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Abstract: In this article we provide a sufficient condition for a continuous-state branching process with immigration (CBI process) to not hit its boundary, i.e. for non-extinction. Our result applies to arbitrary dimension $d \geq 1$ and is formulated in terms of an integrability condition for its immigration and branching mechanisms F and R. The proof is based on a suitable comparison with one-dimensional CBI processes and an existing result for one-dimensional CBI processes. The same technique is also used to provide a sufficient condition for transience of multi-type CBI processes.

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1 Introduction

Continuous-state branching processes with immigration (shorted as CBI processes) form a class of time-homogeneous Markov processes with state space

$$\mathbb{R}_+^d = \{x \in \mathbb{R}^d \mid x_1, \dots, x_d \geq 0\}, \quad d \in \mathbb{N},$$

whose Laplace transform is an exponentially affine function of the initial state variable, i.e., CBI processes are affine processes in the sense of [DFS03, Definition 2.6]. They have been first studied in dimension $d = 1$ in [Fel51], [Lam67] and [SW73], where it was shown that they arise as scaling limits of Galton-Watson branching processes. For an introduction to such type of processes in

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arbitrary dimension we refer to [Kyp06], [Par16] and [Li11], where superprocesses were also discussed. Although these processes are initially used to describe populations of multiple spices, they have also various applications in mathematical finance, see, e.g., [Alf15] and [DFS03] and the references therein. At this point we would like to mention only some recent results on the long-time behavior of CBI processes. Namely, convergence of supercritical CBI processes was recently studied in [BPP18a] and [BPP18b] while convergence in the total variation distance for affine processes on convex cones (including subcritical CBI processes) was recently studied in [MSV18]. Results applicable to the class of affine processes on the canonical state space $\mathbb{R}_+^d \times \mathbb{R}^n$ were obtained in [FJR18c], [GZ18] and [JKR18].

Let us describe CBI processes in more detail.

Definition 1.1. *The tuple (c, β, B, ν, μ) is called admissible if*

(i) $c = (c_1, \dots, c_d) \in \mathbb{R}_+^d$.

(ii) $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}_+^d$.

(iii) $B = (b_{kj})_{k,j \in \{1,\dots,d\}}$ *is such that, for* $k, j \in \{1, \dots, d\}$ *with* $k \neq j$, *one has*

$$b_{kj} - \int_{\mathbb{R}_+^d} z_k \mu_j(dz) \geq 0.$$

(iv) ν *is a Borel measure on* $\mathbb{R}_+^d$ *satisfying* $\int_{\mathbb{R}_+^d} (1 \wedge |z|) \nu(dz) < \infty$ *and* $\nu(\{0\}) = 0$.

(vi) $\mu = (\mu_1, \dots, \mu_d)$, *where, for each* $j \in \{1, \dots, d\}$, μ_j *is a Borel measure on* $\mathbb{R}_+^d$ *satisfying*

$$\int_{\mathbb{R}_+^d} \left(|z| \wedge |z|^2 + \sum_{k \in \{1,\dots,d\} \setminus \{j\}} z_k \right) \mu_j(dz) < \infty, \quad \mu_j(\{0\}) = 0. \tag{1.1}$$

Note that this definition is a special case of [DFS03, Definition 2.6]. Here we consider the state space $\mathbb{R}_+^d$, exclude killing and require the measures $\mu_1, \dots, \mu_d$ to satisfy the additional integrability condition $\sum_{j=1}^d \int_{|z|>1} |z| \mu_j(dz) < \infty$, see also [BLP15, Remark 2.3] for additional comments. These conditions together imply that the multi-type CBI process introduced below is conservative.

Let (c, β, B, ν, μ) be admissible parameters. It was shown in [DFS03, Theorem 2.7] (see also [BLP15, Remark 2.5]), that there exists a unique conservative Feller transition semigroup $(P_t)_{t \geq 0}$ acting on the Banach space of continuous functions vanishing at infinity with state space $\mathbb{R}_+^d$ such that its generator has core $C_c^\infty(\mathbb{R}_+^d)$ and is, for $f \in C_c^2(\mathbb{R}_+^d)$, given by

$$(Lf)(x) = \sum_{j=1}^d c_j x_j \frac{\partial^2 f(x)}{\partial x_j^2} + \langle \beta + Bx, (\nabla f)(x) \rangle + \int_{\mathbb{R}_+^d} (f(x+z) - f(x)) \nu(dz) \tag{1.2}$$

$$+ \sum_{j=1}^d x_j \int_{\mathbb{R}_+^d} \left(f(x+z) - f(x) - \langle z, (\nabla f)(x) \rangle \right) \mu_j(dz), \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^d$. The corresponding Markov process with generator L is called multi-type CBI process. Moreover, the Laplace transform of its transition kernel $P_t(x, dy)$ has representation

$$\int\limits_{\mathbb{R}_+^d} e^{-\langle \xi, y \rangle} P_t(x, dy) = \exp\left(-\langle x, v(t, \xi) \rangle - \int\limits_0^t F(v(s, \xi)) ds \right), \quad x, \xi \in \mathbb{R}_+^d, \quad t \geq 0,$$

where, for any $\xi \in \mathbb{R}_+^d$, the continuously differentiable function $t \longmapsto v(t, \xi) \in \mathbb{R}_+^d$ is the unique locally bounded solution to the system of differential equations

$$\frac{\partial v(t, \xi)}{\partial t} = -R(v(t, \xi)), \quad v(0, \xi) = \xi. \tag{1.4}$$

Here F and R are of Lévy-Khinchine form

$$F(\xi) = \langle \beta, \xi \rangle + \int\limits_{\mathbb{R}_+^d} \left(1 - e^{-\langle \xi, z \rangle} \right) \nu(dz),$$

$$R_j(\xi) = c_j \xi_j^2 - \langle B e_j, \xi \rangle + \int\limits_{\mathbb{R}_+^d} \left(e^{-\langle \xi, z \rangle} - 1 + \langle \xi, z \rangle \right) \mu_j(dz), \qquad j \in \{1, \ldots, d\},$$

and $e_1, \ldots, e_d$ denote the canonical basis vectors in $\mathbb{R}^d$. Most of the results obtained for multi-type CBI processes are based on a detailed study of the generalized Riccati equation (1.4), where F and R are called the immigration and branching mechanisms, respectively.

The possibility to describe a multi-type CBI process as a strong solution to a stochastic differential equation was studied in [BLP15]. Below we provide such a pathwise description. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Consider the following objects defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$:

(A1) A d-dimensional $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $W = (W(t))_{t \geq 0}$.

(A2) $(\mathcal{F}_t)_{t \geq 0}$-Poisson random measures $N_1, \ldots, N_d$ on $\mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+$ with compensators

$$\widehat{N}_j(ds, dz, dr) = ds \mu_j(dz) dr, \qquad j \in \{1, \ldots, d\}.$$

(A3) A $(\mathcal{F}_t)_{t \geq 0}$-Poisson random measure N_ν on $\mathbb{R}_+ \times \mathbb{R}_+^d$ with compensator $\widehat{N}_\nu(ds, dz) = ds \nu(dz)$.

The objects $W, N_\nu, N_1, \ldots, N_d$ are supposed to be mutually independent. Denote by $\widetilde{N}_j = N_j - \widehat{N}_j$, $j \in \{1, \ldots, d\}$, and $\widetilde{N}_\nu = N_\nu - \widehat{N}_\nu$ the corresponding compensated Poisson random measures. Then it was shown in [BLP15, Theorem 4.6] that, for each $x \in \mathbb{R}_+^d$ there exists a unique $\mathbb{R}_+^d$-valued strong solution to

$$X(t) = x + \int_0^t (\beta + BX(s))\, ds + \sum_{k=1}^d \sqrt{2c_k} e_k \int_0^t \sqrt{X_k(s)} dW_k(s) + \int_0^t \int_{\mathbb{R}_+^d} z N_\nu(ds, dz) \tag{1.5}$$

$$+ \sum_{j=1}^d \int_0^t \int_{|z|\le 1} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \le X_j(s-)\}} \widetilde{N}_j(ds, dz, dr)$$

$$+ \sum_{j=1}^d \int_0^t \int_{|z|>1} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \le X_j(s-)\}} N_j(ds, dz, dr) - \sum_{j=1}^d \int_0^t \left(\int_{|z|>1} z \mu_j(dz) \right) X_j(s) ds.$$

An application of the Itô-formula shows that X solves the martingale problem with generator (1.2), i.e., X is a multi-type CBI process. Conversely, the law of a multi-type CBI process can be obtained from (1.5), see [BLP15] for additional details.

Smoothness of transition probabilities for one-dimensional CBI processes was recently studied in [CLP18], where very precise results have been obtained. In [FJR18a] (see also [FMS13] for related results) we have studied existence of transition densities for multi-type CBI processes. It was shown that, under appropriate conditions, such a density exists on the interior of its state space, i.e. on $\Gamma = \{x \in \mathbb{R}_+^d \mid x_1, \dots, x_d > 0\}$. In this work we provide conditions under which the corresponding multi-type CBI process is supported on Γ, i.e. $\mathbb{P}[X(t) \in \Gamma, \quad t \ge 0] = 1$. Such property simply states that the population described by X does not get extinct. As a consequence, it has, under the conditions of [FJR18a] and those presented in this work, a density on the whole state space $\mathbb{R}_+^d$.

The study of boundary behavior, recurrence and transience for CBI processes has, in dimension $d = 1$, a long history where we would like to mention the works [Gre74] and [FFS85]. More recent works, still in dimension $d = 1$, include [CPGUB13], [DFM14], [FUB14a], and [FUB14b]. Based on these results we provide sufficient conditions for non-extinction and transience of multi-type CBI processes applicable in arbitrary dimension $d \ge 1$.

This work is organized as follows. In Section 2 we state and discuss the main results of this work. These results are then proved in Section 3, while some technical computations are given in the appendix.

2 Statement of the results

Here and below we denote by X a multi-type CBI process with admissible parameters (c, β, B, ν, μ) obtained from (1.5). We start with the simple case where one component of the multi-type CBI process has bounded variation.

Proposition 2.1. *Suppose that there exists $k \in \{1, \dots, d\}$ such that*

$$c_k = 0 \qquad \text{and} \qquad \int_{|z|\le 1} z_k \mu_k(dz) < \infty. \tag{2.1}$$

Then X_k has bounded variation and

$$X_k(t) \geq \begin{cases} e^{\theta_k t} x_k + \beta_k \frac{e^{\theta_k t} - 1}{\theta_k}, & \text{if } \theta_k \neq 0 \\ x_k + \beta_k t, & \text{if } \theta_k = 0 \end{cases}, \qquad t \geq 0, \tag{2.2}$$

where $\theta_k = b_{kk} - \int_{\mathbb{R}^d_+} z_k \mu_k(dz) \in \mathbb{R}$.

The proof of this result is given in the appendix. From this we easily obtain the following corollary.

Corollary 2.2. *Let $k \in \{1, \dots, d\}$ and suppose that* (2.1) *holds. If either $x_k > 0$ or $\beta_k > 0$, then $\mathbb{P}[X_k(t) > 0, \quad t \geq 0] = 1$.*

The next proposition gives a multi-dimensional analogue of this result. For $x, y \in \mathbb{R}^d$ we will write $x \leq y$ to mean that $x_i \leq y_i$ for all $i = 1, \dots, d$.

Proposition 2.3. *Suppose that* (2.1) *holds for all $k \in \{1, \dots, d\}$. Then X has bounded variation and it holds that*

$$X(t) \geq e^{tG} x + \int_0^t e^{sG} \beta ds, \tag{2.3}$$

where $G = (g_{kj})_{k,j \in \{1,\dots,d\}}$ is given by

$$g_{kj} = b_{kj} - \int_{\mathbb{R}^d_+} z_k \mu_j(dz), \qquad k, j \in \{1, \dots, d\}. \tag{2.4}$$

The proof of this statement is given in the appendix. In view of this estimate we restrict our further analysis to the case where (2.1) does not hold, i.e., the process has unbounded variation. In this case we define, for $k \in \{1, \dots, d\}$, the projected immigration and branching mechanisms $F^{(k)}, R^{(k)} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F^{(k)}(\xi) = \beta_k \xi + \int_{\mathbb{R}^d_+} \left(1 - e^{-\xi z_k}\right) \nu(dz),$$

$$R^{(k)}(\xi) = -b_{kk}\xi + c_k \xi^2 + \int_{\mathbb{R}^d_+} \left(e^{-\xi z_k} - 1 + \xi z_k\right) \mu_k(dz).$$

Then we obtain the following result.

Theorem 2.4. *Suppose that there exists $k \in \{1, \dots, d\}$ and $\kappa > 0$ such that $R^{(k)}(\xi) > 0$ for $\xi \geq \kappa$. If $c_k > 0$ or $\int_{|z| \leq 1} z_k \mu_k(dz) = \infty$, and it holds that*

$$\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right) \frac{1}{R^{(k)}(\xi)} d\xi = \infty, \tag{2.5}$$

then $\mathbb{P}[X_k(t) > 0, \quad t \geq 0] = 1$, provided $x_k > 0$.

From this we directly deduce the following corollary.

Corollary 2.5. *If for each $k \in \{1,\dots,d\}$ the conditions of Theorem 2.4 are satisfied, then $\mathbb{P}[X(t) \in \Gamma, \quad t \geq 0] = 1$, provided $x \in \Gamma = \{x \in \mathbb{R}_+^d \mid x_1,\dots,x_d > 0\}$.*

We close this subsection with a sufficient condition for (2.5).

Remark 2.6. *Suppose that for some $k \in \{1,\dots,d\}$ the following conditions are satisfied:*

(i) There exists $M_0 > 0$ such that $R^{(k)}(\xi) > 0$ for $\xi \geq M_0$.

(ii) There exists $\gamma_k \in (0,1]$ and $M_1, C_1 > 0$ such that $F^{(k)}(\xi) \geq C_1 \xi^{\gamma_k}$ for $\xi \geq M_1$.

(iii) There exist $\alpha_k \in (1,2]$ and $M_2, C_2 > 0$ such that $R^{(k)}(\xi) \leq C_2 \xi^{\alpha_k}$ for $\xi \geq M_2$.

Then (2.5) is satisfied, provided one of the following conditions holds:

(a) $\alpha_k \in (1, 1+\gamma_k)$.

(b) $\alpha_k = 1 + \gamma_k$ and $\gamma_k \leq \frac{C_1}{C_2}$.

The proof of this remark is given in the appendix. Note that, if $\beta_k > 0$, then $F^{(k)}(\xi) \geq \beta_k \xi$ and hence $\gamma_k = 1$. However, this corollary also applies in the particular case where $\beta_1 = \cdots = \beta_d = 0$.

Finally we close our considerations with one sufficient condition for transience.

Theorem 2.7. *Let $k \in \{1,\dots,k\}$ and suppose that $R^{(k)}(\xi) > 0$ holds for all $\xi > 0$. Then $\mathbb{P}[\lim_{t\to\infty} X_k(t) = \infty] = 1$, provided one of the following conditions is satisfied:*

(a) $b_{kk} > 0$.

(b) $b_{kk} \leq 0$ and

$$\int_0^1 \exp\left(-\int_\xi^1 \frac{F^{(k)}(u)}{R^{(k)}(u)} du \right) \frac{d\xi}{R^{(k)}(\xi)} < \infty. \tag{2.6}$$

From this we easily conclude that, if the assumptions of Theorem 2.7 hold for each $k \in \{1,\dots,d\}$, then X is transient.

Let us close this section with one particlar example. The multi-type CBI process X with admissible parameters $(c = 0, \beta, B, \nu, \mu)$, where $\mu = (\mu_1,\dots,\mu_d)$ are, for $\alpha_1,\dots,\alpha_d \in (1,2)$, given by

$$\mu_j(dz) = \mathbb{1}_{\mathbb{R}_+}(z_j) \frac{dz_j}{z_j^{1+\alpha_j}} \otimes \prod_{k \neq j} \delta_0(dz_k), \tag{2.7}$$

is called d-dimensional anisotropic $(\alpha_1,\dots,\alpha_d)$-root process.

Theorem 2.8. *Let X be the anisotropic $(\alpha_1,\dots,\alpha_d)$-root process starting from $x \in \mathbb{R}_+^d$. Fix $k \in \{1,\dots,d\}$.*

(a) Suppose that there exist $C, M > 0$ and $\gamma_k \in (0,1]$ such that

$$\beta_k \xi + \int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \nu(dz) \geq C\xi^{\gamma_k}, \qquad \xi \geq M. \tag{2.8}$$

If $x_k > 0$ and $\alpha_k \in (1, 1+\gamma_k)$, then $\mathbb{P}[X_k(t) > 0, \quad t \geq 0] = 1$.

(b) If $b_{kk} > 0$, then $\mathbb{P}[\lim_{t\to\infty} X_k(t) = \infty] = 1$.

Proof. Assertion (b) follows immediately from Theorem 2.7 (a). Let us prove assertion (a). Since $\alpha_1, \dots, \alpha_d \in (1,2)$, it follows that X has unbounded variation. Hence it suffices to show that Theorem 2.4 is applicable. First observe that

$$F^{(k)}(\xi) = \beta_k \xi + \int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \nu(dz),$$

$$R^{(k)}(\xi) = -b_{kk}\xi + \int_0^\infty \left(e^{-\xi z} - 1 + \xi z\right) \frac{dz}{z^{1+\alpha_k}} = -b_{kk}\xi + K\xi^{\alpha_k},$$

where $K = \int_0^\infty \left(e^{-w} - 1 + w\right) \frac{dw}{w^{1+\alpha_k}} > 0$. Next it is easily seen that

$$R^{(k)}(\xi) > 0, \quad \text{whenever } \xi > \left(\frac{\max\{0, b_{kk}\}}{K}\right)^{\frac{1}{\alpha_k - 1}}.$$

Moreover, one finds $R^{(k)}(\xi) \leq (|b_{kk}| + K)\,\xi^{\alpha_k}$ for $\xi \geq 1$, and hence the assertion follows from Remark 2.6 since $\alpha_k \in (1, 1+\gamma_k)$. □

In Remark 2.6, if $\beta_k > 0$, then we may take $\gamma_k = 1$ so that (2.8) is satisfied. However, if $\beta_k = 0$, then (2.8) may be still satisfied as it is shown in the following example.

Example 2.9. *Let $\gamma \in (0,1)$ and set $\nu(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z)\frac{dz}{|z|^{d+\gamma}}$. Then $\int_{\mathbb{R}_+^d}(1 \wedge |z|)\nu(dz) < \infty$ and*

$$\int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \frac{dz}{|z|^{d+\gamma}} = \xi^\gamma \int_{\mathbb{R}_+^d} \left(1 - e^{-w_k}\right) \frac{dw}{|w|^{d+\gamma}}.$$

So (2.8) holds for $\gamma_k = \gamma$. Hence the assumptions of Theorem 2.8 (a) are satisfied, if $\alpha_k \in (1, 1+\gamma)$.

It is worthwhile to mention that there exists a large class of measures which satisfy (2.8) but are not of the form $\nu(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z)\frac{dz}{|z|^{d+\gamma}}$, see, e.g., [KS17], [FJR18a] and [FJR18b].

3 Proofs of main results

3.1 Construction of auxilliary CBI process

Let (c, β, B, ν, μ) be admissible parameters and set

$$\widetilde{b}_{kj} = b_{kj} - \int\limits_{|z|>1} z_k \mu_j(dz) - \mathbb{1}_{\{k \neq j\}} \int\limits_{|z|\leq 1} z_k \mu_j(dz). \tag{3.1}$$

Let $(W, N_\nu, N_1, \ldots, N_d)$ be given as in (A1) – (A3) and consider a process $Y = (Y_1, \ldots, Y_d)$ satisfying, for each $k = 1, \ldots, d$, the stochastic equation

$$\begin{aligned} Y_k(t) = y_k &+ \int\limits_0^t \left(\beta_k + \widetilde{b}_{kk} Y_k(s) \right) ds + \sqrt{2c_k} \int\limits_0^t \sqrt{Y_k(s)} dW_k(s) + \int\limits_0^t \int\limits_{\mathbb{R}_+^d} z_k N_\nu(ds, dz) \\ &+ \int\limits_0^t \int\limits_{|z|\leq 1} \int\limits_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq Y_k(s-)\}} \widetilde{N}_k(ds, dz, dr) + \int\limits_0^t \int\limits_{|z|>1} \int\limits_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq Y_k(s-)\}} N_k(ds, dz, dr), \end{aligned} \tag{3.2}$$

where $y = (y_1, \ldots, y_d) \in \mathbb{R}_+^d$. Finally, define projection mappings $\mathrm{pr}_j : \mathbb{R}_+^d \longrightarrow \mathbb{R}_+$, $\mathrm{pr}_j(z) = z_j$, $j \in \{1, \ldots, d\}$. The next lemma states that the system of equations (3.2) has a unique strong solution which describes a CBI process.

Proposition 3.1. *Let (c, β, B, ν, μ) be admissible parameters and let $(W, N_\nu, N_1, \ldots, N_d)$ be given as in (A1) – (A3). Then the following hold:*

(a) For each $y \in \mathbb{R}_+^d$, there exists a unique $\mathbb{R}_+^d$-valued strong solution Y to (3.2)*.*

(b) For each $j \in \{1, \ldots, d\}$, Y_j is a one-dimensional CBI process with admissible parameters $(c_j, \beta_j, b_{jj}, \widetilde{\nu}_j, \widetilde{\mu}_j)$, where $\widetilde{\nu}_j = \nu \circ \mathrm{pr}_j^{-1}$, $\widetilde{\mu}_j = \mu_j \circ \mathrm{pr}_j^{-1}$.

Proof. Define random measures $M_1(ds, dz, dr), \ldots, M_d(ds, dz, dr)$ on $\mathbb{R}_+^3$ by

$$M_k((a, b] \times A \times B) = N_k((a, b] \times \mathrm{pr}_k^{-1}(A) \times B), \qquad k \in \{1, \ldots, d\},$$

and $N_{\widetilde{\nu}_1}(ds, dz), \ldots, N_{\widetilde{\nu}_d}(ds, dz)$ on $\mathbb{R}_+^2$ by

$$N_{\widetilde{\nu}_k}((a, b] \times A) = N_\nu((a, b] \times \mathrm{pr}_k^{-1}(A)), \qquad k \in \{1, \ldots, d\},$$

where $a < b$, $A, B \in \mathcal{B}(\mathbb{R}_+)$. Then $M_1, \ldots, M_d$ and $N_{\widetilde{\nu}_1}, \ldots, N_{\widetilde{\nu}_d}$ are Poisson random measures with compensators

$$\widehat{M}_k(ds, dz, dr) = ds \widetilde{\mu}_k(dz) dr \quad \text{and} \quad \widehat{N}_{\widetilde{\nu}_k}(ds, dz) = ds \widetilde{\nu}_k(dz), \quad k \in \{1, \ldots, d\}.$$

Moreover, $M_k, N_{\widetilde{\nu}_k}, W_k$ are mutually independent. Let $\widetilde{M}_k(ds,dz,dr) = M_k(ds,dz,dr) - \widehat{M}_k(ds,dz,dr)$ be the corresponding compensated Poisson random measures. Then (3.2) takes the form

$$Y_k(t) = y_k + \int_0^t \left(\beta_k + \widetilde{b}_{kk} Y_k(s)\right) ds + \sqrt{2c_k} \int_0^t \sqrt{Y_k(s)} dW_k(s) + \int_0^t \int_{\mathbb{R}_+} z N_{\widetilde{\nu}_k}(ds, dz)$$
$$+ \int_0^t \int_{(0,1]} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \le Y_k(s-)\}} \widetilde{M}_k(ds, dz, dr) + \int_0^t \int_{(1,\infty)} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \le Y_k(s-)\}} M_k(ds, dz, dr).$$

This equation is now a particular case of (1.5) for dimension $d = 1$, i.e., it has a unique $\mathbb{R}_+$-valued solution which is a CBI process with admissible parameters $(c_k, \beta_k, \widetilde{b}_{kk}, \widetilde{\nu}_k, \widetilde{\mu}_k)$, see also [FL10] for related results. □

We close this section with the observation that Y obtained from (3.2) is actually a CBI process on $\mathbb{R}_+^d$.

Remark 3.2. *Let (c, β, B, ν, μ) be admissible parameters, let $(W, N_\nu, N_1, \dots, N_d)$ be given as in (A1) – (A3), and let Y be the unique solution to* (3.2)*. Then Y is a multi-type CBI process with admissible parameters $(c, \beta, B^Y, \nu, \mu^Y)$, where $B^Y = \operatorname{diag}(b_{11}, \dots, b_{dd})$ and $\mu^Y = (\mu_1^Y, \dots, \mu_d^Y)$ with $\mu_j^Y(dz) = \widetilde{\mu}_j(dz_k) \otimes \prod_{k \neq j} \delta_0(dz_k)$, $j = 1, \dots, d$.*

Since we do not use this result later on, we only sketch the main idea of proof. In view of [BLP15] it suffices to show that the Markov generator of Y takes the desired form. However, this can be shown by direct computation using Itô's formula.

3.2 Comparison with auxiliary CBI process

The next statement is the key estimate for this work.

Proposition 3.3. *Let (c, β, B, ν, μ) be admissible parameters. Consider $(W, N_\nu, N_1, \dots, N_d)$ as in (A1) – (A3), and let X be the multi-type CBI process obtained from* (1.5)*. Let Y be the unique strong solution to* (3.2) *with $y = x$. Then*

$$\mathbb{P}[X_k(t) \ge Y_k(t), \quad t \ge 0] = 1, \qquad k \in \{1, \dots, d\}.$$

Proof. Our proof is based on the method developed in [BLP15, Lemma 4.1]. Define $\Delta_k(t) := Y_k(t) - X_k(t)$ and $\delta_k(r, s-) = \mathbb{1}_{\{r \le Y_k(s-)\}} - \mathbb{1}_{\{r \le X_k(s-)\}}$. Then $\Delta_k(0) = 0$ and we obtain, for each $k \in \{1, \dots, d\}$,

$$\begin{aligned}\Delta_k(t) &= \int_0^t \left(\widetilde{b}_{kk}\Delta_k(s) - \sum_{j\neq k} \widetilde{b}_{kj} X_j(s) \right) ds + \sqrt{2c_k} \int_0^t \left(\sqrt{Y_k(s)} - \sqrt{X_k(s)} \right) dW_k(s) \\ &+ \int_0^t \int_{|z|\leq 1} \int_{\mathbb{R}_+} z_k \delta_k(r, s-) \widetilde{N}_k(ds, dz, dr) + \int_0^t \int_{|z|>1} \int_{\mathbb{R}_+} z_k \delta_k(r, s-) N_k(ds, dz, dr) \\ &- \sum_{j\neq k} \int_0^t \int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq X_j(s-)\}} N_j(ds, dz, dr).\end{aligned}$$

Let $\phi_m : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a sequence of twice continuously differentiable functions with the properties:

(i) $\phi_m(z) \nearrow z_+ := \max\{0, z\}$, as $m \to \infty$ for all $z \in \mathbb{R}$.

(ii) $\phi_m'(z) \in [0, 1]$ for all $m \in \mathbb{N}$ and $z \geq 0$.

(iii) $\phi_m'(z) = \phi_m(z) = 0$ for all $m \in \mathbb{N}$ and $z \leq 0$.

(vi) $\phi_m''(x - y)(\sqrt{x} - \sqrt{y})^2 \leq 2/m$ for all $m \in \mathbb{N}$ and $x, y \geq 0$.

The existence of such a sequence was shown in the proof of [Ma13, Theorem 3.1]. Applying the Itô formula to $\phi_m(\Delta_k(t))$ gives

$$\phi_m(\Delta_k(t)) = \sum_{n=1}^{5} \int_0^t \mathcal{R}^n_{k,m}(s) ds + \mathcal{M}_{k,m}(t), \tag{3.3}$$

where $\mathcal{R}^1_{k,m}, \dots, \mathcal{R}^5_{k,m}$ are given by

$$\begin{aligned}\mathcal{R}^1_{k,m}(s) &= \phi_m'(\Delta_k(s)) \left(\widetilde{b}_{kk}\Delta_k(s) - \sum_{j\neq k} \widetilde{b}_{kj} X_j(s) \right) \\ \mathcal{R}^2_{k,m}(s) &= c_k \phi_m''(\Delta_k(s)) \left(\sqrt{Y_k}(s) - \sqrt{X_k(s)} \right)^2 \\ \mathcal{R}^3_{k,m}(s) &= \int_{|z|\leq 1} \int_{\mathbb{R}_+} \left(\phi_m(\Delta_k(s) + z_k\delta_k(r, s)) - \phi_m(\Delta_k(s)) - z_k\delta_k(r, s)\phi_m'(\Delta_k(s)) \right) dr \mu_k(dz) \\ \mathcal{R}^4_{k,m}(s) &= \int_{|z|>1} \int_{\mathbb{R}_+} \left(\phi_m(\Delta_k(s) + z_k\delta_k(r, s)) - \phi_m(\Delta_k(s)) \right) dr \mu_k(dz) \\ \mathcal{R}^5_{k,m}(s) &= \sum_{j\neq k} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} \left(\phi_m(\Delta_k(s) - z_k \mathbb{1}_{\{r\leq X_j(s)\}}) - \phi_m(\Delta_k(s)) \right) dr \mu_j(dz),\end{aligned}$$

$(\mathcal{M}_{k,m}(t))_{t\geq 0}$ is a local martingale and $\delta_k(r,s) = \mathbb{1}_{\{r\leq Y_k(s)\}} - \mathbb{1}_{\{r\leq X_k(s)\}}$. For $l \in \mathbb{N}$, define the stopping time

$$\tau_l = \inf\{t > 0 \mid \max_{i\in\{1,\dots,d\}} \max\{X_i(t), Y_i(t)\} > l\}.$$

Using the precise form of $\mathcal{M}_{k,m}$ given by Itô's formula combined with similar estimates to [BLP15, Lemma 4.1], one can show that $(\mathcal{M}_{k,m}(t\wedge\tau_l))_{t\geq 0}$ is a martingale for any $l \in \mathbb{N}$. Next we will prove that there exists a constant $C > 0$ such that

$$\sum_{n=1}^{5} \mathcal{R}^n_{k,m}(s) \leq C\Delta_k(s)_+ + \frac{C}{m}. \tag{3.4}$$

Taking then expectations in (3.3), using that $(\mathcal{M}_{k,m}(t\wedge\tau_l))_{t\geq 0}$ is a martingale and estimating as in (3.4), gives

$$\begin{aligned}\mathbb{E}[\phi_m(\Delta_k(t\wedge\tau_l))] &= \sum_{n=1}^{5}\mathbb{E}\left[\int_0^{t\wedge\tau_l}\mathcal{R}^n_{k,m}(s)ds\right] \leq C\mathbb{E}\left[\int_0^{t\wedge\tau_l}\Delta_k(s)_+ds\right] + \frac{C}{m}\mathbb{E}[t\wedge\tau_l]\\ &\leq C\int_0^t \mathbb{E}[\Delta_k(s\wedge\tau_l)_+]ds + \frac{Ct}{m}.\end{aligned}$$

Letting $m \to \infty$ and using property (i) gives

$$\mathbb{E}[\Delta_k(t\wedge\tau_l)_+] \leq C\int_0^t \mathbb{E}[\Delta_k(s\wedge\tau_l)_+]ds.$$

Applying Gronwall lemma shows that, for any $k \in \{1,\dots,d\}$ and $l \in \mathbb{N}$, one has $\mathbb{E}[\Delta_k(t\wedge\tau_l)_+] = 0$. Letting now $l \to \infty$ proves the assertion.

Hence it remains to prove (3.4). In order to estimate $\mathcal{R}^1_{k,m}$ we use properties (ii), (iii), $\widetilde{b}_{kj} \geq 0$ for $k \neq j$ and $X_j(s) \geq 0$ to obtain

$$\mathcal{R}^1_{k,m}(s) = \phi_m'(\Delta_k(s))\widetilde{b}_{kk}\Delta_k(s)_+ - \phi_m'(\Delta_k(s))\sum_{j\neq k}\widetilde{b}_{kj}X_j(s) \leq |\widetilde{b}_{kk}|\Delta_k(s)_+.$$

For $\mathcal{R}^2_{k,m}$ we obtain from (iv) the estimate $\mathcal{R}^2_{k,m}(s) \leq \frac{2c_k}{m}$. Let us now turn to $\mathcal{R}^3_{k,m}$. Using property (iv) we see that, for each $y > 0$, $z \geq 0$ and $m \in \mathbb{N}$, there exists $\vartheta = \vartheta(y,z) \in [0,1]$ such that

$$\phi_m(y+z) - \phi_m(y) - \phi_m'(y)z = \phi_m''(y+\vartheta z)\frac{z^2}{2} \leq \frac{2z^2}{2m(y+\vartheta z)} \leq \frac{z^2}{my}.$$

Next observe that $\delta_k(r,s) > 0$ if and only if $\Delta_k(s) > 0$ and $r \in (X_k(s), Y_k(s)]$. Applying both observations, we obtain

$$\begin{aligned}\mathcal{R}^3_{k,m}(s) &\leq \mathbb{1}_{\{\Delta_k(s)>0\}}\int_{|z|\leq 1}\int_{\mathbb{R}_+}\left(\phi_m(\Delta_k(s) + z_k\delta_k(r,s)) - \phi_m(\Delta_k(s)) - z_k\delta_k(r,s)\phi_m'(\Delta_k(s))\right)dr\mu_k(dz)\\ &\leq \frac{\mathbb{1}_{\{\Delta_k(s)>0\}}}{m\Delta_k(s)}\int_{|z|\leq 1}\int_{\mathbb{R}_+} z_k^2\delta_k(r,s)^2 dr\mu_k(dz) \leq \frac{1}{m}\int_{|z|\leq 1} z_k^2\mu_k(dz),\end{aligned}$$

where we have used $\int_{\mathbb{R}_+} \delta_k(r,s)^2 dr = \Delta_k(s)$ a.s. on $\{\Delta_k(s) > 0\}$. For $\mathcal{R}^4_{k,m}$ we use property (ii), so that

$$\begin{aligned}\mathcal{R}^4_{k,m}(s) &\leq \mathbb{1}_{\{\Delta_k(s)>0\}} \int\limits_{|z|>1} \int\limits_{\mathbb{R}_+} \left(\phi_m(\Delta_k(s) + z_k\delta_k(r,s)) - \phi_m(\Delta_k(s))\right) \mu_k(dz) dr \\ &\leq \mathbb{1}_{\{\Delta_k(s)>0\}} \int\limits_{|z|>1} \int\limits_{\mathbb{R}_+} z_k \delta_k(r,s) \mu_k(dz) dr \leq \Delta_k(s)_+ \int\limits_{|z|>1} z_k \mu_k(dz),\end{aligned}$$

where we have also used $\int_{\mathbb{R}_+} \delta_k(r,s) dr = \Delta_k(s)$. For the last term we use property (ii), so that $\mathcal{R}^5_{k,m}(s) \leq 0$. This proves (3.4) and hence the assertion. $\square$

3.3 Proofs of Theorem 2.4 and Theorem 2.7

We are now prepared to prove our main results of this work. First observe that Proposition 3.3 implies that, for any $k \in \{1, \dots, d\}$,

$$\mathbb{P}[Y_k(t) > 0, \quad t \geq 0] = 1 \ \Rightarrow \ \mathbb{P}[X_k(t) > 0, \quad t \geq 0] = 1,$$

and similarly

$$\mathbb{P}[\lim_{t\to\infty} Y_k(t) = \infty] = 1 \ \Rightarrow \ \mathbb{P}[\lim_{t\to\infty} X_k(t) = \infty] = 1,$$

where X and Y are the unique solutions to (1.5) and (3.2), respectively. In view of Proposition 3.1, Y_k satisfies the conditions of [FUB14a, Corollary 6] or [DFM14, Theorem 2], respectively. Now it is easy to see that the assertions of Theorem 2.4 and Theorem 2.7 are true.

Appendix: Additional proofs

Proof of Proposition 2.1. Observe that under condition (2.1) the process X_k also satisfies

$$\begin{aligned}X_k(t) = x_k &+ \int_0^t \left(\beta_k + \sum_{j=1}^d g_{kj} X_j(s)\right) ds + \int_0^t \int\limits_{\mathbb{R}^d_+} z_k N_\nu(ds,dz) \\ &+ \int_0^t \int\limits_{\mathbb{R}^d_+} \int\limits_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq X_k(s-)\}} N_k(ds,dz,dr) + \sum_{j\neq k} \int_0^t \int\limits_{\mathbb{R}^d_+} \int\limits_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq X_j(s-)\}} N_j(ds,dz,dr),\end{aligned}$$

where g_{kj} is defined in (2.4). This implies that X_k has bounded variation. Let $y(t)$ be the unique solution to $y(t) = x_k + \int_0^t (\beta_k + \theta_k y(s))\, ds$, i.e.,

$$y(t) = \begin{cases} x_k e^{\theta_k t} + \beta_k \frac{e^{\theta_k t}-1}{\theta_k}, & \text{if } \theta_k \neq 0 \\ x_k + \beta_k t, & \text{if } \theta_k = 0 \end{cases}, \qquad t \geq 0.$$

Proceeding exactly as in the proof of Proposition 3.3, we obtain $\mathbb{P}[X_k(t) \geq y(t)] = 1$ for all $t \geq 0$. This proves the assertion. $\square$

Proof of Proposition 2.3. Observe that under (2.1) the process X also satisfies

$$X(t) = x + \int_0^t (\beta + GX(s))\, ds + \int_0^t \int_{\mathbb{R}_+^d} z N_\nu(ds, dz) + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \le X_j(s-)\}} N_j(ds, dz, dr).$$

Let $y(t)$ be the unique solution to $y(t) = x + \int_0^t (\beta + Gy(s))\, ds$ which is given by $y(t) = e^{tG}x + \int_0^t e^{sG}\beta ds$. Proceeding exactly as in the proof of Proposition 3.3, we obtain $\mathbb{P}[X_k(t) \ge y_k(t)] = 1$ for all $t \ge 0$ and $k \in \{1, \dots, d\}$. This proves the assertion. $\square$

Proof of Remark 2.6. Set $\kappa = \max\{M_0, M_1, M_2\}$. If $\alpha_k < 1 + \gamma_k$, then $\frac{F^{(k)}(u)}{R^{(k)}(u)} \ge \frac{C_1}{C_2} u^{\gamma_k - \alpha_k}$, for $u \in [\kappa, \xi]$, and hence

$$\begin{aligned}
\exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du \right) &\ge \exp\left(\frac{C_1}{C_2} \int_\kappa^\xi u^{\gamma_k - \alpha_k} du \right) \\
&= \exp\left(-\frac{C_1}{C_2} \frac{\kappa^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k} \right) \exp\left(\frac{C_1}{C_2} \frac{\xi^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k} \right)
\end{aligned}$$

and

$$\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du \right) \frac{d\xi}{R^{(k)}(\xi)} \ge \frac{\exp\left(-\frac{C_1}{C_2} \frac{\kappa^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k} \right)}{C_2} \int_\kappa^\infty \exp\left(\frac{C_1}{C_2} \frac{\xi^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k} \right) \frac{d\xi}{\xi^{\alpha_k}} = \infty.$$

This proves (2.5) under (a). If $\alpha_k = 1 + \gamma_k$, then we obtain for $\xi \ge \kappa$ and $u \in [\kappa, \xi]$,

$$\exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du \right) \ge \exp\left(\frac{C_1}{C_2} \int_\kappa^\xi u^{\gamma_k - \alpha_k} du \right) = \kappa^{-\frac{C_1}{C_2}} \xi^{\frac{C_1}{C_2}}.$$

Using $\alpha_k \le 1 + \frac{C_1}{C_2}$ gives

$$\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du \right) \frac{d\xi}{R^{(k)}(\xi)} \ge \frac{\kappa^{-\frac{C_1}{C_2}}}{C_2} \int_\kappa^\xi \frac{\xi^{\frac{C_1}{C_2}}}{\xi^{\alpha_k}} d\xi = \infty,$$

and hence proves (2.5) under (b). $\square$

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