

$$\int_0^1 \|T_r(t)f(s) - f(s)\|_{\mathrm{c}_0} \, \mathrm{d}s = \int_0^1 \|f(s-t)\chi_{[s-t \geq 0]} - f(s)\|_{\mathrm{c}_0} \, \mathrm{d}s$$

$$= \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \int_0^1 |\chi_\Omega(s-t) - \chi_\Omega(s)| \, \mathrm{d}s$$

$$= \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \int_0^1 |\chi_{(\Omega+t)\Delta\Omega}(s)| \, \mathrm{d}s$$

$$= \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \lambda^1((\Omega+t)\Delta\Omega) \to 0 \text{ as } t \to 0,$$

where λ^1 is the one-dimensional Lebesgue measure on $[0,1]$ and Δ the symmetric difference of sets defined by $A \Delta B := (A \cup B) \setminus (A \cap B)$. Since every function $f \in \mathrm{L}^1([0,1], \mathrm{c}_0)$ is an increasing limit of linear combinations of functions of the form $\mathbf{x} \cdot \chi_\Omega$ for some $\mathbf{x} \in \mathrm{c}_0$ and measurable set $\Omega \subseteq [0,1]$, the vector-valued version of Beppo–Levis's monotone convergence theorem yields the result. $\qquad \square$

Remark 3.3. Observe that the strong continuity does not depend on the space c_0, hence one could generalise this result to strongly continuous right-translation semigroups on $\mathrm{L}^1([0,1], Y)$ and even to $\mathrm{L}^p([0,1], Y)$ for an arbitrary Banach space Y and $1 \leq p < \infty$.

Lemma 3.4. *The left translation semigroup $(T_l(t))_{t \geq 0}$, defined by*

$$T_l(t)f(s) := \begin{cases} f(s+t), & s+t \leq 1, \\ 0, & s+t > 1, \end{cases}$$

for $t \geq 0$ and $s \in [0,1]$, is bi-continuous on $\mathrm{L}^\infty([0,1], \ell^1)$ with respect to the weak*-topology.

Proof. By [26, Corollary 2.1.7], the dual space Y' of any Banach space Y satisfies Assumption 2.1 for the weak*-topology. Moreover, by [23, Proposition 3.18], the dual semigroup $(T'(t))_{t \geq 0}$ on Y' is bi-continuous with respect to the weak*-topology whenever $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on Y. Thus, [20, Example 1.3], Example 1 and Lemma 3.2 imply the result. $\qquad \square$

We now use the formula for the semigroup which was derived by Dorn [12] and show that it yields a bi-continuous semigroup on $L^\infty([0,1], \ell^1)$. For that we have to check all the assertions from Definition 2.3 which we do in several steps.

Lemma 3.5. *The semigroup $(T(t))_{t \geq 0}$ on $X = L^\infty([0,1], \ell^1)$, defined by*

(3.5) $$T(t)f(s) = \mathbb{B}^n f(t+s-n), \quad n \leq t+s < n+1, \ f \in X, \ n \in \mathbb{N}_0,$$

is strongly continuous with respect to the weak*-topology.

Proof. The semigroup property is easy to verify. Observe that for any $f \in X$, $g \in L^1([0,1], c_0)$, and $t \in (0,1]$ we have

$$|\langle T(t)f - f, g \rangle| = \left| \int_0^1 \langle T(t)f(s) - f(s), g(s) \rangle_X \, \mathrm{d}s \right|$$

$$\leq \left| \int_0^{1-t} \langle f(s+t) - f(s), g(s) \rangle_X \, \mathrm{d}s \right| + \int_{1-t}^1 |\langle \mathbb{B}f(s+t-1) - f(s), g(s) \rangle_X| \, \mathrm{d}s$$

$$\leq \left| \int_0^1 \langle T_l(t)f(s) - f(s), g(s) \rangle_X \, \mathrm{d}s \right| + \int_{1-t}^1 |\langle \mathbb{B}f(s+t-1) - f(s), g(s) \rangle_X| \, \mathrm{d}s.$$

Now, notice that the second summand vanishes since $\lambda^1([1-t,1]) \to 0$ as $t \to 0$. Here, λ^1 is the one-dimensional Lebesgue measure on the unit interval $[0,1]$. By Lemma 3.4, the left translation semigroup is bi-continuous on X, which means, in particular, that it is strongly continuous with respect to the weak*-topology and hence the first summand also vanishes as $t \to 0$. $\qquad\square$

Lemma 3.6. *The semigroup $(T(t))_{t \geq 0}$, defined by (3.5), is a contraction semigroup on X.*

Proof. Let $f \in X$ and $t \geq 0$. Then there exists $n \in \mathbb{N}$ such that $n \leq t < n+1$. This means that for $s \in [0,1]$ one has $n \leq s+t < n+2$. By (3.5), we can make the following estimate.

$$\|T(t)f\|_X = \operatorname*{ess\,sup}_{s \in [0,1]} \|T(t)f(s)\|_{\ell^1}$$

$$\leq \max \left\{ \operatorname*{ess\,sup}_{s \in [0,1]} \|\mathbb{B}^n f(s+t-n)\|_{\ell^1}, \operatorname*{ess\,sup}_{s \in [0,1]} \|\mathbb{B}^{n+1} f(s+t-n-1)\|_{\ell^1} \right\}.$$

Since $\|\mathbb{B}^n\| = \|\mathbb{B}\|^n = 1$, we have

$$\|\mathbb{B}^n f(s+t-n)\|_{\ell^1} \leq \|\mathbb{B}^n\| \cdot \|f\|_X = \|f\|_X$$

and hence, $\|T(t)f\|_X \leq \|f\|_X$. $\qquad\square$

Lemma 3.7. *The semigroup $(T(t))_{t \geq 0}$, defined by (3.5), is locally bi-equicontinuous with respect to the weak*-topology on $X = L^\infty([0,1], \ell^1)$.*

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in X that is $\|\cdot\|_X$-bounded and converges to 0 with respect to the weak*-topology. By Definition 2.3 we need to show that $(T(t)f_n)_{n \in \mathbb{N}}$ also converges to 0 with respect to the weak*-topology uniformly on compact intervals $[0, t_0]$. To this end, fix $t_0 > 0$ and let $m := \lfloor t_0 \rfloor$. Then $0 \leq s+t \leq m+1$ for all $t \in [0, t_0]$, $s \in [0,1]$ and by (3.5) we can estimate as follows.

$$|\langle T(t)f_n, g \rangle| \leq \int_0^1 |\langle T(t)f_n(s), g(s) \rangle_X| \, \mathrm{d}s$$

$$\leq \int_0^1 \max_{0 \leq k \leq m} |\langle \mathbb{B}^k f_n(s+t-k), g(s) \rangle_X| \, \mathrm{d}s$$

$$\leq \int_0^1 \max_{0 \leq k \leq m} \left| \left\langle T_l(t-k)f_n(s), (\mathbb{B}^k)' g(s) \right\rangle_X \right| \, \mathrm{d}s$$

for any $g \in L^1([0,1], c_0)$. Since, by Lemma 3.4, the left translation semigroup $(T_l(t))_{t \geq 0}$ is bi-continuous, hence locally bi-equicontinuous on compact intervals, $|\langle T(t)f_n, g \rangle|$ tends to 0 uniformly on $[0, t_0]$. This finishes the proof. $\qquad\square$

Let us here recall the explicit expression of the resolvent of operator $(A, \mathrm{D}(A))$ defined by (3.3) which was obtained in [12, Theorem 18]. This result does not rely on the Banach space and remains the same by taking $X = L^\infty([0,1], \ell^1)$.

Proposition 3.8. *For* $\mathrm{Re}(\lambda) > 0$ *the resolvent* $R(\lambda, A)$ *of the operator* $(A, \mathrm{D}(A))$ *defined by* (3.3) *is given by*

$$(R(\lambda, A)f)(s) := \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \int_0^1 \mathrm{e}^{-\lambda(t+1-s)} \mathbb{B}^{k+1} f(t) \, \mathrm{d}t + \int_s^1 \mathrm{e}^{\lambda(s-t)} f(t) \, \mathrm{d}t, \quad f \in X, \ s \in [0, 1].$$

We are now in the state prove the first generation theorem.

Theorem 3.9. *The operator* $(A, \mathrm{D}(A))$, *defined in* (3.3), *generates a contraction bi-continuous semigroup* $(T(t))_{t \geq 0}$ *on* X *with respect to the weak*-topology. This semigroup is given by* (3.5).

Proof. By Lemmas 3.5, 3.6, and 3.7, semigroup $(T(t))_{t \geq 0}$ defined by (3.5) is a bi-continuous semigroup with respect to the weak*-topology. It remains to show that $(A, \mathrm{D}(A))$, given in (3.3), is the generator of this semigroup.

Let $(C, \mathrm{D}(C))$ be the generator of $(T(t))_{t \geq 0}$. For $f \in \mathrm{D}(A)$ and $s \in [0, 1]$ we have $T(t)f \in \mathrm{D}(A)$. By (2.1), the resolvent of C is the Laplace transform of the semigroup $(T(t))_{t \geq 0}$, that is, for $\lambda > \omega_0(T)$ we have

$$\begin{aligned}
R(\lambda, C)f(s) &= \int_0^{\infty} \mathrm{e}^{-\lambda t} T(t) f(s) \, \mathrm{d}t \\
&= \int_0^{1-s} \mathrm{e}^{-\lambda t} f(t+s) \, \mathrm{d}t + \sum_{n=1}^{\infty} \int_{n-s}^{n-s+1} \mathrm{e}^{-\lambda t} \mathbb{B}^n f(t+s-n) \, \mathrm{d}t \\
&= \int_s^1 \mathrm{e}^{-\lambda(\xi-s)} f(\xi) \, \mathrm{d}\xi + \sum_{n=1}^{\infty} \int_0^1 \mathrm{e}^{-\lambda(\xi-s-n)} \mathbb{B}^n f(\xi) \, \mathrm{d}\xi
\end{aligned}$$

By Proposition 3.8, the resolvent operators $R(\lambda, A)$ and $R(\lambda, C)$, coincide on the bi-dense set $D(A)$, so we may conclude that $C = A$. $\qquad\square$

Corollary 1. *If all* $c_j = 1$, $j \in J$, *the flow problem* (3.2) *is well-posed on* $X = \mathrm{L}^{\infty}([0, 1], \ell^1)$.

Remark 3.10. All the obtained results also hold for finite networks. If $G = (V, E)$ is a finite network with $|E| = m < \infty$, we have $\ell^1(\{1, \ldots, m\}) \cong \mathbb{C}^m$, hence we consider our semigroups on the space $X = \mathrm{L}^{\infty}([0, 1], \mathbb{C}^m)$.

3.2. **The rationally dependent case.** We now consider the case when the velocities c_j appearing in (3.2) are not all equal to 1 and define on $X := \mathrm{L}^{\infty}([0, 1], \ell^1)$ the operator

$$(3.6) \qquad \begin{aligned}
A_C &:= \mathrm{diag}\left(c_j \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right), \\
\mathrm{D}(A_C) &:= \left\{f \in \mathrm{W}^{1,\infty}([0, 1], \ell^1) \mid f(1) = \mathbb{B}^C f(0)\right\}.
\end{aligned}$$

We assume, however, that the velocities are linearly dependant over $\mathbb{Q}$: $\frac{c_i}{c_j} \in \mathbb{Q}$ for all $i, j \in J$, with a finite common multiplier, that is,

$$(3.7) \qquad \text{there exists } 0 < c \in \mathbb{R} \text{ such that } \ell_j := \frac{c}{c_j} \in \mathbb{N} \text{ for all } j \in J.$$

This enables us to use the procedure that was introduced in the proof of [21, Thm. 4.5] and carried out in detail in [6, Sect. 3]. We construct a new directed graph $\widetilde{G}$ by adding $\ell_j - 1$ vertices on edge e_j for all $j \in J$. The newly obtained edges inherit the direction of the original edge and are parametrised as unit intervals $[0, 1]$. We can thus consider a new problem on $\widetilde{G}$ with corresponding functions $\widetilde{u}_j$ and velocities $\widetilde{c}_j := c$ for each $j \in \widetilde{J}$. After appropriately correcting the initial and boundary conditions the new problem is equivalent to the original one. Since all the velocities on the edges of the new graph are equal, we can treat this case by rescaling to 1 and use the results from Subsection 3.1. Moreover, since (3.1) and (3.7) hold, the procedure described in [6, Sect. 3] for the finite case can be as preformed in the infinite case as well. Hence, we even obtain an isomorphism between the corresponding semigroups.

Proposition 3.11. *Let the assumptions* (3.1) *and* (3.7) *on the velocities c_j hold. Then operator* $(A_C, \mathrm{D}(A_C))$, *defined in* (3.6), *generates a contraction bi-continuous semigroup* $(T_C(t))_{t\geq 0}$ *on X with respect to the weak*-topology. Moreover, there exists an isomorphism $S\colon X \to X$ such that*

$$(3.8) \qquad\qquad T_C(ct)f = ST(t)S^{-1}f,$$

where the semigroup $(T(t))_{t\geq 0}$ is given by (3.5)

Corollary 2. *If the assumptions* (3.1) *and* (3.7) *on the velocities c_j hold, the flow problem* (3.2) *is well-posed on $X = \mathrm{L}^\infty\left([0,1],\ell^1\right)$.*

3.3. **The general case for finite networks.** We finally consider the case of general $c_j \in \mathbb{R}$ but restrict ourselves to *finite* graphs, i.e., we work on the Banach space $X = \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right)$, where m denotes the number of edges in the graph. In [8, Cor. 18.15] the Lumer–Phillips generation theorem for positive strongly continuous semigroups is applied to show that the transport problem with general $c_j \in \mathbb{R}$ is well-posed on $X = \mathrm{L}^1\left([0,1],\mathbb{C}^m\right)$. Since an appropriate variant of this result for the bi-continuous situation is not known, we proceed differently and use the variant of Trotter–Kato approximation theorem given in Theorem 2.8.

Let
$$E_\lambda(s) := \mathrm{diag}\left(\mathrm{e}^{(\lambda/c_j)s}\right), \quad s \in [0,1], \quad \text{and} \quad \mathbb{B}_\lambda^C := E_\lambda(-1)\mathbb{B}^C.$$

By using this notation one can write an explicit expression for the resolvent of operator A_C defined in (3.6).

Lemma 3.12. [8, Prop. 18.12] *For $\mathrm{Re}(\lambda) > 0$ the resolvent $R(\lambda, A_C)$ of operator A_C given in* (3.6) *equals*

$$R(\lambda, A_C) = \left(\mathrm{I}_X + E_\lambda(\cdot)\left(1 - \mathbb{B}_\lambda^C\right)^{-1}\mathbb{B}_\lambda^C \otimes \delta_0\right)R_\lambda,$$

where δ_0 denotes the point evaluation at 0 and

$$\left(R_\lambda f\right)(s) = \int_s^1 E_\lambda(s-t)C^{-1}f(t)\,\mathrm{d}t, \quad s \in [0,1], \ f \in \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right).$$

Theorem 3.13. *The operator $(A_C, \mathrm{D}(A_C))$, defined in* (3.6), *generates a bi-continuous semigroup $(T_C(t))_{t\geq 0}$ on $X = \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right)$.*

Proof. We first show that operator A_C is bi-densely defined. Take any $f \in \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right)$. For $n \in \mathbb{N}$ let $\Omega_n := \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subseteq [0,1]$ and define $f_n\colon [0,1] \to \mathbb{C}^m$ by a linear truncation of f outside Ω_n, i.e.,

$$f_n(x) := \begin{cases} nf\left(\frac{1}{n}\right)x, & x \in \left[0, \frac{1}{n}\right], \\ f(x), & x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ nf\left(1 - \frac{1}{n}\right)(1-x), & x \in \left[1 - \frac{1}{n}, 1\right]. \end{cases}$$

Observe that f_n is Lipschitz for each $n \in \mathbb{N}$ and hence $f_n \in \mathrm{W}^{1,\infty}\left([0,1],\mathbb{C}^m\right)$. Moreover $f_n(1) = f_n(0) = 0$ for each $n \in \mathbb{N}$ implying that $f_n(0) = \mathbb{B}^C f_n(0)$, hence $f_n \in \mathrm{D}(A_C)$. Furthermore one has that $\sup_{n\in\mathbb{N}} \|f_n\| \leq \|f\| < \infty$ and $f_n \to f$ as $n \to \infty$ with respect to the weak*-topology since

$$\left|\int_0^1 \langle(f_n(x) - f(x)), g(x)\rangle_X \, \mathrm{d}x\right| \leq 2\|f\|_\infty \lambda^1\left(\left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right]\right)\|g\|_1 = \frac{4}{n}\|f\|_\infty \|g\|_1$$

for each $g \in \mathrm{L}^1\left([0,1],\mathbb{C}^m\right)$.

We now define a sequence of operators A_n approximating A_C in the following way. For each $c_j \in \mathbb{R}$ there exists a sequence $\left(c_j^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ such that $\lim_{n\to\infty} c_j^{(n)} = c_j$. Since the network is finite, for

each $n \in \mathbb{N}$ the velocities $c_j^{(n)}$, $j \in J$, satisfy condition (3.7) and, by Proposition 3.11 we obtain contraction bi-continuous semigroup $(T_n(t))_{t \geq 0}$ generated by

$$A_n := \mathrm{diag}\left(c_j^{(n)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right),$$

(3.9)

$$\mathrm{D}(A_n) := \left\{f \in \mathrm{W}^{1,\infty}\left([0,1],\mathbb{C}^m\right) \mid f(1) = \mathbb{B}^{C_n}f(0)\right\},$$

where $C_n := \mathrm{diag}\left(c_j^{(n)}\right)$. Moreover, all semigroups $(T_n(t))_{t \geq 0}$, $n \in \mathbb{N}$, are similar and thus uniformly bi-continuous of type 0.

Observe, that the general assumptions of Theorem 2.8 are satisfied. Let us now check the assumptions of assertion (b). Let $R := R(\lambda, A_C)$ and observe that $R : \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right) \to \mathrm{D}(A_C)$ is a bijection. By above, $\mathrm{Ran}(R)$ is bi-dense in $\mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right)$. For every $n \in \mathbb{N}$, replacing c_j by $c_j^{(n)}$ for all $j \in J$, Lemma (3.12) yields an explicit expression for $R(\lambda, A_n)$. It is easy to see that $R(\lambda, A_n)f \overset{\|\cdot\|}{\to} Rf$ for $f \in D(A_C)$ as $n \to \infty$. Applying Theorem 2.8 gives us a bi-continuous semigroup $(T_C(t))_{t \geq 0}$ with generator $(B, \mathrm{D}(B))$. Note that, since in our case $R = R(\lambda, A_C)$ is a resolvent, by Remark 2.9 we have $R = R(\lambda, A_C) = R(\lambda, B)$ for $\lambda \in \rho(A_C)$ and by the uniqueness of the Laplace transform we conclude that $(B, \mathrm{D}(B)) = (A_C, \mathrm{D}(A_C))$. $\qquad\square$

Corollary 3. *The flow problem* (3.2) *is well-posed on* $X = \mathrm{L}^\infty\left([0,1],\mathbb{C}^m\right)$.

Remark 3.14. Observe that in the same manner, by using the original strongly continuous version of the Trotter–Kato Theorem (see [16, Sect. III.4b]), one can deduce the well-posedness of the problem on $X = \mathrm{L}^1\left([0,1],\mathbb{C}^m\right)$.

Acknowledgdement

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