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WELL-POSEDNESS OF INFINITE-DIMENSIONAL NON-AUTONOMOUS PASSIVE BOUNDARY CONTROL SYSTEMS

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Abstract. We study a class of non-autonomous boundary control and observation linear systems that are governed by non-autonomous multiplicative perturbations. This class is motivated by different fundamental partial differential equations, such as controlled wave equations and Timoshenko beams. Our main results give sufficient condition for well-posedness, existence and uniqueness of classical and mild solutions.

Key words: infinite-dimensional non-autonomous control system, evolution family, port-Hamiltonian system, well-posedness

MSC: 93C25, 47D06, 93C20.

1. Introduction

We consider the following non-autonomous partial differential equation with boundary input u and boundary output y

$$\begin{aligned}
\frac{\partial}{\partial t}x(t,\zeta) &= \sum_{k=1}^{N} P_k(t)\frac{\partial^k}{\partial \zeta^k}\big[\mathcal{H}(t,\zeta)x(t,\zeta)\big] + P_0(t,\zeta)\mathcal{H}(t,\zeta)x(t,\zeta), \quad t\geq 0,\ \zeta\in(a,b)\\
x(0,\zeta) &= x(\zeta), \qquad \zeta\in(a,b),\\
u(t) &= W_{B,1}\tau(\mathcal{H}x)(t), \qquad t\geq 0,\\
0 &= W_{B,2}\tau(\mathcal{H}x)(t), \qquad t\geq 0,\\
y(t) &= W_C\tau(\mathcal{H}x)(t), \qquad t\geq 0.
\end{aligned}$$

Here τ denotes the *trace operator* $\tau : H^N((a,b);\mathbb{K}^n)\to\mathbb{K}^{2Nn}$ defined by

$$\tau(x) := \big(x(b), x'(b), \cdots, x^{N-1}(b), x(a), x'(a)\cdots, x^{N-1}(a)\big),$$

$P_k(t)$ is $n\times n$ matrix for all $t\geq 0$, $k=0,1,\cdots,N$, $\mathcal{H}(t,\zeta)\in\mathbb{K}^{n\times n}$ for all $t\geq 0$ and almost every $\zeta\in[a,b]$, $W_{B,1}$ is a $m\times 2nN$-matrix, $W_{B,2}$ is $(nN-m)\times 2nN$-matrix and W_C is a $d\times 2nN$-matrix. Finally, $u(t)\in\mathbb{K}^m$ denotes the input and $y(t)\in\mathbb{K}^d$ is the output at time t.

This partial differential equation is also known as port-Hamiltonian systems, and covers the wave equation, the transport equation, beam equations, coupled beam and wave equations as well as certain networks. Autonomous port-Hamiltonian systems, that is when $\mathcal{H}, P_k$ are time-independent, have been intensively investigated, see e.g., [15, 16, 3, 2, 17, 22, 36, 41]. The existence of mild/classical solutions with non-increasing energy and well-posedness for autonomous port-Hamiltonian systems can in most cases be tested via a simple matrix condition [22, Theorem 4.1]. Well-posedness of linear systems in general is not easy to prove and a necessary condition is that the state operator generates a strongly continuous semigroup. For the class of autonomous port-Hamiltonian systems of first order i.e., $N=1$, this condition is even sufficient under some weak assumptions on $P_1\mathcal{H}$, see [22] or [17, Theorem 13.2.2].

In this paper, we aim to generalize these solvability and well-posedness results to the non-autonomous situation. To our knowledge, in contrast to infinite-dimensional autonomous port-Hamiltonian systems,

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the non-autonomous counterpart has not been discussed so far. Motivated by this class we start a systematic study of non-autonomous linear boundary control and observation systems, and in particular those of the following form

$$\dot{x}(t) = \mathfrak{A}(t)M(t)x(t), \quad x(0) = x_0, \ t \geq 0 \tag{1}$$

$$\mathfrak{B}M(t)x(t) = u(t), \tag{2}$$

$$\mathfrak{C}M(t)(t)x(t) = y(t), \tag{3}$$

which we denote by $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. Here $\mathfrak{A}(t) : \mathfrak{D} \subset X \to X$ is a linear operator, $u(t) \in U$, $y(t) \in Y$, the *boundary operators* $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ and $\mathfrak{C} : \mathfrak{D} \subset X \to Y$ are linear such that $\mathfrak{D} \subset D(\mathfrak{B})$, X, U and Y are complex Hilbert spaces and $M(t) \in \mathcal{L}(X)$ for all $t \geq 0$. Setting

$$\mathfrak{A}(t)x = \sum_{k=0}^{N} P_k(t)\frac{\partial^k}{\partial \zeta^k}x, \quad \mathfrak{B}x := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \tau(x), \quad \mathfrak{C}x := W_C\tau(x), \text{ and } \mathcal{H}(t, \cdot) := M(t)$$

we see that the non-autonomous port-Hamiltonian system is in fact a special class of non-autonomous systems of the form (1)-(3).

A pair (x, y) is a *classical solution of* (1)-(3) if $x \in C^1((0, \infty); X) \cap C([0, \infty); X), y \in C([0, \infty); Y)$ and $x(t) \in D(\mathfrak{A}(t)M(t))$ for all $t \geq 0$ such that x, y satisfy (1)-(2) and (3), respectively. The system $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called *well-posed* if for each (classical) solution (x, y) and any *final time* $\tau > 0$, the operator mapping the input functions u and to the initial state x_0 to $x(\tau)$ and the output functions y is bounded, i.e.

$$\|x(\tau)\|^2 + \int_0^\tau \|y(s)\|^2 ds \leq m_\tau\big(\|x_0\|^2 + \int_0^\tau \|u(s)\|^2 ds\big)$$

for some constant $m_\tau > 0$ independent of x_0 and u.

Our approach for the solvability of $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is based on a non-autonomous version of the *Fattorini's trick*, the theory of *evolution families* together with an idea of Schnaubelt and Weiss [29, Section 2].

Evolution families are a generalization of strongly continuous semigroups, and are often used to describe the solution of an *abstract non-autonomous Cauchy problem.* In Section 2, we therefore review the concept of evolution families and that of C^1-well posed non-autonomous Cauchy problems. Furthermore, we provide several abstract results which are crucial for the analysis of our non-autonomous boundary control and observation systems.

Fattorini's trick is well known for autonomous boundary control systems [17, 12, 9]. The basic idea of this approach is to reformulate the state and the control equation into an abstract inhomogeneous Cauchy problem on X. A brief description of the autonomous situation is given in Subsection 3.1. In Subsection 4.1 we provide a generalization to non-autonomous boundary control systems (see Proposition 4.2). This generalization and the results of Section 2 are then used to prove our main classical solvability results: Theorem 4.8 and Theorem 6.5.

The second main purpose of this paper is the study of the well-posedness for non-autonomous boundary and observation systems $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. However, we will restrict ourselves to the case where for every $t \geq 0$ the (unperturbed) autonomous system $\Sigma_{N,\mathrm{id}}(\mathfrak{A}(t), \mathfrak{B}, \mathfrak{C})$ is $(R(t), P(t), J(t))$-*scattering passive* i.e., when

$$2\,\mathrm{Re}(\mathfrak{A}(t)x \,|\, P(t)x)_X \leq (R(t)u \,|\, \mathfrak{B}x)_U - (\mathfrak{C}x \,|\, J(t)\mathfrak{C}(x)_Y$$

for all x in an appropriate subspace of $X \times U$ where $P(t), R(t)$ and $J(t)$ are bounded linear operators. A precise definition and a characterization of scattering passive autonomous and non-autonomous systems is the subject of Subsection 3.2 and Subsection 4.2, respectively. Under additional conditions we then prove in Theorem 4.8 that the perturbed system $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is well-posed. In particular, we deduce in Theorem 6.5 that well-posedness for a large class of non-autonomous port-Hamiltonian systems can be checked via a simples matrix condition.

In the literature most attention has been devoted to autonomous control systems. However, in view of applications, the interest in non-autonomous systems has been rapidly growing in recent years, see e.g., [13, 26, 7, 29, 19, 6, 18, 28] and the references therein. In particular, a class of scattering passive linear non-autonomous linear systems of the form

$$\dot{x}(t) = A_{-1}M(t)x(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{4}$$

$$y(t) = CM(t)x(t) + Du(t) \tag{5}$$

has been studied by R. Schnaubelt and G. Weiss in [29]. Here $(A, D(A))$ generates a strongly continuous semigroup on X, $A_{-1} \in \mathcal{L}(X, X_{-1})$ is a bounded extension of $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(Z, Y)$ and $D \in \mathcal{L}(U, Y)$, where X_{-1} is the extrapolation space corresponding to A, and $Z := D(A) + (\alpha - A)^{-1}BU$ for some $\alpha \in \rho(A)$.
The control part (1)-(2) of the nonautonomous boundary control system $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ can be rewritten in the (standard) abstract formulation (4), however, in the particular case where $\mathfrak{A}(t) = \mathfrak{A}$ is constant which for non-autonomous port-Hamiltonian system correspond to the case where the matrices $P_k, k = 1, \cdots, N$, are constant with respect to time variable. On the other hand, when $\mathfrak{A}(t) = \mathfrak{A}$ the output part (3) could be also written into (5) using the concept of *system nodes*. Indeed, well-posed autonomous port-Hamiltonian system fit into the framework of *compatible system nodes* [37, Theorem 10]. This can be also easily generalized for boundary control and observation systems defined in Definition 3.2. Since we do not follow the approach of [29], this topic will not be discussed in this paper and we refer to [31, 34] for more details on system nodes.

For the general case, that is when $\mathfrak{A}$ is not constant, then A_{-1}, B, C, D and Z will be time dependent. Thus, the abstract results in [29] cannot be immediately applied to deduce classical solvability and well-posedness for (1)-(3). We expect that the results in [29] can be generalized to include this general case. However, for the class of boundary control systems defined in Definition 3.2 we deal directly with (1)-(2) in combination with Fattorini's trick instead of its corresponding system (4)-(5). Our method is indeed much simpler. Moreover, in general it is not clear how the solution of (4)-(5) can be related to that of (1)-(3) even for the special case where $\mathfrak{A}(t) = \mathfrak{A}$ is constant. In the autonomous case this relationship is quite simple as we can see in Section 3. The reason is that C_0-semigroups can be always extended to the extrapolation space. The situation is more delicate for the non-autonomous setting. Indeed, a general extrapolation theory for evolution families is still missing. Moreover, the extrapolation space may also depend on the time variable. In Section 5 we deal with this question by associating a *mild solution* to the control part (1)-(2) of the nonautonomous boundary control system $\Sigma_{N,M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.
Finally, we apply our abstract results to non-autonomous port-Hamiltonian systems, in particular to the time-dependent vibrating string and the time-dependent Timoschenko beam.

2. Background on evolution families and preliminary results

Throughout this section $(X, \|\cdot\|)$ is a Banach space. Let $\mathcal{A} := \{A(t) \,|\, t \geq 0\}$ be a family of linear, closed operators with domains $\{D(A(t)) \,|\, t \geq 0\}$. Consider the *non-autonomous Cauchy problem*

$$\dot{u}(t) = A(t)u(t) \quad \text{on } [s, \infty), \; u(s) = x_s, (s > 0). \tag{6}$$

A continuous function $u : [s, \infty) \to X$ is called a *classical solution* of (6) if $u(t) \in D(A(t))$ for all $t \geq s, u \in C^1((s, \infty), X)$ and u satisfies (6).

Definition 2.1. The non-autonomous Cauchy problem (6) is called C^1-*well posed* if there is a family $\{Y_t \,|\, t \geq 0\}$ of dense subspaces of X such that:

(a) $Y_t \subseteq D(A(t))$ for all $t \geq 0$.
(b) For each $s \geq 0$ and $x_s \in Y_s$ the Cauchy problem (6) has a unique classical solution $u(\cdot, s, x_s)$ with $u(t, s, x_s) \in Y_t$ for all $t \geq s$.
(c) The solutions depend continuously on the initial data s, x_s.

If we want to specify the *regularity subspaces* Y_t, $t \geq 0$, we also say (6) is C^1-*well posed* on Y_t.

In the autonomous case, i.e., if $A(t) = A$ is constant, then it is well known that the associated Cauchy problem is well-posed if and only if A generates a C_0-semigroup $(T(t))_{t\geq 0}$. In this case, for each $x \in D(A)$ the unique classical solution is given by $T(\cdot)x$. The following definition provides a natural generalization of operator semigroups for non-autonomous evolution equations.

Definition 2.2. A family $\mathcal{U} := \{U(t,s) \,|\, (t,s) \in \Delta\} \subset \mathcal{L}(X)$ where $\Delta := \{t, s \geq 0 \,|\, t \geq s\}$ is called *an evolution family* if:

(*i*) $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for every $0 \leq s \leq r \leq t$,
(*ii*) $U(\cdot,\cdot) : \Delta \to \mathcal{L}(X)$ is strongly continuous.

The evolution family $\mathcal{U}$ is said to be generated by $\mathcal{A}$, if there is a family $\{Y_t \,|\, t \geq 0\}$ of dense subspaces of X with $Y_t \subset D(A(t))$ and

(*iii*) For every $x_s \in Y_s$, the function $t \to U(t,s)x_s$ is the unique classical solution of (6).

The Cauchy problem (6) is then C^1-well posed if and only if $A(t)$, $t \geq 0$, generates a unique evolution family, see [10, Proposition 9.3] or [24, Proposition 3.10]. Clearly, if $(T(t))_{t\geq 0}$ is a C_0-semigroup in X with generator $(A, D(A))$, then $U(t,s) := T(t-s)$ yields an evolution family on X with regularity spaces $Y_t = D(A)$.

2.1. **Similar evolution families.** Let $\mathcal{U} := \{U(t,s) \,|\, (t,s) \in \Delta\}$ be an evolution family on X and let $\{Q(t) \,|\, t \geq 0\} \subset \mathcal{L}(X)$ be a family of isomorphisms on X such that Q and Q^{-1} are strongly continuous on $[0,\infty)$. Define the two parameters operator family $\mathcal{W} := \{W(t,s) \,|\, (t,s) \in \Delta\}$ by

$$W(t,s) = Q^{-1}(t)U(t,s)Q(s) \quad \text{for } (t,s) \in \Delta. \tag{7}$$

It is well known that if S is a C_0-semigroup on X with generator A and $Q \in \mathcal{L}(X)$ is an isomorphism, then $T(\cdot) := Q^{-1}S(\cdot)Q$ is again a C_0-semigroup on X, called similar C_0-semigroup to S, and its generator is given by $Q^{-1}AQ$, where

$$D(Q^{-1}AQ) = D(AQ) = \{x \in X \,|\, Qx \in D(A)\} = Q^{-1}D(A).$$

The purpose of this section is to generalize the concept of similar semigroups to evolution families.

Lemma 2.3. *The two parameters family $\mathcal{W}$, defined by* (7), *defines an evolution family on X.*

Proof. Clearly, the evolution law (i) in Definition 2.2 is fulfilled. It remains to prove the strong continuity of $\mathcal{W}$ in Δ. Let $x \in X$ and $T > 0$, and set $\Delta_T := \{(t,s) \in [0,T]^2 \,|\, t \geq s\}$. Let (t,s), $(t_n, s_n) \in \Delta_T$ for $n \in \mathbb{N}$ such that $(t_n, s_n) \to (t,s)$. Then $\{Q^{-1}(t_n) \,|\, n \in \mathbb{N}\}$ is bounded by the uniform boundedness theorem. Since

$$\begin{aligned}\|Q^{-1}(t_n)U(t_n,s_n)x - Q^{-1}(t)U(t,s)x\| \leq{}& \|Q^{-1}(t_n)\|\|U(t_n,s_n)x - U(t,s)x\| \\ &+ \|[Q^{-1}(t_n) - Q^{-1}(t)]U(t,s)x\|,\end{aligned}$$

we deduce that $(t,s) \mapsto Q^{-1}(t)U(t,s)x$ is continuous on Δ_T. Thus, using a similar argument for $Q(s)$ and $Q^{-1}(t)U(t,s)$ we obtain that $(t,s) \mapsto W(t,s)x$ is continuous on Δ_T. Since $T > 0$ is arbitrary, this proves the assertion. □

In contrast to semigroups, the evolution law (i) and the strong continuity (ii) do not guarantee that the given evolution family is generated by some family of linear closed operators.

Proposition 2.4. *Assume that $Q(\cdot)$ is in addition strongly C^1-differentiable. Then $\mathcal{U}$ is generated by a family $\mathcal{A}$ with regularity spaces $\{Y_t \,|\, t \geq 0\}$ if and only if $\mathcal{W}$ is generated by $\mathcal{A}_Q := \{Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)\dot{Q}(t) \,|\; t \geq 0\}$ with regularity spaces $\{\tilde{Y}_t \,|\; t \geq 0\}$ where*

$$\tilde{Y}_t := \{x \in X \,|\, Q(t)x \in Y_t\}.$$

Proof. (i) Assume that $\mathcal{U}$ is generated by $\mathcal{A}$ with regularity spaces $\{Y_t \,|\; t \geq 0\}$. We first remark that $\tilde{Y}_t$ is a dense subspace of X and

$$\tilde{Y}_t = Q^{-1}(t)Y_t \subset Q^{-1}(t)D(A(t)) = D(A(t)Q(t)) = D(A_Q(t)) \tag{8}$$

for every $t \geq 0$, where $A_Q(t) := Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)\dot{Q}(t)$. Next, let $x_s \in \tilde{Y}_s$. Then $Q(s)x_s \in Y_s$ and by assumption $U(\cdot,s)Q(s)x_s$ is the unique classical solution of

$$\dot{u}(t) = A(t)u(t) \;\text{ on } [s,\infty),\; u(s) = Q(s)x_s, (s > 0). \tag{9}$$

It follows that $W(t,s)x \in Y_t \subset D(A_Q(t))$ by (8) and

$$\frac{d}{dt}W(t,s)x_s = [\frac{d}{dt}Q(t)^{-1}]U(t,s)Q(s)x_s + Q(t)^{-1}\frac{d}{dt}U(t,s)Q(s)x_s$$

$$= -Q(t)^{-1}\dot{Q}(t)Q(t)^{-1}U(t,s)Q(s)x_s + Q(t)^{-1}A(t)U(t,s)Q(s)x_s \tag{10}$$

$$= \big[Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)\dot{Q}(t)\big]W(t,s)x_s. \tag{11}$$

Since Q is strongly C^1-differentiable, it now follows from (10)-(11) that $W(\cdot,s)x_s \in C^1((s,\infty),X)$ and $W(\cdot,s)x_s$ solves the non-autonomous problem

$$\dot{u}(t) = A_Q(t)u(t) \quad \text{on } [s,\infty), \; u(s) = x_s. \tag{12}$$

Clearly, $W(\cdot,s)x_s$ is the unique classical solution of (12). We conclude that $\mathcal{W}$ is generated by $\{A_Q(t) \mid t \geq 0\}$ with regularity space $\{\tilde{Y}_t \,|\, t \geq 0\}$.
(ii) Conversely, assume that $\mathcal{A}_Q$ generates the evolution family $\mathcal{W}$ with some regularity spaces $\{\tilde{Y}_t \mid t \geq 0\}$. Since Q^{-1} is C^1-strongly continuous we obtain by (i) that the family $(\mathcal{A}_Q)_{Q^{-1}} = \mathcal{A}$ generates the evolution $\mathcal{V}$ defined by

$$V(t,s) := Q(t)W(t,s)Q^{-1}(s) = U(t,s), \quad (t,s) \in \Delta$$

with regularity space $Y_t = Q(t)\tilde{Y}_t$. This completes the proof. □

If $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup and $B \in \mathcal{L}(X)$, then the perturbed operator $\tilde{A} := A + B$ is again the generator of a C_0-semigroup, see e.g., [10, Section 1.2] or [35]. This perturbation results fails to be true in general for non-autonomous evolution equations [10, Example 9.2]. Thus one cannot conclude from Proposition 2.4 that the family $\{Q^{-1}(t)A(t)Q(t) \,|\, t \geq 0\}$ generates an evolution family. Nevertheless, inspired by an idea of Schnaubelt and Weiss [29], using Proposition 2.4 we show that a positive answer can be given under some additional regularity assumptions.
For this we first need to introduce the following definition.

Definition 2.5. (Kato's class)

(1) A family $\mathcal{A}$ is said to be *Kato-stable* if for each $t \geq 0$ there exists a norm $\|\cdot\|_t$ on X equivalent to the original one such that for each $T \geq 0$ there exists a constant $c_T \geq 0$ with

$$|\|x\|_t - \|x\|_s| \leq c_T|t-s|\|x\|_s, \quad x \in X, t,s \in [0,T] \tag{13}$$

and $A(t)$ generates a contractive C_0-semigroup on $X_t := (X, \|\cdot\|_t)$ for all $t \geq 0$.
(2) A family $\mathcal{A}$ is said to belong to *Kato's class* if it is Kato-stable and the operators $A(t)$, $t \geq 0$, have a common domain D such that $A(\cdot) : [0,\infty) \to \mathcal{L}(D,X)$ is strongly C^1-differentiable.

It is known that Kato-stability is a sufficient condition for C^1-well posedness of (hyperbolic) non-autonomous evolution equations [21, 35, 32]. In particular, each non-autonomous evolution equation that is governed by a Kato-class family is C^1-well posed.

Obviously, $\mathcal{A}$ is Kato-stable if each operator $A(t)$ generates a contractive C_0-semigroup, as one can simply choose $\|\cdot\|_t = \|\cdot\|$, $t \geq 0$. In this case we say that $\mathcal{A}$ belongs to the *elementary Kato class*. Starting from this simple case many less trivial Kato-stable families can be constructed.

Example 2.6. Assume that $(H, \|\cdot\|_H)$ is a Hilbert space. Let $M : [0,\infty) \to \mathcal{L}(H)$ be *self-adjoint and uniformly coercive*, i.e., $M(t)^* = M(t)$ and $(M(t)x|x)_H \geq \beta\|x\|_H^2$ for some constant $\beta > 0$ and all $t \geq 0$. If M is strongly C^1-continuous and M^{-1} is strongly continuous, then for each $t \in [0,\infty)$ the function

$$x \mapsto \|x\|_t := \sqrt{(M(t)x|x)} = \|M^{1/2}(t)x\| \tag{14}$$

defines a norm on H which is equivalent to the norm $\|\cdot\|_H$ and satisfies (13). Moreover, if $\mathcal{A}$ has a common domain D and for each $t \geq 0$ the operator $(A(t), D)$ generates a contraction C_0-semigroup in H, then $(A(t)M(t), D(A(t)M(t))$ and $(M(t)A, D(A(t)))$ generate contractive C_0-semigroups on H_t, and thus both families $\{A(t)M(t) \,|\, t \geq 0\}$ and $\{M(t)A(t) \,|\, t \geq 0\}$ are Kato-stable. We refer to [17, Lemma 7.2.3] and to the proof of [29, Proposition 2.3] for precise details. Finally, if $P : [0,\infty) \to \mathcal{L}(X)$ is a locally uniformly bounded function, then $\{M(t)A(t) + P(t) \,|\, t \geq 0\}$ and $\{A(t)M(t) + P(t) \,|\, t \geq 0\}$ are Kato-stable [32, Propositions 4.3.2 and 4.3.3].

Proposition 2.7. *Let $\mathcal{A}$ belong to the Kato-class and let D denote the common domain of $A(t)$, $t \geq 0$. Assume that $Q(\cdot)$ is strongly C^2-continuous. Then $\{Q^{-1}(t)A(t)Q(t) \mid t \geq 0\}$ generates a unique evolution family $\mathcal{W}$ with regularity spaces $Y_t = Q{-}1(t)D$, $t \geq 0$. Moreover, for each $F \in C^1([0,\infty);X)$ and $x_s \in Q^{-1}(s)D$ the inhomogeneous non-autonomous Cauchy problem*

$$\dot{x}(t) = Q^{-1}(t)A(t)Q(t)x(t) + F(t) \quad \text{a.e. on } [s,\infty),\ x(s) = x_s, s > 0, \tag{15}$$

has a unique classical solution given by

$$x(t) = W(t,s)x_s + \int_s^t W(t,r)F(r)\mathrm{d}r \quad t \geq s. \tag{16}$$

Proof. It is not difficult to verify that (13) implies that $\|x\|_t \leq e^{c_T|t-s|}\|x\|_s$ for all $x \in X$, $t,s \in [0,T]$ and $T > 0$. Using [32, Propositions 4.3.2 and 4.3.3] and [32, Corollary of Theorem 4.4.2] we obtain that $\{A(t) + \dot{Q}(t)Q^{-1}(t) \,|\, t \geq 0\}$ generates a unique evolution family $\mathcal{U}$ on X. Thus the first assertion follows from Proposition 2.4. Next, let $F \in C^1([0,\infty);X)$ and $x_s \in Q(s)D$. By [32, Theorem 4.5.3] the inhomogeneous Cauchy problem

$$\dot{u}(t) = A(t)u(t) + \dot{Q}(t)Q^{-1}(t)u(t) + Q(t)F(t) \quad \text{a.e. on } [s,\infty), \tag{17}$$

$$u(s) = Q^{-1}(s)x_s, s > 0. \tag{18}$$

has a unique classical solution x given by

$$u(t) = U(t,s)Q^{-1}(s)x_s + \int_s^t U(t,r)Q(r)F(r)\mathrm{d}r \quad t \geq s. \tag{19}$$

On the other hand, arguing as in the proof of Proposition 2.4 we see that $x := Q^{-1}(\cdot)u$ is a classical solution of (15). The uniqueness of classical solutions of (15) follows from the uniqueness of classical solutions of (17). Finally, (16) follows from (19) and (7). ☐

Using Example 2.6 and Proposition 2.7 one can formulate the following two corollaries.

Corollary 2.8. *Assume that X is a Hilbert space. Assume that $\mathcal{A}$ belongs to the elementary Kato class and denote by D the common of $A(t)$, $t \geq 0$. Let $M : [0,\infty) \to \mathcal{L}(X)$ and $P : [0,\infty) \to \mathcal{L}(X)$ be self-adjoint and uniformly coercive such that M is strongly C^2-continuous while P is strongly C^1-differentiable. Then $\{A(t)M(t) + P(t) \mid t \geq 0\}$ generates a unique evolution family $\mathcal{W}$ with regularity spaces $Y_t = M^{-1}(t)D$, $t \geq 0$. Moreover, for each $F \in C^1([0,\infty);X)$ and $x_s \in M^{-1}(s)D$ the inhomogeneous non-autonomous Cauchy problem*

$$\dot{x}(t) = A(t)M(t)x(t) + P(t)x(t) + F(t) \quad \text{a.e. on } [s,\infty),\ x(s) = x_s, s > 0. \tag{20}$$

has a unique classical solution given by (16).

Proof. For the proof we just have to apply Proposition 2.7 for $M(t)A(t) + M(t)P(t)M^{-1}(t)$ instead of $A(t)$ and $M(t)$ instead of $Q(t)$. ☐

Corollary 2.9. *Let X be a Hilbert space and let $(A, D(A))$ be generator of a contractive C_0-semigroup on X. Let $M : [0,\infty) \to \mathcal{L}(X)$ and $P : [0,\infty) \to \mathcal{L}(X)$ be as in Corollary 2.8. Further, let $R : [0,\infty) \to \mathcal{L}(X)$ be self-adjoint and uniformly coercive such that R is strongly C^1-continuous and commute with M i.e.*

$$R(t)M(t) = M(t)R(t) \qquad \textit{for all } t \geq 0. \tag{21}$$

Then the family $\{R(t)AM(t) + P(t) \mid t \geq 0\}$ generates a unique evolution family $\mathcal{W}$ with regularity spaces $Y_t = M^{-1}(t)D(A)$, $t \geq 0$. Moreover, for each $F \in C^1([0,\infty);X)$ and $x_s \in M^{-1}(s)D(A)$ the inhomogeneous non-autonomous Cauchy problem

$$\dot{x}(t) = R(t)AM(t)x(t) + P(t)x(t) + F(t) \quad \text{a.e. on } [s,\infty),\ x(s) = x_s, s > 0. \tag{22}$$

has a unique classical solution given by (16).

Proof. From Example 2.6 we deduce that the family $\{M(t)R(t)A \,|\, t \geq 0\}$, and therefore $\{M(t)R(t)A + M(t)P(t)M^{-1}(t) \,|\, t \geq 0\}$, belongs to Kato's class. In fact, using (22) we see that $M(\cdot)R(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is selfadjoint and uniformly coercive. Now, applying Proposition 2.7 for $M(t)R(t)A+M(t)P(t)M^{-1}(t)$ instead of $A(t)$ and $M(t)$ instead of $Q(t)$ concludes the proof. ☐

Remark 2.10. *Corollary 2.8 has been proved in* [29, Proposition 2.8-(a)] *using a slightly different method for $A(t) = A$ and $F = P = 0$.*

2.2. **Backward evolution families.** Let X be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

Definition 2.11. A family $\mathcal{V} := \{V(t,s) \,|\; (t,s) \in \Delta\} \subset \mathcal{L}(X)$ is called a *backward evolution family* if

(i) $V(t,t) = I$ and $V(r,s)V(t,r) = V(t,s)$ for every $0 \le s \le r \le t$,

(ii) $V(\cdot,\cdot) : \Delta \to \mathcal{L}(X)$ is strongly continuous.

A family $A(t) : D(A(t)) \subset X \to X$, $t \ge 0$, of linear operators generates a backward evolution equation $\mathcal{V}$ if there is a family $\{Y_t \,|\, t \ge 0\}$ of dense subspaces of X with $Y_t \subset D(A(t))$ and

$$V(t,s)Y_t \subset Y_s \text{ for all } \; 0 \le s \le t, \tag{23}$$

$V(t,\cdot)x_t \in C^1([0,t],X)$ for every $x_t \in Y_t$ and $V(t,\cdot)x_t$ solves uniquely the *backward* non-autonomous problem

$$\dot{u}(s) = -A(s)u(s) \quad \text{on} \;\; 0 \le s \le t, \;\; u(t) = x_t, (t > 0). \tag{24}$$

Lemma 2.12. *(1) Assume that $\mathcal{A} = \{A(t) \,|\, t \ge 0\}$ belongs to the elementary Kato-class. Then $\mathcal{A}$ generates a backward evolution family.*

(2) Assume that $\mathcal{A}$ generates an evolution family $\mathcal{U}$. If the adjoint operators $\mathcal{A}^* := \{A^*(t) \,|\, t \ge 0\}$ generate a backward evolution family $\mathcal{U}_* := \{U_*(t,s) \,|\; (t,s) \in \Delta\}$, then for $(t,s) \in \Delta$ we have

$$U(t,s) = [U_*(t,s)]'. \tag{25}$$

Proof. (i) Let $T > 0$ be fixed and set $\mathcal{A}_T := \{A(T-t) \,|\, t \in [0,T]\}$. Then, obviously $\mathcal{A}_T$ belongs to the Kato-class and thus generates an evolution family $\mathcal{U}_T := \{U_T(t,s) \,|\, 0 \le s \le t \le T\}$ [35, Theorem 4.8] such that for each $x \in D$ and $0 \le s \le t \le T$

$$\frac{\mathrm{d}}{\mathrm{d}t}U_T(t,s)x = A_T(t)U_T(t,s)x, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}U_T(t,s)x = -U_T(t,s)A_T(s)x. \tag{27}$$

It is easy to see that $S(t,s) := U_T(T-s, T-t)$ for each $0 \le s \le t \le T$ defines a backward evolution family with generator $\{A(t) \,|\, t \in [0,T]\}$. This completes the proof since T is arbitrary.
(ii) Denote by Y_t and $Y_{t,*}, t \ge 0$ the regularity spaces corresponding to $\mathcal{A}$ and $\mathcal{A}^*$, repectively. Let $t > s \ge 0$ and let $x_s \in Y_s$ and $y_t \in Y_{t,*}$. Then for $s \ge r \ge t$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}(x_s \,|\, [U(r,s)]'U_*(t,r)y_t) &= \frac{\mathrm{d}}{\mathrm{d}r}(U(r,s)x_s \,|\, U_*(t,r)y_t) \\ &= (A(r)U(r,s)x_s \,|\, U_*(t,r)y_t) - (U(r,s)x_s \,|\, A^*(r)U_*(t,r)y_t) \\ &= 0. \end{aligned}$$

Integrating over $[s,t]$ and using that Y_s and $Y_{t,*}$ are dense in X yield the desired identity. □

3. Review on Autonomous boundary control and observation systems

Many systems governed by linear partial differential equations with inhomogeneous boundary conditions are described by an abstract boundary system of the form

$$\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0, \; t \ge 0, \tag{28}$$

$$\mathfrak{B}x(t) = u(t), \tag{29}$$

$$\mathfrak{C}x(t) = y(t). \tag{30}$$

Here $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$ is a linear operator, $u(t) \in U$, $y(t) \in Y$, the *boundary operators* $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ and $\mathfrak{C} : D(\mathfrak{A}) \subset X \to Y$ are linear such that $D(\mathfrak{A}) \subset D(\mathfrak{B})$, and X, U and Y are complex Hilbert spaces. We shall call X the *state space*, U the *input space* and Y the *output space* of the system.
In this section, we recall some well-known results on *well-posedness* of these system which are needed throughout this paper.

Definition 3.1. Let $x_0 \in X$ and $u : [0,\infty) \to U$ be given.

(i) x is called a *classical solution* of (28)-(29), if $x \in C^1([0,\infty), X)$, $x(t) \in D(\mathfrak{A})$ for all $t \geq 0$ and x satisfies (28)-(29).
(ii) A pair (x, y) is called a *classical solution* of (28)-(30), if x is a classical solution of (28)-(29), $y \in C([0,\infty); Y)$ and y satisfies (30).
(iii) The system $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called *well-posed*, if for any *final time* $\tau > 0$ there exists $m_\tau > 0$ such that for all classical solution of (28)-(30) we have

$$\|x(\tau)\|^2 + \int_0^\tau \|y(s)\|^2 ds \leq m_\tau\big(\|x(0)\|^2 + \int_0^\tau \|u(s)\|^2 ds\big). \tag{31}$$

Remark that, if $\mathfrak{C} \in \mathcal{L}(D(\mathfrak{A}), Y)$, then (x, y) is a classical solution of (28)-(30) if and only if x is a classical solution of (28)-(29).

3.1. **Existence of classical solutions.** In order to study existence of classical solutions it is often useful to write the boundary control system (28)-(29) as a C^1-well posed (inhomogeneous) autonomous Cauchy problem. We introduce the following definition which is based on Curtain and Zwart [9, Definition 3.3.2].

Definition 3.2. The linear (autonomous) system (28)-(30) is called *a boundary control and observation autonomous system*, and we write $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a BCO-system, if the following assertions hold:

(i) The operator $A : D(A) \subset X \to X$, called *the main operator*, defined by

$$D(A) : = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$$
$$Ax : = \mathfrak{A}x \quad \text{for } x \in D(A)$$

generates a strongly continuous semigroup on X.

(ii) There exists a linear operator $\tilde{B} \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have

$$\tilde{B}u \in D(\mathfrak{A}), \mathfrak{A}\tilde{B} \in \mathcal{L}(U, X) \ \text{ and } \ \mathfrak{B}\tilde{B}u = u.$$

(iii) $\mathfrak{C} : D(\mathfrak{A}) \subset X \to Y$ is a linear bounded operator, where $D(\mathfrak{A})$ is equipped with the graph norm.

In the following $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is assumed to be a BCO-system. The following remark will be very useful for non-autonomous boundary control systems.

Remark 3.3. *Let $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a BCO-system. Then for each $(x, u) \in X \times U$ we have*

$$x \in D(\mathfrak{A}) \text{ and } \mathfrak{B}x = u \Longleftrightarrow x - \tilde{B}u \in D(A).$$

This is an easily consequence of Definition 3.2.

We denote by X_{-1} the extrapolation space associated to A, i.e., the completion of X with respect to the norm $x \mapsto \|(\beta I - A)^{-1}x\|$ for some arbitrary $\beta \in \rho(A)$. Let A_{-1} be the extension of A to X_{-1}. It is well known that A_{-1} with domain X generates a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} and for all $t \geq 0$ the operator $T_{-1}(t)$ is the unique continuous extension of $T(t)$ to X_{-1}. We associate with $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ the linear operator $B \in \mathcal{L}(U, X_{-1})$ called *control operator* defined by

$$B := \mathfrak{A}\tilde{B} - A_{-1}\tilde{B}. \tag{32}$$

It turns out, that for sufficiently smooth initial data and inputs the two Cauchy problems

$$\dot{w}(t) = Aw(t) + \mathfrak{A}\tilde{B}u(t) - \tilde{B}\dot{u}(t), \quad t \geq 0, \quad w(0) = x_0 - \tilde{B}u(0), \tag{33}$$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{34}$$

and the BCO-system (28)-(30) are equivalent. More precisely, we have

Proposition 3.4. *Let $(x_0, u) \in D(\mathfrak{A}) \times W^{2,2}([0,\infty); U)$ such that $\mathfrak{B}x_0 = u(0)$. Then* (28)-(29) *has a unique classical solution x given by*

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds \qquad t \geq 0, \tag{35}$$

$$= T(t)(x_0 - \tilde{B}u(0)) + \int_0^t T(t-s)\big[\mathfrak{A}\tilde{B}u(s) - \tilde{B}\dot{u}(s)\big]ds + \tilde{B}u(t) \tag{36}$$

$$y(t) = \mathfrak{C}T(t)x_0 + \mathfrak{C}\int_0^t T_{-1}(t-s)Bu(s)ds \qquad t \geq 0. \tag{37}$$

Therefore, x is the unique classical solution of (34) *and $w := x - \tilde{u}$ is the unique classical solution of* (33) *with initial value $w_0 = x_0 - \tilde{B}u(0)$.*

Proof. The proof follows from a combination of [17, Theorem 11.1.2] (see also [9, Theorem 3.3.3]) and [17, Corollary 10.1.4] taking Remark 3.3 into account. □

3.2. **Scattering passive BCO-systems.** Let $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a BCO-system on (X, U, Y) and let $P = P^* \in \mathcal{L}(X)$, $R = R^* \in \mathcal{L}(U)$ and $J = J^* \in \mathcal{L}(Y)$. The *admissible space* $\mathcal{V} \subset X \times U$ is defined by

$$\mathcal{V} := \left\{(x, u) \in X \times U \;\middle|\; x \in D(\mathfrak{A}) \text{ and } \mathfrak{B}x = u\right\}.$$

Definition 3.5. We say that $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is (P, R, J)*-scattering passive* if

$$\frac{\mathrm{d}}{\mathrm{d}t}(Px(t) \,|\, x(t))_X \le (Ru(t) \,|\, u(t))_U - (y(t) \,|\, Jy(t))_Y, \quad \text{for all } t \ge 0 \tag{38}$$

and all classical solutions (x, y) of (28)-(30). Further, $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called *(R,P,J)-scattering energy preserving* if equality holds in (38). If $P = I, R = I$ and $J = I$, then we simply say that $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is *scattering passive* (or *dissipative*).

Each (P, R, J)-scattering passive boundary system $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is well-posed if P and J are invertible. This can be seen by using Gronwall's Lemma (see the proof of Lemma 4.5). The following lemma characterizes (P, R, J)-scattering passive BCO-systems. A comparable results has been proved in [23, Theorem 3.2, Proposition 5.2] for *systems nodes*.

Lemma 3.6. *The BCO-system $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is (P, R, J)-scattering passive if and only if for each $(x_0, u_0) \in \mathcal{V}$ we have*

$$2\,\mathrm{Re}(\mathfrak{A}x_0 \,|\, Px_0)_X \le (Ru_0 \,|\, \mathfrak{B}x_0)_U - (\mathfrak{C}x_0 \,|\, J\mathfrak{C}x_0)_Y \tag{39}$$

or equivalently,

$$2\,\mathrm{Re}(\mathfrak{A}x_0 \,|\, Px_0)_X \le (Ru_0 \,|\, u_0)_U - (\mathfrak{C}x_0 \,|\, J\mathfrak{C}x_0)_Y. \tag{40}$$

Then the BCO-system $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is (P, R, J)-energy preserving if and only if equality holds in (39)*, or equivalently in* (40)*.*

Proof. Obviously, the inequalities (39) and (40) are equivalent since $\mathfrak{B}x_0 = u_0$ for each $(x_0, u_0) \in \mathcal{V}$. Assume that $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is (R, P, J)-scattering passive. Let $(x_0, u_0) \in \mathcal{V}$ and $u : [0, \infty) \to U$ such that $u(0) = u_0$. Assume that (x, y) is a classical solution of (28)-(30) corresponding to (x_0, u). Then $(x(t), u(t)) \in \mathcal{V}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(Px(t) \,|\, x(t)) = 2\,\mathrm{Re}(\dot{x}(t) \,|\, Px(t)) = 2\,\mathrm{Re}(\mathfrak{A}x(t) \,|\, Px(t))$$

for all $t \ge 0$. Inserting this into (38) yields

$$2\,\mathrm{Re}(\mathfrak{A}x(t) \,|\, Px(t)) \le (Ru(t) \,|\, \mathfrak{B}x(t))_U - (\mathfrak{C}x(t) \,|\, J\mathfrak{C}x(t))_Y \tag{41}$$

for all $t \ge 0$. The previous inequality implies (39) by taking $t = 0$. The converse implication and the last assertion can be proved similarly. □

4. Non-autonomous boundary and observation systems

In this section, our aim is to extend the results of Section 3 to the more general case where $\mathfrak{A}, \mathfrak{B}$, and $\mathfrak{C}$ are time dependent. Let X, U and Y be Hilbert spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. For each $t \ge 0$ we consider the linear operators $\mathfrak{A}(t) : D(\mathfrak{A}(t)) \subset X \to X$, $\mathfrak{B}(t) : D(\mathfrak{B}(t)) \subset X \to U$ and $\mathfrak{C}(t) : D(\mathfrak{A}(t)) \subset X \to Y$ such that $D(\mathfrak{A}(t)) \subset D(\mathfrak{B}(t))$ for each $t \ge 0$.
We consider the following abstract non-autonomous boundary system

$$\dot{x}(t) = \mathfrak{A}(t)x(t), \quad t \ge s, \quad x(s) = x_s, \ (s \ge 0) \tag{42}$$

$$\mathfrak{B}(t)x(t) = u(t), \quad t \ge s, \tag{43}$$

$$\mathfrak{C}(t)x(t) = y(t), \quad t \ge s, \tag{44}$$

which we denote by $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

Definition 4.1. Let $s \geq 0, x_s \in X$ and $u : [0,\infty) \to U$ be given.

(*i*) A function $x : [s,\infty) \to X$ is called a *classical solution* of (42)-(43), if $x \in C^1([s,\infty), X)$, $x(t) \in D(\mathfrak{A}(t))$ for all $t \geq s$ and x satisfies (42)-(43).

(*ii*) A pair (x, y) is a *classical solution* of (42)-(44), if x is classical solution of (42)-(43), $y \in C([s,\infty); Y)$ and (x, y) satisfies (42)-(44).

(*iii*) $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a *non-autonomous boundary control and observation system*, and we write NBCO-systems, if for each $t \geq 0$ the autonomous system $\Sigma(\mathfrak{A}(t), \mathfrak{B}(t), \mathfrak{C}(t))$ is a BCO-system such that the family $\{A(t) \,|\, t \geq 0\}$ of main operators generates an evolution family.

(*iv*) The non-autonomous system $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called *well-posed* if for any *final time* $\tau > 0$ there exists a constant $m_\tau > 0$ such that for all classical solution of (42)-(44) we have

$$\|x(\tau)\|_X^2 + \int_s^\tau \|y(r)\|_Y \mathrm{d}r \leq m_\tau \Big(\|x(s)\|_X^2 + \int_s^\tau \|u(r)\|_U^2 \mathrm{d}r \Big).$$

4.1. **Existence of classical solutions.** Let $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a NBCO-system. In this subsection, we study existence and uniqueness of classical solutions of $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ without output, i.e., classical solution of (42)-(43). In the previous section we have seen in the autonomous case that (42)-(43) can be equivalently written as a C^1-well-posed inhomogeneous Cauchy problem (in X) for sufficiently smooth initial data and inputs. This idea can be extended to the non-autonomous setting.
For each $t \geq 0$, we denote by $A(t) : D(A(t) \subset X \to X$ the main operator of $\Sigma(\mathfrak{A}(t), \mathfrak{B}(t), \mathfrak{C}(t))$, and by $\mathcal{U}$ the evolution family generated by $\{A(t) \,|\, t \geq 0\}$. Further, according to Definition 4.1-(*iii*) there exists $\{\tilde{B}(t) \,|\, t \geq 0\} \subset \mathcal{L}(U, X)$ such that for all $t \geq 0$ we have

$$\tilde{B}(t)U \subset D(\mathfrak{A}(t)), \ \mathfrak{A}(t)\tilde{B}(t) \in \mathcal{L}(U, X) \ \text{ and } \ \mathfrak{B}(t)\tilde{B}(t) = I_U. \tag{45}$$

We also consider the time-dependent admissible spaces $\mathcal{V}(t), t \geq 0$, i.e,

$$\mathcal{V}(t) := \{(x, u) \in X \times U \,|\, x \in D(\mathfrak{A}(t)) \text{ and } \mathfrak{B}(t)x = u\}.$$

Since $\{A(t) \,|\, t \geq 0\}$ generates an evolution family $\mathcal{U}$ on X, for a given $f \in L^1_{Loc}([0,\infty); X)$ the inhomogeneous non-autonomous Cauchy problem

$$\dot{v}(t) = A(t)v(t) + f(t), \quad t \geq s, \ (s \geq 0), \tag{46}$$
$$v(s) = v_s, \tag{47}$$

has at most one classical solution given by

$$v(t) = U(t,s)v_s + \int_s^t U(t,r)f(r)\mathrm{d}r,$$

see e.g., [35, Section 5.5.1]. Thus the following proposition provides a generalization of [9, Theorem 3.3.3] (see also Proposition 3.6).

Proposition 4.2. *Assume that $u \in C^1([0,\infty); U)$, $\tilde{B}(\cdot)u_0 \in C^1([0,\infty); X)$ and $\mathfrak{A}(\cdot)\tilde{B}(\cdot)u_0 \in L^1([0,\infty); X)$ for each $u_0 \in U$. Let $x_s \in X$ such that $(x_s, u_s) \in \mathcal{V}(s)$. Then x is a classical solution of (42)-(43) if and only if $v := x - \tilde{B}u$ is a classical solution of (46)-(47) with inhomogeneity*

$$f(t) = \mathfrak{F}_u(t) := \mathfrak{A}(t)\tilde{B}(t)u(t) - \frac{\mathrm{d}}{\mathrm{d}t}\big[\tilde{B}(t)u(t)\big] \tag{48}$$

and initial data $v_s = x_s - \tilde{B}(s)u(s)$. Therefore, (42)-(43) has at most one classical solution x given by

$$x(t) = U(t,s)[x_s - \tilde{B}(s)u(s)] + \tilde{B}(t)u(t) + \int_s^t U(t,r)\mathfrak{F}_u(r)\mathrm{d}r \tag{49}$$

for each $t \geq s$.

Proof. Let $s \geq 0$. Clearly $x \in C^1([s,\infty);X)$ if and only if $v \in C^1([s,\infty);X)$. Assume now that x is a classical solution of (42)-(43). Then $v(t) \in \mathcal{V}_t \subset D(A(t))$ for every $t \geq s$ by Remark 3.3 and

$$\begin{aligned}\dot v(t) &= \dot x(t) - \dot{\tilde B}(t)u(t) - \tilde B(t)\dot u(t)\\ &= \mathfrak{A}(t)x(t) - \mathfrak{A}(t)\tilde B(t)u(t) + \mathfrak{A}(t)\tilde B(t)u(t) - \dot{\tilde B}(t)u(t) - \tilde B(t)\dot u(t)\\ &= A(t)[x(t) - \tilde B(t)u(t)] + \mathfrak{A}(t)\tilde B(t)u(t) - \dot{\tilde B}(t)u(t) - \tilde B(t)\dot u(t)\\ &= A(t)v(t) + \mathfrak{A}(t)\tilde B(t)u(t) - \frac{\mathrm{d}}{\mathrm{d}t}\big[\tilde B(t)u(t)\big].\end{aligned}$$

Thus v is a classical solution of (46) with f given by (48). The converse implication can be proved similarly. Finally, (49) follows by the above the remark. □

4.2. **Scattering passive NBCO-systems.** Let $R : [0,\infty) \to \mathcal{L}(U)$, $P : [0,\infty) \to \mathcal{L}(X)$ and $J : [0,\infty) \to \mathcal{L}(Y)$ be continuous functions such that P is strongly differentiable and $R(t)^* = R(t)$, $P(t)^* = P(t)$, $J(t)^* = J(t)$ for all $t \geq 0$.

Definition 4.3. Let (x,y) be classical solution of (42)-(44). Then $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is called *(R,P,J)-scattering passive* if for all $t \geq s$

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)x(t)\,|\,x(t)) + (y(t)\,|\,J(t)y(t))_Y \leq (u(t)\,|\,R(t)u(t))_U + (\dot P(t)x(t)\,|\,x(t)). \tag{50}$$

Further, $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is called *(R,P,J)-scattering energy preserving* if equality holds in (50). If $P = I, R = I$ and $J = I$ then $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is called *scattering passive*, and *scattering energy preserving* if we have equality in (50).

We have seen in Section 3.2 that for autonomous BCO-systems $\Sigma(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ the (R,P,J)-scattering passivity can be characterized in terms of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ and (R,P,J)-scattering passivity is a sufficient condition for well-posedness, if additionally P and J are invertible. Proposition 4.4 and Lemma 4.5 generalize this facts for non-autonomous boundary control and observation systems.

Proposition 4.4. *The following assertion are equivalent.*

(*i*) $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *is (R,P,J)-scattering passive.*

(*ii*) *For each* $t \geq 0$ *and all* $(x,u) \in \mathcal{V}(t)$ *we have*

$$2\,\mathrm{Re}(\mathfrak{A}(t)x\,|\,P(t)x)_X \leq (R(t)u\,|\,\mathfrak{B}(t)x)_U - (\mathfrak{C}(t)x\,|\,J(t)\mathfrak{C}(t)x)_Y. \tag{51}$$

(*iii*) *For each* $t \geq 0$, *the autonomous BCO-system* $\Sigma(\mathfrak{A}(t),\mathfrak{B}(t),\mathfrak{C}(t))$ *is* $(R(t),P(t),J(t))$*-scattering passive.*

Proof. The equivalence of (*ii*) and (*iii*) has been proved in Proposition 3.6. It remains to prove the equivalence of (*i*) and (*ii*). Assume that (*i*) holds and let $s \geq 0$ and let $(x_s,u_s) \in \mathcal{V}(s)$. Let $u : [s,\infty) \to U$ such that $u(s) = u_s$. If (x,y) is a classical solution of (42)-(44) corresponding to (x_s,u) then $(x(t),u(t)) \in \mathcal{V}(t)$, $y(t) = \mathfrak{C}(t)x(t)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)x(t)\,|\,x(t)) - (\dot P(t)x(t)\,|\,x(t)) = 2\,\mathrm{Re}(\dot x(t)\,|\,P(t)x(t)) \tag{52}$$

$$= 2\,\mathrm{Re}(\mathfrak{A}(t)x(t)\,|\,P(t)x(t)) \tag{53}$$

for all $t \geq s$. Inserting this into (50) yields

$$2\,\mathrm{Re}(\mathfrak{A}(t)x(t)\,|\,P(t)x(t)) \leq (R(t)u(t)\,|\,\mathfrak{B}(t)x(t))_U - (\mathfrak{C}(t)x(t)\,|\,J(t)\mathfrak{C}(t)x(t))_Y$$

for all $t \geq s$. The last inequality (*ii*) by taking $t = s$. Conversely, assume that (*ii*) holds and let (x,y) be a classical solution of (42)-(44). Then $(x(t),u(t)) \in \mathcal{V}(t)$ and (52)-(53) holds for all $t \geq s$. This together with (51) imply (50), which completes the proof. □

Lemma 4.5. *Let* $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *be* (R,P,J)*-scattering passive such that* $J \geq 0$. *Assume that* P *is strongly* C^1*-continuous and uniformly coercive with*

$$(P(t)x|x) \geq \beta\|x\|^2, \ \text{for all } t \geq 0, x \in X \tag{54}$$

for some constant $\beta > 0$. Then each classical solution of (42)-(44) *satisfies the following inequality*

$$\beta\|x(t)\|^2 + \int_s^t (y(r)\,|\,J(r)y(r))\mathrm{d}r \le c_{t,s} e^{\frac{1}{\beta}\int_s^t \|\dot P(r)\|\mathrm{d}r}\Big[\int_s^t (u(r)\,|\,R(r)u(r))\mathrm{d}r + \|x(s)\|^2\Big] \tag{55}$$

where $c_{t,s} = \max\{1, \max_{r\in[s,t]} \|P(r)\|\}$. Therefore, $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is well-posed provided that J is uniformly coercive and $R \in L^\infty_{Loc}([0,\infty);\mathcal{L}(U))$.

Proof. For the proof we follow a similar argument as in fourth steps of the proof of [29, Theorem 4.1]. Assume that $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,P,J)-scattering passive. Clearly (50) holds if and only if

$$\begin{aligned}(P(t)x(t)\,|\,x(t)) + \int_s^t (y(r)\,|\,J(r)y(r))\mathrm{d}r \le \int_s^t (u(r)\,|\,R(r)u(r))\mathrm{d}r \\ + \int_s^t (\dot P(r)x(r)\,|\,x(r))\mathrm{d}r + (P(s)x_s\,|\,x_s)\end{aligned} \tag{56}$$

for all $t \ge s \ge 0$. Thus using (54) and that $J \ge 0$ we obtain

$$\begin{aligned}\beta\|x(t)\|^2 + \int_s^t (y(r)\,|\,J(r)y(r))\mathrm{d}r &\le \int_s^t (u(r)\,|\,R(r)u(r))\mathrm{d}r + \|P(s)\|\|x(s)\|^2 \\ &\quad + \int_s^t \|\dot P(r)\|\|x(r)\|^2\mathrm{d}r \\ &\le \int_s^t (u(r)\,|\,R(r)u(r))\mathrm{d}r + \|P(s)\|\|x(s)\|^2 \\ &\quad + \int_s^t \frac{1}{\beta}\|\dot P(r)\|\Big[\beta\|x(r)\|^2 + \int_s^r (y(\zeta)\,|\,J(\zeta)y(\zeta))\mathrm{d}\zeta\Big]\mathrm{d}r.\end{aligned}$$

Applying Gronwall's Lemma yields

$$\beta\|x(t)\|^2 + \int_s^t (y(r)\,|\,J(r)y(r))\mathrm{d}r \le e^{\frac{2}{\beta}\int_s^t \|\dot P(r)\|\mathrm{d}r}\Big[\int_s^t (u(r)\,|\,R(r)u(r))\mathrm{d}r + \|P(s)\|\|x(s)\|^2\Big], \tag{57}$$

which implies (55). This completes the proof. □

4.3. **Multiplicative perturbed of NBCO-systems.** We will adopt the same notations of the previous sections. The main purpose of this section is the study of some classes of NBCO-systems which are governed by a time-dependent multiplicative perturbation. More precisely, let $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a NBCO-system such that the boundary operators are constant, that is $\mathfrak{C}(t) = \mathfrak{C}$ and $\mathfrak{B}(t) = \mathfrak{B}$ for all $t \ge 0$. Thus the domain $\mathfrak{A}(t)$ should also be constant and we set $\mathcal{D}(\mathfrak{A}(t)) = \mathfrak{D}$ for all $t \ge 0$.
Further, throughout this section we assume that the following assumption holds:

Assumption 4.6. (1) $M : [0,\infty) \to \mathcal{L}(X)$and $R : [0,\infty) \to \mathcal{L}(X)$ be two self-adjoint and uniformly coercive functions.
(2) $M(\cdot)x \in C^2([0,\infty);X)$ and $M^{-1}(\cdot)x \in C([0,\infty);X)$ for each $x \in X$.
(3) $L(\cdot)x \in C^1([0,\infty);X)$ for each $x \in X$ such that L and M commute.

For each $t \ge 0$ we set

$$\begin{aligned}\mathfrak{A}_M(t) &:= \mathfrak{A}(t)M(t) \\ \mathfrak{C}_M(t) &:= \mathfrak{C}M(t) \quad \text{and} \quad \mathfrak{B}_M(t) := \mathfrak{B}M(t).\end{aligned}$$

We consider the following perturbed system

$$\dot x(t) = \mathfrak{A}_M(t)x(t), \quad x(0) = x_0, \tag{58}$$

$$\mathfrak{B}_M(t)x(t) = u(t), \tag{59}$$

$$\mathfrak{C}_M(t)x(t) = y(t), \tag{60}$$

which we denote by $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C}) = \Sigma_N(\mathfrak{A}M,\mathfrak{B}M,\mathfrak{C}M)$. Let $\tilde B(t)$ be operators associated with $\Sigma_M(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ provided by Definition 3.2-(*ii*). Then $\tilde B_M(t) := M^{-1}(t)\tilde B(t)$ satisfies for each $t \ge 0$ all properties listed in Definition 3.2-(ii). Moreover, the main operators associated with $\Sigma_{M,N}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ are given by $\{A(t)M(t)\,|\,t\geq 0\}$, where $D(A(t)M(t))=M^{-1}(t)(\mathfrak{D}\cap\ker(\mathfrak{B}))$ for each $t\geq 0$.

Lemma 4.7. *The perturbed system $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,P,J)-scattering passive if and only if $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,PM^{-1},J)-scattering passive.*

Proof. For each $t\geq 0$ we set

$$\mathcal{V}_M(t):=\{(x,u)\in X\times U\,|\,x\in D(\mathfrak{A}_M(t))\text{ and }\mathfrak{B}_M(t)x=u\}.$$

Then, $(x,u)\in\mathcal{V}_M(t)$ if and only if $(M(t)x,u)\in\mathcal{V}(t)$ for all $t\geq 0$. Assume now that $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is $(R,M^{-1}P,J)$-scattering passive and let $(x,u)\in\mathcal{V}_M(t)$. Using Proposition 4.4 we obtain

$$\begin{aligned}2\operatorname{Re}\big(\mathfrak{A}_M(t)x\,|\,P(t)x\big)&=2\operatorname{Re}\big(\mathfrak{A}(t)M(t)x\,|\,P(t)M^{-1}(t)M(t)x(t)\big)\\&\leq(R(t)u\,|\,\mathfrak{B}M(t)x)_U-(\mathfrak{C}M(t)x(t)\,|\,J(t)\mathfrak{C}M(t)x(t))_Y\\&=(R(t)u\,|\,\mathfrak{B}_M(t)x)_U-(\mathfrak{C}_M(t)x\,|\,J(t)\mathfrak{C}_M(t)x)_Y.\end{aligned}$$

This implies, again by Proposition 4.4, that $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,P,J)-scattering passive.
Conversely, assume that $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,P,J)-scattering passive. This means that $\Sigma_N(\mathfrak{A}M,\mathfrak{B}M,\mathfrak{C}M)$ is $(R,PM^{-1}M,J)$-scattering passive. Applying the first part of the proof yields that

$$\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})=\Sigma_{N,M^{-1}}(\mathfrak{A}M,\mathfrak{B}M,\mathfrak{C}M)$$

is (R,PM^{-1},J)-scattering passive. This completes the proof. ☐

In particular, the system $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,M,J)-scattering passive if and only if the unperturbed system $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is (R,I,J)-scattering passive. According to the above assumptions, we remark that (x,y) is a classical solution of (58)-(60) if and only if x is a classical solution of (58)-(59).

Now we can formulate the first main result of this section.

Theorem 4.8. *Assume that the following additional assumptions holds*

- (a) *$\mathfrak{A}:[0,\infty)\to\mathcal{L}(\mathfrak{D},X)$ is strongly C^1-continuous.*
- (b) *The main operators $A(t):\mathfrak{D}\cap\ker(\mathfrak{B})\to X$, $t\geq 0$ generate contraction C_0-semigroups.*
- (c) *$\tilde{B}(\cdot)u\in C^2([0,\infty);U)$ for each $u\in U$.*

Then the perturbed system $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is a NBCO-system on (X,U,Y). Furthermore, if we denote by $\mathcal{W}$ the associated evolution family, then for each $s\geq 0$ and $(x_s,u)\in X\times C^2([0,\infty);U)$ with $(M(s)x_s,u(s))\in\mathcal{V}(s)$ the system (58)-(60) has a unique classical solution (x,y) given by

$$x(t)=W(t,s)x_s+\int_s^t W(t,r)\mathfrak{A}(r)\tilde{B}(r)u(r)\mathrm{d}r-\int_s^t W(t,r)\frac{\mathrm{d}}{\mathrm{d}r}\big[\tilde{B}_M(r)u(r)\big]\mathrm{d}r,\quad t\geq s,$$

$$y(t)=\mathfrak{C}_M(t)W(t,s)x_s+\mathfrak{C}_M(t)\int_s^t W(t,r)\mathfrak{A}(r)\tilde{B}(r)u(r)\mathrm{d}r-\mathfrak{C}_M(t)\int_s^t W(t,r)\frac{\mathrm{d}}{\mathrm{d}r}\big[\tilde{B}_M(r)u(r)\big]\mathrm{d}r,\quad t\geq s.$$

The system $\Sigma_{N,M}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is well-posed if in addition

$$2\operatorname{Re}(\mathfrak{A}(t)x_0\,|\,x_0)_X\leq(R(t)u_0\,|\,\mathfrak{B}x_0)_U-(\mathfrak{C}x_0\,|\,J(t)\mathfrak{C}x_0)_Y\tag{61}$$

for all $t\geq 0$ and $(x_0,u_0)\in\mathcal{V}(t)$ where $R=R^*\in L^\infty_{Loc}([0,\infty);\mathcal{L}(U))$ and $J=J^*$ is uniformly coercive, where

$$\mathcal{V}=\{(x,u)\in X\times U\,|\,x\in D(\mathfrak{A})\text{ and }\mathfrak{B}x=u\}.$$

Proof. The first and the second assertion follow from Proposition 4.2 and Corollary 2.8, whereas the last assertion follows from Lemma 4.7, Proposition 4.4 and Lemma 4.5. ☐

Next we consider the case where $\mathfrak{A}(t)=L(t)\mathfrak{A}$ with $L(t)$ is as in Assumption 4.6 and such that $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is an autonomous BCO-system. This implies that $(L(t)\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is again an autonomous BCO-system for each $t\geq 0$ such that the associated operator $\tilde{B}$ is time-independent. In fact, if $\tilde{B}$ denotes the operator associated with the autonomous BCO-system $\Sigma(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, then it is easy to see that $\tilde{B}$ satisfies all properties listed in Definition 3.2-(ii) corresponding to $(L(t)\mathfrak{A},\mathfrak{B},\mathfrak{C})$. We consider the following perturbed system

$$\dot{x}(t) = L(t)\mathfrak{A}M(t)x(t), \quad x(0) = x_0, \tag{62}$$
$$\mathfrak{B}_M(t)x(t) = u(t), \tag{63}$$
$$\mathfrak{C}_M(t)x(t) = y(t), \tag{64}$$

which we denote by $\Sigma_{N,M,L}(\mathfrak{A},\mathfrak{B},\mathfrak{C}) = \Sigma_N(L\mathfrak{A}M,\mathfrak{B}M,\mathfrak{C}M)$.

Clearly, the main operators associated with $\Sigma_{M,N,L}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ are given by $\{L(t)AM(t)\,|\,t\geq 0\}$.

Theorem 4.9. *Assume that the main operators $A:\mathfrak{D}\cap\ker(\mathfrak{B})\to X$ generate a contraction C_0-semigroup on X. Then the perturbed system $\Sigma_{N,M,L}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is a NBCO-system on (X,U,Y). Furthermore, if we denote by $\mathcal{W}$ the associated evolution family, then for each $s\geq 0$ and $(x_s,u)\in X\times C^2([0,\infty);U)$ with $(M(s)x_s,u(s))\in\mathcal{V}(s)$ the system (62)-(64) has a unique classical solution (x,y) given by*

$$x(t) = W(t,s)x_s + \int_s^t W(t,r)L(r)\mathfrak{A}\tilde{B}u(r)\mathrm{d}r - \int_s^t W(t,r)\frac{\mathrm{d}}{\mathrm{d}r}\big[M^{-1}(r)\tilde{B}u(r)\big]\mathrm{d}r, \quad t\geq s,$$

$$y(t) = \mathfrak{C}_M(t)W(t,s)x_s + \mathfrak{C}_M(t)\int_s^t W(t,r)L(r)\mathfrak{A}\tilde{B}u(r)\mathrm{d}r - \mathfrak{C}_M(t)\int_s^t W(t,r)\frac{\mathrm{d}}{\mathrm{d}r}\big[M^{-1}(r)\tilde{B}u(r)\big]\mathrm{d}r, \quad t\geq s.$$

The system $\Sigma_{N,M,L}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is well-posed if in addition

$$2\,\mathrm{Re}(L(t)\mathfrak{A}x_0\,|\,x_0)_X \leq (R(t)u_0\,|\,\mathfrak{B}x_0)_U - (\mathfrak{C}x_0\,|\,J(t)\mathfrak{C}x_0)_Y \tag{65}$$

for all $t\geq 0$ and $(x_0,u_0)\in\mathcal{V}$ where $R=R^*\in L^\infty_{Loc}([0,\infty);\mathcal{L}(U))$ and $J=J^*$ is uniformly coercive.

Proof. The first and the second assertion follow from Proposition 4.2 and Corollary 2.9, whereas the last assertion follows from Lemma 4.7, Proposition 4.4 and Lemma 4.5. □

Remark 4.10. *Theorem 4.9 is not a special case of Theorem 4.8 since we do not assume that $P(t)A$ generates a contractive C_0-semigroup on X.*

5. Mild solutions for NBC-systems

As mentioned in Section 3, for an autonomous BCO-system $\Sigma(\mathfrak{A},\mathfrak{B},\mathfrak{C})$, for smooth input u and initial data x_0, the classical solution of the corresponding boundary control system can be formulated as

$$x(t) = T(t)x_s + \int_s^t T_{-1}(t-s)Bu(s)ds, \qquad t\geq s. \tag{66}$$

We recall that $B\in\mathcal{L}(U,X_{-1})$ is given by (32). If $x_s\in X$ and $u\in L^2([0,\infty);U)$, then the above formula makes sense and it is called the mild solution in X_{-1} of (28)-(30). Moreover, it is well known that the mild solution belongs to $C([0,\infty);X)$ if B is admissible for the semigroup $(T(t))_{t\geq 0}$, i.e., if for some $\tau>0$ one has

$$\int_s^\tau T_{-1}(\tau-s)Bu(s)ds\in X,$$

see, e.g., [33, Proposition 4.2.4].

The main purpose of this section is to extend the conceps of mild solutions to non-autonomous boundary control and observation systems $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$. In contrast to the autonomous case, this is more delicate. In fact, firstly we remark that the extrapolation spaces $X_{-1,t}$ associated with the family $\{A(t)\,|\,t\geq 0\}$ of the main operators are in general time-dependent. Secondly, in contrast to semigroups, it is not clear whether the evolution family $\mathcal{U}$ generated by $\{A(t)\,|\,t\geq 0\}$ can be extended to the extrapolation space even if the spaces $X_{-1,t}$ are constant. However, if the latter condition holds, then we can still use the adjoint problem, i.e, $A^*(t)$, $t\geq 0$, and the associated backward evolution family to extend $\mathcal{U}$ to $\mathcal{L}(X_{-1})$. The idea to use a duality argument can be found in [7, 25, 29] to study some classes of non-autonomous systems.

We will adopt here the notations of the previous sections. Let $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a NBCO-system. Then the main operators $\{A(t)\,|\,t\geq 0\}$ generate, by definition, an evolution family $\mathcal{U}=\{U(t,s)\,|\,(s,t)\in\Delta\}$ with regularity space Y_t, $t \geq 0$. We restrict ourselves to case where $\{A(t) \mid t \geq 0\}$ have a common extrapolation space X_{-1}, i.e.,

$$X_{-1} := X_{-1}(t) = X_{-1}(s) \text{ for all } t, s \geq 0. \tag{67}$$

According to [33, Proposition 2.10.2], (67) holds if and only if $D(A^*(t)) = D(A^*(s))$ for $t, s \in [0, \infty)$ and the corresponding graph norms are locally uniformly equivalent. In fact, $X_{-1}(t)$ is the dual space of $D(A^*(t))$ with respect to the pivot space X. This condition holds, if for instance $A(t) = AM(t)$ or $A(t) = A + M(t)$ and $M \in C^1([0, \infty); \mathcal{L}(X))$.
In the following we denote $\mathfrak{D}_* := D(A^*(0))$ equipped with the graph norm and by $\langle \cdot, \cdot \rangle$ the duality between X_{-1} and $\mathfrak{D}_*$. Recall from (32)

$$B(t) = \mathfrak{A}(t)\tilde{B}(t) - A_{-1}(t)\tilde{B}(t), \quad t \geq 0. \tag{68}$$

Proposition 5.1. *Assume that $\mathcal{A}^* := \{A^*(t) \mid t \geq 0\}$ generates a backward evolution family $\mathcal{U}_*$. Then $U(t, s)$ has a unique extension $U_{-1}(t, s) \in \mathcal{L}(X_{-1})$ for each $(t, s) \in \Delta$ and for each $T > 0$ there is $c_T > 0$ such that*

$$\sup_{0 \leq s \leq t \leq T} \|U_{-1}(t, s)\|_{\mathcal{L}(X_{-1})} < c_T. \tag{69}$$

Moreover, if the assumptions of Proposition 4.2 hold, then each classical solution x of the boundary control system (42)-(43) *satisfies*

$$x(t) = U(t, s)x_s + \int_s^t U_{-1}(t, r)B(r)u(r)\mathrm{d}r, \ t \geq s \geq 0. \tag{70}$$

Proof. By [33, Proposition 2.9.3-(b)] we obtain that for each $(t, s) \in \Delta$ the operator $U(t, s)$ has a unique extension $U_{-1}(t, s) \in \mathcal{L}(X_{-1})$ since $[U(t, s)]^*\mathfrak{D}_* = U_*(t, s)\mathfrak{D}_* \subset \mathfrak{D}_*$. Next, similar to the proof of [29, Proposition 2.7-(c)] we show the uniform boundedness of $\mathcal{U}_{-1}$ on compact intervals. Next, we claim that for each $y \in \mathfrak{D}_*, x \in X_{-1}$ we have

$$\langle U_{-1}(t, s)x, y \rangle = \langle x, U_*(t, s)y \rangle \tag{71}$$

In fact, this equality holds for $x \in X$ by Lemma 2.12-(ii) since

$$\langle x, U_*(t, s)y \rangle = (x \mid U_*(t, s)y) = (U(t, s)x \mid y) = \langle U_{-1}(t, s)x, y \rangle.$$

Remark that $U_*(t, s)y \in \mathfrak{D}_*$, thus the claim follows since X is dense in X_{-1}.
Using again Lemma 2.12 and (71), we obtain for each $y \in \mathfrak{D}_*$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\langle U(t, s)\tilde{B}(s)u(s), y \rangle &= \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{B}(s)u(s), U_*(t, s)y) \\
&= (\frac{\mathrm{d}}{\mathrm{d}s}[\tilde{B}(s)u(s)], U_*(t, s)y) - (\tilde{B}(s)u(s), A^*(s)U_*(t, s)y) \\
&= (\frac{\mathrm{d}}{\mathrm{d}s}[\tilde{B}(s)u(s)], U_*(t, s)y) - \langle U_{-1}(t, s)A_{-1}(s)\tilde{B}(s)u(s), y \rangle \\
&= (U(t, s)\frac{\mathrm{d}}{\mathrm{d}s}[\tilde{B}(s)u(s)], y) - \langle U_{-1}(t, s)A_{-1}(s)\tilde{B}(s)u(s), y \rangle.
\end{aligned}$$

Integrating over $[s, t]$, we obtain

$$\int_s^t U_{-1}(t, r)A_{-1}(r)\tilde{B}(r)u(r)\mathrm{d}r = -\tilde{B}(t)u(t) + U(t, s)\tilde{B}(s)u(s) + \int_s^t U(t, r)\frac{\mathrm{d}}{\mathrm{d}r}[\tilde{B}(r)u(r)]\mathrm{d}r. \tag{72}$$

Inserting this equality in (49), we obtain that a classical solution x of (42)-(43) satisfies (70). □

If the assumptions of Proposition 5.1 hold, then for $x_s \in X$ and $u \in L^2_{Loc}([0, \infty); U)$ we see that (70) is well defined with value in X_{-1} provided $B(\cdot)u(\cdot) \in L^1_{Loc}([0, \infty); X_{-1})$. In fact, (69) guaranties that the integral term on the right hand side of (73) is well defined. Thus the following definition makes sense.

Definition 5.2. Let $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a NBCO-system and let $\mathcal{U}$ and $\{B(t)\,|\,t\geq 0\}$ the associated evolution family and control operators, respectively. Let $(x_s,u)\in X\times L^2_{Loc}([0,\infty);U)$. If $U(t,s)$ has a unique extension $U_{-1}(t,s)\in\mathcal{L}(X_{-1})$ for each $(t,s)\in\Delta$ such that $U_{-1}(t,\cdot)B(\cdot)u(\cdot)\in L^1_{Loc}([0,\infty);X_{-1})$, then the function

$$x(t)=U(t,s)x_s+\int_s^t U_{-1}(t,r)B(r)u(r)\mathrm{d}r,\ t\geq s\geq 0, \tag{73}$$

is called the *mild solution of* (42)-(44) *in* X_{-1}. Further, (73) is called a *mild solution of* (42)-(44) *(in* X*)*, if in addition

$$\Phi_{t,s}u:=\int_s^t U_{-1}(t,r)B(r)u(r)\mathrm{d}r\in X,\ \text{for all } (t,s)\in\Delta, \tag{74}$$

and $x\in C([s,\infty);X)$.

This definition is related to the notion of *admissibility* for non-autonomous linear systems. More precisely, recall that a family $\{B(t)\,|\,t\geq 0\}\subset\mathcal{L}(U,X_{-1})$ is L^2-admissible for a given evolution family $\mathcal{U}$ that admit an extension to $\mathcal{L}(X_{-1})$ if $U_{-1}(t,\cdot)B(\cdot)u(\cdot)\in L^1_{Loc}([0,\infty);X_{-1})$, (74) holds and for each $T>0$ there exists $c_T>0$ such that

$$\Big\|\int_s^t U_{-1}(t,r)B(r)u(r)\mathrm{d}r\ \Big\|_X^2\leq c_T\int_s^t \|u(r)\|_U^2\mathrm{d}r \tag{75}$$

for each $u\in L^2_{Loc}([0,\infty);U)$ and all $0\leq s\leq t\leq T$ [28, Definition 3.3]. For L^2-admissible control operators we have that $(t,s)\mapsto\Phi_{t,s}u$ is continuous on Δ with value in X [28, Proposition 3.5-(2)].

Proposition 5.3. *Assume that* $\mathcal{A}^*:=\{A^*(t)\,|\,t\geq 0\}$ *belongs to the Kato-class and* $\{B(t)\,|\,t\geq 0\}$ *is* L^2*-admissible. Then for each* $(x_s,u)\in X\times L^2_{Loc}([0,\infty);U)$ *with* $B(\cdot)u(\cdot)\in L^1_{Loc}([0,\infty);X_{-1})$ *the system* (42)-(44) *has a unique mild solution in* X.

Proof. The proof follows from Lemma 2.12-(i) and Proposition 5.1. □

If $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is a well-posed NBCO-system and the classical solutions is given by (70), then the corresponding family $\{B(t)\,|\,t\geq 0\}$ is L^2-admissible provided

$$U_{-1}(t,\cdot)B(\cdot)L^2_{Loc}([0,\infty);U)\subset L^1_{Loc}([0,\infty);X_{-1}).$$

Thus the following corollary follows from Proposition 5.3, Lemma 4.5 and (69).

Corollary 5.4. *Assume* $\Sigma_N(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *is* (R,P,J)*-scattering passive such that* $R\in L^\infty_{Loc}([0,\infty);\mathcal{L}(U))$ *and* J,P *are uniformly coercive. In addition, we assume that* $\mathcal{A}^*:=\{A^*(t)\,|\,t\geq 0\}$ *belongs to the Kato-class. Then for each* $(x_s,u)\in X\times L^2_{Loc}([0,\infty);U)$ *with* $B(\cdot)u(\cdot)\in L^1_{Loc}([0,\infty);X_{-1})$ *the system* (42)-(44) *has a unique mild solution in* X.

Finally, if Assumption 4.6 holds such that $A(t)$ generates contractive C_0-semigroup on X for each $t\geq 0$ then we can follow [29, Section 2, page 8] to deduce that the extrapolation spaces corresponding to $A(t)M(t)$, $t\geq 0$ can be all identified with X_{-1} and that $[A(t)M(t)]_{-1}=A_{-1}(t)M(t)$ for every $t\geq 0$.

Corollary 5.5. *Assume that Assumption 4.6 holds such that the adjoint operators* $\{A(t)^*\,|\,t\geq 0\}$ *have a common domain. Then the perturbed system* (58)-(59) *has a unique mild solution in* X *if the unperturbed system* $\Sigma_{N,\mathrm{id}}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *is* (R,I,J)*-scattering passive with* $R\in L^\infty_{Loc}([0,\infty);\mathcal{L}(U))$ *and* J,P *are uniformly coercive.*

Proof. The proof is an easy consequence of Corollary 5.4 and Lemma 4.7. □

6. Application to non-autonomous Port-Hamiltonian systems

Let $N, n \in \mathbb{N}$ be fixed and let $X := L^2([a,b];\mathbb{K}^n)$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this section we investigate the well-posedness of the linear non-autonomous *port-Hamiltonian systems of order* $N \in \mathbb{N}$, given by the boundary control and observation system

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=1}^{N} P_k(t)\frac{\partial^k}{\partial \zeta^k}\big[\mathcal{H}(t,\zeta)x(t,\zeta)\big] + P_0(t,\zeta)\mathcal{H}(t,\zeta)x(t,\zeta), \quad t \geq 0,\ \zeta \in (a,b) \tag{76}$$

$$\mathcal{H}(0,\zeta)x(0,\zeta) = x_0(\zeta), \qquad \zeta \in (a,b), \tag{77}$$

$$u(t) = W_{B,1}\tau(\mathcal{H}x)(t), \qquad t \geq 0, \tag{78}$$

$$0 = W_{B,2}\tau(\mathcal{H}x)(t), \qquad t \geq 0, \tag{79}$$

$$y(t) = W_C\tau(\mathcal{H}x)(t), \qquad t \geq 0. \tag{80}$$

Here τ denotes the *trace operator* $\tau : H^N((a,b);\mathbb{K}^n) \to \mathbb{K}^{2Nn}$ defined by

$$\tau(x) := \big(x(b), x'(b), \cdots, x^{N-1}(b), x(a), x'(a) \cdots, x^{N-1}(a)\big),$$

$P_k(t)$ is $n \times n$ matrix for all $t \geq 0$, $k = 0, 1, \cdots, N$, $\mathcal{H}(t,\zeta) \in \mathbb{K}^{n\times n}$ for all $t \geq 0$ and almost every $\zeta \in [a,b]$, $W_{B,1}$ is a $m \times 2nN$-matrix, $W_{B,2}$ is $(nN-m) \times 2nN$-matrix and W_C is a $d \times 2nN$-matrix. Finally, $u(t) \in U := \mathbb{K}^m$ denotes the input and $y(t) \in Y := \mathbb{K}^d$ is the output at time t.

Set $W_B := \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right]$, $\Sigma := \left[\begin{smallmatrix} 0 & I \\ I & 0\end{smallmatrix}\right]$ and for each $t \geq 0$ we set

$$Q(t) := \begin{bmatrix} P_1(t) & P_2(t) & \cdots & & P_N(t) \\ -P_2(t) & -P_3(t) & \cdots & -P_N(t) & 0 \\ \vdots & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ (-1)^{N-1}P_N(t) & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

$R_{ext}(t) := \left[\begin{smallmatrix} Q(t) & -Q(t) \\ \mathrm{I} & \mathrm{I}\end{smallmatrix}\right]$ and $W_B(t) := W_B R_{ext}^{-1}(t), W_C(t) := W_B R_{ext}^{-1}(t)$.
In this section we assume the following assumptions:

Assumption 6.1.

- W_B has full rank and $W_B(t)\Sigma W_B^*(t) \geq 0$ for all $t \geq 0$.
- $P_N(t)$ is invertible and $P_k^*(t) = (-1)^{k-1}P_k(t)$ for all $k \geq 1$, $t \geq 0$,
- $P_k \in C^1([0,\infty); L^\infty(a,b;\mathbb{C}^{n\times n}))$ for all $t \geq 0$ and $k = 0, 1, \cdots N$.
- $\mathcal{H} \in C^2([0,\infty); L^\infty([a,b];\mathbb{C}^{n\times n}))$ and there exist $m, M \geq 0$ such that

$$m \leq \mathcal{H}(t,\xi) = \mathcal{H}^*(t,\xi) \leq M, \quad \text{a.e. } \xi \in [a,b], t \geq 0.$$

Under these assumptions, the port-Hamiltonian system (76)-(80) can be written as a non-autonomous boundary control and observation system in the sense of Definition 4.1-(ii). In fact, on the Hilbert space X we consider the (maximal) *port-Hamiltonian operators*

$$\mathfrak{A}(t)x = \sum_{k=0}^{N} P_k(t)\frac{\partial^k}{\partial\zeta^k}x \quad \text{with domain} \quad D(\mathfrak{A}(t)) = \big\{H^N([a,b];\mathbb{K}^n) \,|\, W_{B,2}\tau(x) = 0\big\} \tag{81}$$

Then $(\mathfrak{A}(t), D(\mathfrak{A}(t)))$ is a closed and densely defined operator and its graph norm $\|\cdot\|_{D(\mathfrak{A}(t))}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^N((a,b);\mathbb{K}^n)}$ as $P_N(t)$ is invertible. Moreover, for each $t \geq 0$ the operator $A(t) : D(A(t)) \subset X \to X$ defined by

$$A(t)x = \mathfrak{A}(t)x \quad x \in D(A(t)) \tag{82}$$

$$D(A(t)) = \Big\{x \in H^N((a,b);\mathbb{K}^n) \,|\, W_B\tau(x) = 0\Big\} \tag{83}$$

generates a contractive C_0-semigroup on X. Further, we define the input operator $\mathfrak{B}$ and output operator $\mathfrak{C}$ a follows

$$\mathfrak{B} : H^N((a,b);\mathbb{K}^n) \to U,\ \mathfrak{B}x := W_{B,1}\tau(x),$$

and

$$\mathfrak{C} : H^N((a,b);\mathbb{K}^n) \to Y,\ \mathfrak{C}x := W_C\tau(x).$$

The operator $\mathfrak{C}$ is a linear and bounded operator from $D(\mathfrak{A}(t))$ to Y, since the trace operator τ is bounded and the norm graph norm of $D(\mathfrak{A})$ is equivalent to the $H^N((a,b);\mathbb{K}^n)$-norm. Moreover, Lemma 6.2 below shows that there exists an operator $\tilde{B} \in \mathcal{L}(U,X)$ which is independent of $t \geq 0$ satisfying the assumption (ii) of Definition 3.2. The proof of this fact follows by a minor modification of the proof of [17, Theorem 11.3.2] and that of [2, Lemma 3.2.19] (see also the second step of the proof of [22, Theorem 4.2]).

Lemma 6.2. *There exists a linear operator $\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ such that $\tilde{B}\mathbb{K}^m \subset D(\mathfrak{A}(t))$, $\mathfrak{A}(t)\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ for each $t \geq 0$ and $\mathfrak{B}\tilde{B} = \mathrm{I}_{\mathbb{K}^m} = I_U$.*

Proof. Since the $nN \times 2nN$-matrix W_B has full rank nN there exists a $2nN \times nN$-matrix S such that

$$W_BS = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} S = \begin{bmatrix} I_{\mathbb{K}^m} & 0 \\ 0 & 0 \end{bmatrix}. \tag{84}$$

In fact, one can choose S as follows

$$S = W_B^*(W_BW_B^*)^{-1} \begin{bmatrix} I_{\mathbb{K}^m} & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us write $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ \vdots & \vdots \\ S_{(2nN)1} & S_{(2nN)2} \end{bmatrix} =: \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \end{bmatrix}$, where $S_{j1}, j = 1, \cdots, 2nN$, are $1 \times m$ matrices.

Next, let $\{e_j\}_{j=1}^{2Nn}$ be the standard orthogonal basis in $\mathbb{K}^{2nN}$. For each $j = 1, 2, \cdots 2nN$ we take $f_j \in H^N(a,b;\mathbb{K}^n)$ such that $\tau(f_j) = e_j$ [37, Lemma A.3], and we define the operator $\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ by

$$\tilde{B}u := \sum_{j=1}^{2nN} S_{j1}uf_j \quad u \in \mathbb{K}^m. \tag{85}$$

Thus $B \in \mathcal{L}(\mathbb{K}^m, H^N((a,b);\mathbb{K}^n))$. Furthermore, (84) implies that $W_{B,2}\tilde{S}_1 = 0$ and thus

$$\begin{aligned} W_{B,2}\tau(\tilde{B}u) &= W_{B,2}\sum_{j=1}^{2nN} S_{j1}u\tau(f_j) \\ &= W_{B,2}\sum_{j=1}^{2nN} S_{j1}ue_j = W_{B,2}\tilde{S}_1u = 0 \end{aligned}$$

for every $u \in \mathbb{K}^m$. We deduce that $\tilde{B}\mathbb{K}^m \subset D(\mathfrak{A}(t))$ for all $t \geq 0$. It follows that $\Sigma(\mathfrak{A}(t), \mathfrak{B}, \mathfrak{C})$ is for each $t \geq 0$ a BCO-system on $(L^2([a,b];\mathbb{K}^n), \mathbb{K}^m, \mathbb{K}^d)$. Using (84) once more, we obtain that

$$\mathfrak{B}\tilde{B}u = W_{B,1}\tau(\tilde{B}u) = W_{B,1}\tilde{S}_1u = u$$

for all $u \in \mathbb{K}^m$. This completes the proof. □

Moreover, if in addition the following assumption holds

Assumption 6.3.

- $nN = m = d$ (and thus $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$),
- $R = \{R(t) \mid t \geq 0\}$ and $J = \{J(t) \mid t \geq 0\}$ are bounded and self adjoint operators on $\mathbb{K}^n$,
- $\mathrm{Re}\, P_0(t,\zeta) \leq 0$ for all $t \geq 0$ and a.e. $\zeta \in [a,b]$,
- the matrix W_C has full rank,

then we obtain:

Lemma 6.4. *Under assumptions 6.3 and 6.1 for each $t \geq 0$ the autonomous port-Hamiltonian system $\Sigma(\mathfrak{A}(t), \mathfrak{B}, \mathfrak{C})$ is $(R(t), \mathrm{I}, J(t))$-scattering passive if*

$$P_{W_B,W_C}(t) := \left(\begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix} \Sigma \begin{bmatrix} W_B^*(t) & W_C^*(t) \end{bmatrix} \right)^{-1} \leq \begin{bmatrix} 2R(t) & 0 \\ 0 & -2J(t) \end{bmatrix}. \tag{86}$$

Proof. Using [2, Lemma 3.2.13] we obtain

$$\operatorname{Re}(\mathfrak{A}(t)x \mid x) = \operatorname{Re}(R_{ext}(t)\tau(x) \mid \Sigma R_{ext}(t)\tau(x)) + \operatorname{Re}(P_0(t)x \mid x). \tag{87}$$

Inserting

$$\begin{bmatrix} W_B\tau(x) \\ W_C\tau(x) \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} R_{ext}^{-1}(t)R_{ext}(t)\tau(x) = \begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix} R_{ext}(t)\tau(x)$$

into (87) we obtain that

$$2\operatorname{Re}(\mathfrak{A}(t)x \mid x) \leq \left\langle \begin{bmatrix} W_B\tau(x) \\ W_C\tau(x) \end{bmatrix} \,\middle|\, \begin{bmatrix} W_B(t)^* & W_C(t)^* \end{bmatrix}^{-1} \Sigma \begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix}^{-1} \begin{bmatrix} W_B\tau(x) \\ W_C\tau(x) \end{bmatrix} \right\rangle_{\mathbb{K}^{2nN}} \quad t \geq 0, \tag{88}$$

holds for every $x \in H^N([a,b];\mathbb{K}^n)$, since $ReP_0(t,\zeta) \leq 0$. Now the claim follows by Lemma 3.6. □

Finally, the assumption on $\mathcal{H}$ ensures that the family of operators $M(t) := \mathcal{H}(t)(\cdot) := \mathcal{H}(t,\cdot)$ as matrix multiplication operators on $L^2(a,b;\mathbb{K}^n)$ satisfies all assumptions of Section 4.3.

Our abstract results in the previous sections hence yield the following main result.

Theorem 6.5. *If Assumption 6.1 holds, then the port-Hamiltonian system* (76)-(80) *is a non-autonomous boundary control and observation system. Furthermore, there exists a unique evolution family $\mathcal{W}$ in $L^2([a,b];\mathbb{K}^n)$ such that for each $x_0 \in H^N((a,b);\mathbb{K}^n)$ and $u \in C^2([a,b];\mathbb{K}^m)$ with, $W_{B,1}\tau(x_0) = u(0)$ and $W_{B,2}\tau(x_0) = 0$ we have*

$$x(t) = W(t,0)\mathcal{H}^{-1}(0,\zeta)x_0 + \int_0^t W(t,r)\mathfrak{A}(r)\tilde{B}u(r)\mathrm{d}r + \int_0^t W(t,r)\mathcal{H}^{-1}(r)\dot{\mathcal{H}}(r)\mathcal{H}^{-1}(r)\tilde{B}u(r)\mathrm{d}r - \int_0^t W(t,r)\mathcal{H}^{-1}(r)\tilde{B}\dot{u}(r)\mathrm{d}r, \quad t \geq 0,$$

$$y(t) = \mathfrak{C}\mathcal{H}(t)W(t,0)\mathcal{H}^{-1}(0,\zeta)x_0 + \mathfrak{C}\int_0^t \mathcal{H}(t)W(t,r)\Big[\mathfrak{A}(r) - \mathcal{H}^{-1}(r)\dot{\mathcal{H}}(r)\mathcal{H}^{-1}(r)\Big]\tilde{B}u(r)\mathrm{d}r - \mathfrak{C}\int_0^t \mathcal{H}(r)W(t,r)\mathcal{H}^{-1}(r)\tilde{B}\dot{u}(r)\mathrm{d}r, \quad t \geq 0.$$

is the unique classical solution of (76)-(80). *If in addition Assumption 6.3 and* (86) *hold, then* (76)-(80) *is $(R,\mathcal{H},J)$-scattering passive and the classical solution (x,y) satisfies the balance inequality*

$$m\|x(t)\|^2 + \int_s^t (y(r) \mid J(r)y(r))\mathrm{d}r \leq c_{t,s}e^{\frac{1}{m}\int_s^t \|\dot{\mathcal{H}}(r)\|\mathrm{d}r}\Big[\int_s^t (u(r) \mid R(r)u(r))\mathrm{d}r + \|x(s)\|^2\Big] \tag{89}$$

where $c_{t,s} = \max\{1, \max_{r\in[s,t]}\|\mathcal{H}(r)\|\}$. Moreover, (76)-(80) *is well posed if in addition J is uniformly coercive and $R \in L^\infty_{Loc}([0,\infty);\mathcal{L}(\mathbb{K}^n))$.*

Finally, we give a result on the existence of mild solution of the non-autonomous port-Hamiltonian system. For that we assume that $nN = m = d$. Then it is known [22, Lemma A1] (see also [17, Section 7.3]) that there exist a matrix $V \in \mathbb{K}^{nN\times nN}$ and an invertible matrix $S \in \mathbb{K}^{nN\times nN}$ such that

$$W_B = S\begin{bmatrix} I+V & I-V \end{bmatrix}$$

with $VV^* \geq$ I. Further, we have $\ker W_B = \operatorname{Ran}\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$. For each $t \geq 0$, the adjoint operator $A^*(t) : D(A^*(t)) \to X$ of (82)-(83) is given by

$$A^*(t)x = -\mathfrak{A}(t)x \quad x \in D(A^*(t)) \tag{90}$$

$$D(A^*(t)) = \Big\{x \in H^N(a,b;\mathbb{K}^n) \mid \begin{bmatrix} I-V^* & -I-V^* \end{bmatrix}\begin{bmatrix} Q(t) & 0 \\ 0 & -Q(t) \end{bmatrix}\tau(x) = 0\Big\} \tag{91}$$

see e.g., [36, Theorem 2.24], [2, Proposition 3.4.3]. We deduce that the domain of $A^*(t)$ are time-independent if for instance all matrices $P_k, k = 1,2,\cdots N$ are constant. Thus using Corollary 5.5 we obtain the followin proposition.

Proposition 6.6. *Assume that Assumption 6.1 and Assumption 6.3 hold with $P_k, k = 1, 2, \cdots N$ are constant and J is uniformly coercive and $R \in L^\infty_{Loc}([0,\infty); \mathcal{L}(\mathbb{K}^n))$. If* (86) *holds, then the non-autonomous system* (76)-(79) *has a unique mild solution.*

We closed this section by some examples of physical systems which can be modelled as a non-autonomous port-Hamiltonian system. Then the existence of classical and mild solutions as well as well-posedness can be checked by a simple application of the abstracts results presented in this section. Here we will present just two relevant examples, however various other control systems fit into the framework of port-Hamiltonian system and into the general class of NBCO-systems.

6.1. **Vibrating string.** Let us consider the model of vibrating string on the compact interval $[a, b]$. The string is fixed at the left end point a and at the right end point b a damper is attached. The Young's modulus and the mass density of the string are assumed to be time- and spatial dependent. Let us denote by $\omega(t,\zeta)$ the vertical position of the string at position $\zeta \in [a, b]$ and time $t \geq 0$. Then the evolution of the controlled vibrating string can be modelled by a non-autonomous wave equation of the form

$$\frac{\partial}{\partial t}\Big(\alpha(t)\rho(t,\zeta)\frac{\partial w}{\partial t}(t,\zeta)\Big) = \frac{1}{\alpha(t)}\frac{\partial}{\partial \zeta}\Big(T(t,\zeta)\frac{\partial w}{\partial \zeta}(t,\zeta)\Big),\ \zeta \in [a,b],\ t \geq 0, \tag{92}$$

$$T(b,t)\frac{\partial w}{\partial \zeta}(t,b) + k\alpha(t)\frac{\partial w}{\partial t}(t,b) = u_1(t), \tag{93}$$

$$\frac{\partial w}{\partial t}(t,a) = u_2(t). \tag{94}$$

We assume that $k \geq 0$ and $T, \rho \in C^2([0,\infty); L^\infty(a,b)) \cap C_b([0,\infty); L^\infty(a,b))$ such that for some $m > 0$, for a.e $\zeta \in [a,b]$ and all $t \geq 0$ we have $m^{-1} \leq \rho(t,\zeta), T(t,\zeta) \leq m$, moreover, $\alpha \in C^1([0,\infty))$ is strictly positive. We take as state variable the momentum-strain couple $x := (\alpha\rho\frac{\partial w}{\partial t}, \frac{\partial w}{\partial \zeta})$. Then the first equation can be equivalently written as follows

$$\frac{\partial}{\partial t}x(t,\zeta) = \mathfrak{A}(t)\mathcal{H}(t,\zeta)x(t,\zeta) \tag{95}$$

where

$$\mathfrak{A}(t) := \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix}\frac{\partial}{\partial \zeta} \quad \text{and } \mathcal{H}(t,\zeta) := \begin{bmatrix} \frac{1}{\rho(t,\zeta)} & 0 \\ 0 & T(t,\zeta) \end{bmatrix}.$$

Indeed, we have

$$\begin{aligned}
\mathfrak{A}(t)\mathcal{H}(t,\zeta)x(t,\zeta) &= \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\begin{bmatrix} \frac{1}{\rho(t,\zeta)} & 0 \\ 0 & T(t,\zeta) \end{bmatrix}\begin{bmatrix} \alpha(t)\rho(t,\zeta)\frac{\partial w}{\partial t}(t,\zeta) \\ \frac{\partial w}{\partial \zeta}(t,\zeta) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\begin{bmatrix} \alpha(t)\frac{\partial w}{\partial t}(t,\zeta) \\ T(t,\zeta)\frac{\partial w}{\partial \zeta}(t,\zeta) \end{bmatrix} \\
&= \begin{bmatrix} 1/\alpha(t)\frac{\partial}{\partial \zeta}\big(T(t,\zeta)\frac{\partial w}{\partial \zeta}(t,\zeta)\big) \\ \frac{\partial}{\partial \zeta}\frac{\partial w}{\partial t}(t,\zeta) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial t}\Big(\alpha(t)\rho(t,\zeta)\frac{\partial w}{\partial t}(t,\zeta)\Big) \\ \frac{\partial}{\partial t}\frac{\partial w}{\partial \zeta}(t,\zeta) \end{bmatrix} = \frac{\partial}{\partial t}x(t,\zeta).
\end{aligned}$$

Moreover, the boundary conditions (94)-(93) with $u = (u_1, u_2) = 0$ can be equivalently written as follows

$$W_B\begin{bmatrix} \mathcal{H}(t,b)x(t,b) \\ \mathcal{H}(t,a)x(t,a) \end{bmatrix} := \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \mathcal{H}(t,b)x(t,b) \\ \mathcal{H}(t,a)x(t,a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The 2×4 matrix W_B has full rank. Next,

$$W_B(t) = W_B\begin{bmatrix} 0 & \alpha(t) & 1 & 0 \\ \alpha(t) & 0 & 0 & 1 \\ 0 & -\alpha(t) & 1 & 0 \\ -\alpha(t) & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha(t) & k\alpha(t) & k & 1 \\ 0 & -\alpha(t) & 1 & 0 \end{bmatrix}$$

and $W_B(t)\Sigma W_B^*(t) = \begin{bmatrix} 4k\alpha(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. The corresponding matrices $W_{B,1}, W_{B,2}$ and the corresponding boundary operator $\mathfrak{B}$ can be defined as follows:
Case $u_2 = 0 : W_{B,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ and

$$\begin{aligned} \mathfrak{B} :& H^N((a,b);\mathbb{K}^2) \to U = \mathbb{K}, \\ \mathfrak{B}x :&= W_{B,1}\tau(x) := \begin{bmatrix} k & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix}. \end{aligned}$$

Case $u_2 \neq 0 : W_{B,2} = 0$ and

$$\mathfrak{B} : H^N((a,b);\mathbb{K}^2) \to U = \mathbb{K}^2, \ \mathfrak{B}x := \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix}$$

For each $S, V \in \mathbb{K}^{2\times 2}$ such that S is invertible and $VV^* \geq I$ we can we take

$$y(t) = S \begin{bmatrix} \mathrm{I} + V & \mathrm{I} - V \end{bmatrix} \begin{bmatrix} \mathcal{H}(t,b)x(t,b) \\ \mathcal{H}(t,a)x(t,a) \end{bmatrix} \tag{96}$$

as an output of (94)-(93). Thus, we are in the position to apply Theorem 6.5. However, Proposition 6.6 concerning mild solutions can be applied only if $\alpha(t) \equiv \alpha > 0$ is constant.

Proposition 6.7. *Under the conditions on the physical parameters T, α, ρ, k listed above we have:*

(1) The abstract linear system associated with the controlled vibrating string (92), (94) *with output* (96) *yields a non-autonomous boundary control and observation system on $(L^2([a,b];\mathbb{K}^2), \mathbb{K}^1, \mathbb{K}^2)$ if $u_2 = 0$, i.e., when the string is clamped at the end point a, and in $(L^2([a,b];\mathbb{K}^2), \mathbb{K}^2, \mathbb{K}^2)$ if $u_2 \neq 0$.*

(2) Let $\omega_0, \omega_1 \in H^1(a,b;\mathbb{K})$ be such that $k\omega_0(b) + \omega_1(b) = u_1(0)$ and $\omega_0(a) = u_2(0)$. Then (92)-(94) *with output equation* (96) *and initial conditions*

$$\alpha(0)\rho(0,\cdot)\frac{\partial w}{\partial t}(0,\cdot) = \omega_0, \qquad \frac{\partial \omega}{\partial t}(0,\cdot) = \omega_1$$

has a unique solution (ω, y) such that $y \in C([0,\infty);\mathbb{K}^2)$ and

$$t \longmapsto \begin{bmatrix} \alpha(t)\frac{\partial w}{\partial t} \\ T(t,\cdot)\frac{\partial w(t,\cdot)}{\partial \zeta} \end{bmatrix} \in C^1\big((0,\infty);L^2(a,b;\mathbb{K}^2)\big) \cap C\big([0,\infty);L^2(a,b;\mathbb{K}^2)\big).$$

(3) Let $u_2 \neq 0$. Let $R(t), J(t)$ be self adjoint 2×2-matrices such that $R \in L^\infty_{Loc}([0,\infty);\mathcal{L}(\mathbb{K}^2))$ and $c_0^{-1} \leq J(t) \leq c_0$ for all $t \geq 0$ and some constant $c_0 > 0$. Choose V, S in (96) *such that* (86) *holds for all $t \geq 0$. Then the linear system associated with the non-autonomous controlled vibrating string* (92)-(94) *and* (96) *is a well-posed non-autonomous boundary control and observation system.*

(4) Assume that $\alpha(t) \equiv \alpha > 0$ is constant such that the assumptions in (3) *hold. Let $\omega_0, \omega_1 \in L^2(a,b;\mathbb{C})$. Then* (92)-(94) *with initial conditions*

$$\alpha(0)\rho(0,\cdot)\frac{\partial w}{\partial t}(0,\cdot) = \omega_0, \qquad \frac{\partial \omega}{\partial t}(0,\cdot) = \omega_1$$

has a unique (mild) solution ω such that

$$t \longmapsto \begin{bmatrix} \alpha(t)\frac{\partial w}{\partial t} \\ T(t,\cdot)\frac{\partial w(t,\cdot)}{\partial \zeta} \end{bmatrix} \in C\big([0,\infty);L^2(a,b;\mathbb{K}^2)\big).$$

6.2. **Timoschenko beam.** Consider the following model of the Timoshenko beam with time-dependent coefficient and time dependent boundary control

$$\frac{\partial}{\partial t}\big(\tilde\rho(t)\rho(t,\zeta)\frac{\partial w}{\partial t}(t,\zeta)\big) = \frac{1}{\tilde\rho(t)}\frac{\partial}{\partial \zeta}\Big[K(t,\zeta)\Big(\frac{\partial}{\partial \zeta}w(t,\zeta) + \phi(t,\zeta)\Big)\Big] \tag{97}$$

$$\frac{\partial}{\partial t}\big(\tilde I_\rho(t) I_\rho(t,\zeta)\frac{\partial \phi}{\partial t}(t,\zeta)\big) = \frac{1}{\tilde I_\rho(t)}\frac{\partial}{\partial \zeta}\Big(EI(t,\zeta)\frac{\partial^2}{\partial \zeta}\phi(t,\zeta)\Big) + \frac{1}{\tilde\rho(t)}K(t,\zeta)\Big(\frac{\partial}{\partial \zeta}w(t,\zeta) - \phi(t,\zeta)\Big) \tag{98}$$

$$\frac{\partial w}{\partial t}(t,a) = u_1, \qquad t \geq 0 \tag{99}$$

$$\frac{\partial \phi}{\partial t}(t,a) = u_2, \qquad t \geq 0 \tag{100}$$

$$K(t,b)\Big[\frac{\partial w}{\partial \zeta}(t,b) - \phi(t,b)\Big] + \alpha_1 \tilde{\rho}(t)\frac{\partial w}{\partial t}(t,b) = u_3, \qquad t \geq 0 \tag{101}$$

$$EI(t,b)\frac{\partial \phi}{\partial \zeta}(t,b) + \alpha_2 \tilde{I}_\rho(t)\frac{\partial \phi}{\partial t}(t,b) = u_4, \qquad t \geq 0 \tag{102}$$

for some positive constants $\alpha_1, \alpha_2 \geq 0$. Here $\zeta \in (a,b)$, $t \geq 0$, $w(t,\zeta)$ is the transverse displacement of the beam and $\phi(t,\zeta)$ is the rotation angle of the filament of the beam. We assume that K, ρ, EI, $I_\rho \in C^2([0,\infty); L^\infty(a,b)) \cap C_b([0,\infty); L^\infty(a,b))$ and there exists $m>0$ such that for a.e $\zeta \in [a,b]$ and all $t \geq 0$ we have

$$m^{-1} \leq \rho(t,\zeta), K(t,\zeta), EI, I_\rho \leq m,$$

where $\rho(t,\zeta)$ and I_ρ are strictly positive. Moreover, $\tilde{\rho}, \tilde{I}_\rho \in C^1([0,\infty))$ are strictly positive.
Indeed, taking as state variable $x := (\frac{\partial w}{\partial \zeta} - \phi, \tilde{\rho}\rho\frac{\partial w}{\partial t}, \frac{\partial \phi}{\partial \zeta}, \tilde{I}_\rho I_\rho \frac{\partial \phi}{\partial t})$ one can easily see that (97)-(98) can be written as a system of the form (76)-with

$$P_1 = \begin{bmatrix} 0 & \tilde{\rho}^{-1} & 0 & 0 \\ \tilde{\rho}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{I}_\rho^{-1} \\ 0 & 0 & \tilde{I}_\rho^{-1} & 0 \end{bmatrix}, \; P_0 = \begin{bmatrix} 0 & 0 & 0 & -\tilde{I}_\rho^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{I}_\rho^{-1} & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathcal{H} = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \rho^{-1} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & I_\rho^{-1} \end{bmatrix}.$$

The boundary condition can be formulated as follows

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha_2 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H}(t,b)x \\ \mathcal{H}(t,a)x \end{bmatrix} =: W_B \begin{bmatrix} \mathcal{H}(t,b)x \\ \mathcal{H}(t,a)x \end{bmatrix}$$

Thus W_B has full rank and the corresponding 4×8 matrix $W_B(t)$ is given by

$$W_B(t) = W_B \begin{bmatrix} P_1^{-1}(t) & I \\ -P_1^{-1}(t) & I \end{bmatrix} = \begin{bmatrix} -\tilde{\rho}(t) & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\tilde{I}_\rho(t) & 0 & 0 & 0 & 0 & 1 \\ \alpha_1\tilde{\rho}(t) & \tilde{\rho}(t) & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \tilde{I}_\rho(t) & \tilde{I}_\rho(t) & 0 & 0 & 1 & \alpha_2 \end{bmatrix}.$$

Thus $W_B(t)\Sigma W_B^*(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\alpha_1\tilde{\rho}(t) & 0 \\ 0 & 0 & 0 & \alpha_2 \tilde{I}_\rho(t) \end{bmatrix} \geq 0$. As in Example 6.1, the output equation can be choosing similarly as (96). Thus the above Timoshenko beam fit into the framework of port-Hamiltonian system and thus one obtain a similar results to that presented in Proposition 6.7.

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