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# A new methodology to create valid time-dependent correlation matrices via isospectral flows

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## Abstract

In many areas of finance and of risk management it is interesting to know how to specify time-dependent correlation matrices. In this work we propose a new methodology to create valid time-dependent instantaneous correlation matrices, which we called correlation flows. In our methodology one needs only an initial correlation matrix to create these correlation flows based on isospectral flows. The tendency of the time-dependent matrices can be controlled by requirements. An application example is presented to illustrate our methodology.

***Keywords*** *time-dependent correlation matrix, isospectral flow, matrix differential equation*

## 1 Introduction

In finance and risk management it is very interesting to know how to specify time-dependent instantaneous correlation matrices using real market data. We should naturally recover the real-world correlation matrices. However, the task is not as easy as it might seem, even only for specifying a constant correlation matrix. It is well known that a valid correlation matrix is a real symmetric matrix with the following constraints (i.e., properties):

1) all diagonal elements are equal to one and absolute values of all non-diagonal elements are less than one,

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2) non-negative eigenvalues (positive semidefinite).

For example, the estimated correlation using stock data over a period of time may fail to be semidefinite due to some missing data. In particular, a risk manager wishes to assess the effect on a portfolio of adjusting the correlations between underlying assets, which can be different with those estimated from the historical data. The negative eigenvalues can thus be brought in. In the literature there are numerous methods solving this problem. The basic idea to find a nearest valid correlation matrix, which should approximate the true correlation matrix "perfectly" as well. The technique proposed in [Finger, 1997] is to increase portions of the correlation matrix. However, as commented in [Rebonato and Jäckel, 2000], the drawback is that other portions of the matrix can be changed in an uncontrolled fashion. The shrinkage method proposed by Kupiec in [Kupiec, 1998] has the main drawback that "there is no way of determining to what extent the resulting matrix is optimal in any easily quantifiable sense", see [Rebonato and Jäckel, 2000]. Furthermore, the hyperspherical decomposition method and the unconstrained convex optimization approach are proposed in [Rebonato and Jäckel, 2000] and [Qi and Sun, 2010], respectively. Some Newton-based methods can be found in [Boyd and Xiao, 2005, Malick, 2004, Qi and Sun, 2006]. And many others, see e.g., [Bhansali and Wise, 2001, Kercheval, 2008, León et al., 2002, Dash, 2004, Rapisarda et al., 2007, Turkay et al., 2003, Higham, 2002]. Note that all of the mentioned methods can address the constraints 1) and 2). Some of those methods can also address more constraints, e.g., some correlations with specified indices $(i, j)$ in the current matrix (estimated based on the historical data) must be kept in the target correlation matrix as well during the correlation stress testing.

In this work we develop a methodology based on isospectral flows to create valid time-dependent instantaneous correlation matrices (correlation flows), i.e., the correlation matrices at each time point satisfy constraints 1) and 2). The specification of valid time-dependent instantaneous correlation matrices is still an important application in finance. In [Teng et al., 2015a, Teng et al., 2015b, Teng et al., 2016a] an instantaneous time-dependent correlation function is proposed, with applications to finance the authors have shown that a nonconstant correlation is more realistic. If one uses the proposed correlation function to construct time-dependent correlation matrices, the constraint 1) will be fulfilled automatically for each time point. However, in this way, it cannot be guaranteed that the constraint 2) are enabled for all the time points.

With our new methodology we are able to create correlation flows starting from an initial correlation matrix. Furthermore, in our methodology we cannot only control the tendency of the correlation flows but also let the flows to provide the assigned correlation values at some time points. One possible application of our methodology is the specification of correlation flows in a time interval only with known correlation values at a few time points. For example, if one knows the correlation matrix between underlying assets in a portfolio today (at the time $t$) and is also aware of (or expect) the correlation matrix at the future time $T$. Our methodology can tell us how the correlation flows move from $t$ to $T$, i.e., we can obtain valid correlation matrices for all time points.

In the next section we start with an introduction to the isospectral flows and show in Section 3 how to create valid time-dependent instantaneous covariance matrices which are called covariance flows based on the isospectral flows. In Section 4, we show the possible practical applications of our methodology. Finally, Section 5 concludes this work.

## 2 Isospectral Flows

We use $\mathcal{G}(n)$ to denote the Lie group of all nonsingular matrices in $\mathbb{R}^{n\times n}$, for this we refer to e.g., [Curtis, 1979, Helgason, 1978, Warner, 1983]. We then define an isospectral surface

$$\mathcal{M}(X_0) := \{Z^{-1}X_0Z | Z \in \mathcal{G}(n)\} \tag{1}$$

with the given $X_0 \in \mathbb{R}^{n\times n}$. Note that the matrices in $\mathcal{M}(X_0)$ are similar to $X_0$ and thus have the same kind of geometric multiplicity as $X_0$. Suppose that $Z(t)$, with $Z(0) = 1$, represents a differential curve on the manifold $\mathcal{G}(n)$, one thus obtains

$$X(t) := Z(t)^{-1}X_0Z(t) \tag{2}$$

which defines a differentiable curve on the surface $\mathcal{M}(X_0)$. Clearly, $X(0) = X_0$. The curve $X(t)$ is the solution of the initial value problem [Chu, 1992]

$$\begin{cases} \frac{dX(t)}{dt} = [X(t), k(t)], \quad t \geq 0 \\ X(0) = X_0, \end{cases} \tag{3}$$

where $[X(t), k(t)] := X(t)k(t) - k(t)X(t)$ denotes the Lie bracket and $k(t)$ is defined by

$$k(t) := Z(t)^{-1}\frac{dZ(t)}{dt}. \tag{4}$$

Conversely, if $k(t) \in \mathbb{R}^{n\times n}$ is known, one can find that the solution of (3) can be formulated in the form of (2), where $Z(t)$ satisfies

$$\begin{cases} \frac{dZ(t)}{dt} = Z(t)k(t), \quad t \geq 0 \\ Z(0) = I. \end{cases} \tag{5}$$

Therefore, (5) is called the dual problem of (3), see [Chu, 1992]. Note that different isospectral curves can be defined by (3) with different values of $k(t)$, the asymptotic behavior of $X(t)$ on the surface $\mathcal{M}(X_0)$ is related to that of the corresponding $Z(t)$ on the manifold $\mathcal{G}(n)$.

## 3 Covariance and Correlation Flows

Firstly, our purpose is to create covariance flows $P(t)$, which must be positive semi-definite for all $t \geq 0$. Applying the singular value decomposition (SVD) one obtains

$$P(t) = Q(t)^\top S(t)Q(t), \tag{6}$$

where $Q(t)$ is an unitary matrix consisting of the singular vectors and $S(t)$ is a diagonal matrix consisting of the singular values of $P(t)$. In fact, without loss of generality, we can assume that $Q(t)$ is a rotation matrix whose determinant is always 1, i.e., $|Q(t)| = 1$. Since if $|Q(t)| = -1$, one can rewrite (6) into

$$P(t) = \frac{Q(t)}{|Q(t)|}^{\top} S(t) \frac{Q(t)}{|Q(t)|}, \tag{7}$$

where $\frac{Q(t)}{|Q(t)|}$ is a rotation matrix.

If one replaces $\mathcal{G}(n)$ by its subgroup $\mathcal{O}(n)$ of all orthogonal matrices, i.e.,

$$\widetilde{\mathcal{M}}(X_0) := \{Q^{\top} X_0 Q | Q \in \mathcal{O}(n)\}. \tag{8}$$

Upon differentiation, it is clear that the covariance flows

$$P(t) = Q(t)^{\top} P_0 Q(t) \tag{9}$$

is the solution of the initial value problem

$$\begin{cases} \frac{dP(t)}{dt} = [P(t), k(t)], \quad t \geq 0 \\ P(0) = Q(0)^{\top} S(0) Q(0) := Q_0^{\top} S_0 Q_0 := P_0, \end{cases} \tag{10}$$

with

$$\begin{cases} \frac{dQ(t)}{dt} = Q(t) k(t), \quad t \geq 0 \\ Q(0) = I. \end{cases} \tag{11}$$

### 3.1 In the Commutative Case

Clearly, whenever the matrices $k(t)$ and $\int_0^t k(s)\,ds$ commute, i.e., $\left[k(t), \int_0^t k(s)\,ds\right] = 0$, the unique solution of (11) is

$$Q(t) = e^{\int_0^t k(s)\,ds}. \tag{12}$$

Since $Q(t)$ are orthogonal matrices, actually rotation matrices for any $t$, $\int_0^t k(s)\,ds$ must thus be skew-symmetric.

In the following we show how to control the flow $P(t)$, e.g., by

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} Q(t)^{\top} P_0 Q(t) := P^*. \tag{13}$$

In this work, $P^*$ is supposed to be the target covariance matrix whose singular values must be equal to $S_0$, since the covariance flows are modelled as isospectral flows. This is to say that the SVD of $P^*$ must hold as

$$P^* = Q^{*\top} S^* Q^* \stackrel{!}{=} Q^{*\top} S_0 Q^*, \tag{14}$$

The problem becomes how to construct $Q(t)$. By combining (13) and (14) we see that we need to choose $Q(t)$ such that

$$\lim_{t\to\infty} Q_0 Q(t) = Q^*. \tag{15}$$

Together with (12) we actually need to find $k(t)$ such that

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} e^{\int_0^t k(s)\,ds} := \lim_{t\to\infty} e^{B(t)} := e^B = Q_0^\top Q^*, \tag{16}$$

where $B(0) = \mathbf{0}$ due to $Q(0) = I$. Since $Q_0$ and $Q^*$ both are rotation matrices, there exists a skew-symmetric matrix $B$ which fulfills $e^B = Q_0^\top Q^*$.

Given $Q_0$ and $Q^*$, which are obtained from the initial and target matrices, repectively, we have $B = \log(Q_0^\top Q^*)$. For the covariance flows we then need to find suitable models for $B(t)$, namely $k(t)$. For example, one might find a function $f(t)$ with $f(0) = 0$ such that

$$B(t) = Bf(t) \text{ and } k(t) = Bf(t)', \tag{17}$$

where $B(t)$ and $k(t)$ satisfy all the properties mentioned above. The corresponding correlation flows can be obtained by converting the covariance flow.

### 3.2 In the Non-commutative Case

Generally, the matrices $k(t)$ and $\int_0^t k(s)\,ds$ do not commute, i.e., $\left[k(t), \int_0^t k(s)\,ds\right] \neq 0$. One can solve (11) numerically, e.g., using the methods based on the Magnus series Expansion [Magnus, 1954]. The solution of (11) can be given by $Q(t) = e^{\Omega(t)}$ with $\Omega(t)$ defined by

$$\frac{d\Omega}{dt} = d\exp_\Omega^{-1}(k(t)), \quad \Omega(0) = 0. \tag{18}$$

where $\|\Omega\| < \pi$ and $d\exp_\Omega^{-1}(k(t)) := \sum_{j\geq 0} \frac{\mathcal{B}_j}{j!}[\Omega(t), k(t)]$ converges, $\mathcal{B}_j$ are the Bernoulli numbers.

However, for our purposes, the analytical solution of (11) in a closed form is desired as well in the non-commutative case. For this we use Ascoli-type solution [Ascoli, 1952], see also [Martin, 1967]. When $\left[k(t), \int_0^t k(s)\,ds\right] \neq 0$, there exists a constant matrix $C$ such that $\left[\tilde{B}(t), k(t)\right] = 0$, with $\tilde{B}(t) = C + \int_0^t k(s)\,ds$. The solution of (11) can thus be given by

$$Q(t) = \mathrm{e}^{\tilde{B}(t)} \cdot K \tag{19}$$

for some constant matrix $K$. Clearly, we need to set $K$ to be $\mathrm{e}^{-C}$ so that $Q(0) = I$. Finally, the solution of (10) is given by

$$P(t) = \left(\mathrm{e}^{\tilde{B}(t)}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{\tilde{B}(t)}\mathrm{e}^{-C}\right). \tag{20}$$

For example, we choose

$$\int_0^t k(s)\,ds = \tanh\left(at + \tanh(bt)\right) \tag{21}$$

and thus

$$k(t) = \left(a + b\,\mathrm{sech}^2(bt)\right)\mathrm{sech}^2\left(at + \tanh(bt)\right). \tag{22}$$

Similar to the way described in the previous section, we can control the covariance flows to the given target matrices as $t \to \infty$ by specifying the matrices $a, b$ and $C$ as parameters. In practice, one needs to control the covariance and correlation flows to the given target matrices at some time points, this can be done analogously as well and will be considered in the following sections.

# 4 Practical Applications

In this section we show an example of how to use our methodology. Suppose that a risk analyst retrieves from the middle office reporting system the correlation matrix of underlyings at $t = 0$ (initial correlation matrix). Moreover, the analyst is aware of how the relations between underlyings will develop. This means that the analyst can be aware of the correlation matrices at a few time points, e.g., at $t = T/2, T$ (target correlation matrices). Then, the question is how to create the valid time-dependent instantaneous correlation matrices for the time interval $[0, T]$ using historical data.

## 4.1 Benchmark using Historical Data

We use the historical prices of S&P 500 index (GSPC), the German stock index (DAX) and the Dow Jones Industrial Average (DJIA) from Jan 04, 2016 to Mai 26, 2017. We compute moving correlations with the windows size of 100 days and obtain moving correlations from May 27, 2016 to May 26, 2017, which are plotted in Figure 1. Naturally, the moving correlations are not appropriate to be the benchmark for our correlation flows. For a sensible benchmark we firstly employ the stochastic correlation process proposed in [Teng et al., 2016d], see also [Teng et al., 2016b, Teng et al., 2016c, Teng et al., 2018], for modelling correlation

$$\frac{d\rho_t}{1-\rho^2} = \left(\kappa(\mu - \mathrm{artanh}(\rho_t)) - \rho_t\sigma^2\right)\, dt + \sigma\, dW_t. \tag{23}$$

We then apply the approach proposed in [Teng et al., 2016d] to estimate the parameters in (23) using the historical data in Figure 1. The results are reported in Table 1, whereas we have taken the first historical correlation values as the initial values. The stochastic correlation process $\rho_t$ given by (23) has the conditional probability density [Teng et al., 2016d]

$$f_\rho(\tilde\rho_{s+\Delta t}|\tilde\rho_s, \kappa, \mu, \sigma) = \sqrt{\frac{a}{b}} \cdot \frac{1}{1-\tilde\rho^2_{s+\Delta t}} \cdot \mathrm{e}^{\frac{-\kappa(\mathrm{artanh}(\tilde\rho_{s+\Delta t}) - \mathrm{artanh}(\tilde\rho_s)\mathrm{e}^{-\kappa\Delta t} - \mu c)^2}{\sigma^2 b}}, \quad s < t \tag{24}$$

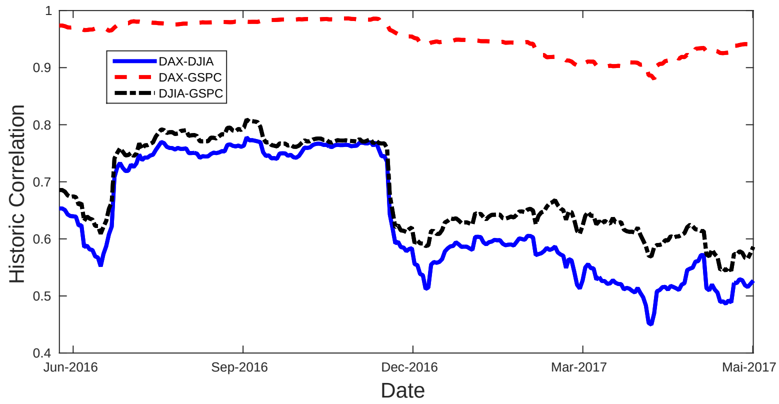


Figure 1: The 100-day historical correlations between GSPC, DAX and DJIA, source of data: www.yahoo.com

| | $\rho_0$ | $\kappa$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|
| DAX-DJIA | 0.653 | 1.322 | 0.620 | 0.345 |
| DAX-GSPC | 0.974 | 0.829 | 1.47 | 0.443 |
| DJIA-GSPC | 0.686 | 1.615 | 0.743 | 0.329 |

Table 1: Estimated stochastic correlation process parameters with the historical data in Figure 1.

with

$$a = \frac{\kappa}{\pi\sigma^2}, \quad b = (1 - \mathrm{e}^{-2\kappa\Delta t}) \quad \text{and} \quad c = (1 - \mathrm{e}^{-\kappa\Delta t}). \tag{25}$$

Therefore, the mean values at each time points can be computed by

$$E[\rho_t] = \int_{-1}^{1} \tilde{\rho} f_\rho(\tilde{\rho}_{s+\Delta t}|\tilde{\rho}_s, \kappa, \mu, \sigma)\, d\tilde{\rho}, \tag{26}$$

which are used as our benchmark. Using the estimated parameter values in Table 1 we plot the computed expected correlations in Figure 2. Note that the 3-dimensional expected correlation matrices theoretically cannot be guaranteed to be positive semidefinite.

### 4.2 Preparation for the construction

For the initial matrix we use again the first historical correlation matrix

$$\mathcal{R}(0) := \mathcal{R}_0 = \begin{pmatrix} 1 & 0.6533 & 0.9738 \\ 0.6533 & 1 & 0.6855 \\ 0.9738 & 0.6855 & 1 \end{pmatrix}, \tag{27}$$

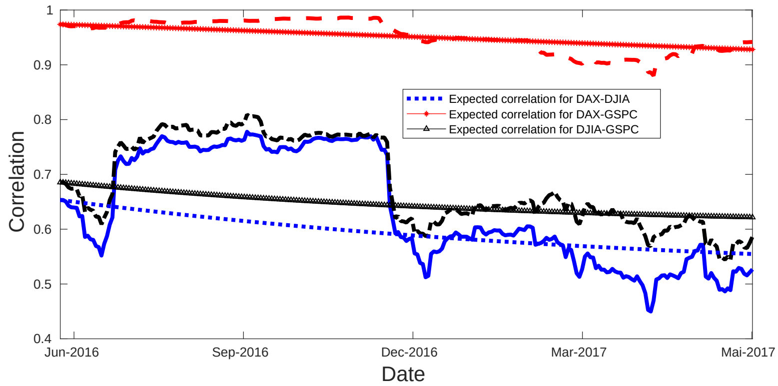


Figure 2: The expected correlations between GSPC, DAX and DJIA computed by (26) with parameter values in Table 1.

which is positive semidefinite. We let the expected correlation matrices at $t = 0.5$ and $t = T = 1$ to be the target matrices

$$\mathcal{R}(0.5) := \mathcal{R}^m = \begin{pmatrix} 1 & 0.5895 & 0.9516 \\ 0.5895 & 1 & 0.6419 \\ 0.9516 & 0.6419 & 1 \end{pmatrix} \tag{28}$$

and

$$\mathcal{R}(1) := \mathcal{R}^* = \begin{pmatrix} 1 & 0.5547 & 0.9280 \\ 0.5547 & 1 & 0.6217 \\ 0.9280 & 0.6217 & 1 \end{pmatrix}, \tag{29}$$

which are both positive semidefinite. In the following, based on those given matrices and the historical data in Figure 1 we create correlation flows applying the proposed methodology in the previous sections. And we will compare the correlation flows to the benchmark, namely expected correlation matrices in Figure 2.

We estimate the covariance matrix of the whole historical data

$$\overline{\Sigma} = \begin{pmatrix} 0.0502\text{e}{-3} & 0.0501\text{e}{-3} & 0.0574\text{e}{-3} \\ 0.0501\text{e}{-3} & 0.0536\text{e}{-3} & 0.0625\text{e}{-3} \\ 0.0574\text{e}{-3} & 0.0625\text{e}{-3} & 0.1534\text{e}{-3} \end{pmatrix} \tag{30}$$

whose SVD reads

$$\overline{\Sigma} = \overline{Q}^\top \overline{SQ}, \tag{31}$$

where

$$\overline{Q} = \begin{pmatrix} -0.4084 & -0.4338 & -0.8031 \\ -0.5825 & -0.5535 & 0.5952 \\ -0.7028 & 0.7109 & -0.0266 \end{pmatrix} \tag{32}$$

and

$$\overline{S} = \begin{pmatrix} 0.2165\text{e}{-3} & 0 & 0 \\ 0 & 0.0391\text{e}{-3} & 0 \\ 0 & 0 & 0.0017\text{e}{-3} \end{pmatrix}. \tag{33}$$

Note that the matrix $\overline{Q}$ is an orthogonal matrix, $\overline{S}$ is a diagonal matrix where the elements are singular values and sorted in descending order. Since the covariance flows is modelled as the isospectral flows, i.e., they must have the same singular values for all the time. Thus, it may be meaningful to keep the singular values of all the covariance matrices be equal to those in $\overline{S}$, which has been computed based on the whole historical data. This criteria allows us to compute the initial and the target covariance matrices using from the correlation matrices, because we need to find such covariance matrices which has singular values $\overline{S}$ and can also be converted to the correlation matrices (27), (28) and (29). More precisely, we need to find the corresponding standard deviations which are needed to compute the initial and target covariance matrices by converting the correlation matrices (27), (28) and (29), whereas all the computed covariance matrices must have the same singular values as those in $\overline{S}$.

One can use e.g., an optimization procedure. We denote the searched covariance matrices by $P_0, P^m$ and $P^*$ which can be determined by minimizing the corresponding errors

$$\epsilon_1 = \|\overline{S} - S_0\|_2^2 = \sum_{ij} \left(\overline{s}_{ij} - s_{0,ij}\right)^2, \tag{34}$$

$$\epsilon_2 = \|\overline{S} - S^m\|_2^2 = \sum_{ij} \left(\overline{s}_{ij} - s_{ij}^m\right)^2, \tag{35}$$

$$\epsilon_3 = \|\overline{S} - S^*\|_2^2 = \sum_{ij} \left(\overline{s}_{ij} - s_{ij}^*\right)^2, \tag{36}$$

by varying the parameters $\sigma_0 = (\sigma_{0,1}\ \sigma_{0,2}\ \sigma_{0,3})^\top, \sigma^m = (\sigma_1^m\ \sigma_2^m\ \sigma_3^m)^\top$ and $\sigma^* = (\sigma_1^*\ \sigma_2^*\ \sigma_3^*)^\top$, respectively. $\overline{s}_{ij}, s_{0,ij}, s_{ij}^m$ and $s_{ij}^*$ are used to denote the elements in the corresponding matrices. Note that all the matrices $\overline{S}, S_0, S^m$ and $S^*$ are diagonal matrices, they are thus just simple optimization problems. Furthermore, since the singular values in $\overline{S}$ are very small, for the optimizations we need to scale these values by multiplying e.g., by 1000. Note that the factor 1000 used to scale in this example works very well. However, depending on the historical data one may need another scale values for the optimizations. In our experiments, the covariance matrices are found as

$$P_0 = \begin{pmatrix} 0.0478\text{e}{-3} & 0.0459\text{e}{-3} & 0.0695\text{e}{-3} \\ 0.0459\text{e}{-3} & 0.1031\text{e}{-3} & 0.0717\text{e}{-3} \\ 0.0695\text{e}{-3} & 0.0717\text{e}{-3} & 0.1062\text{e}{-3} \end{pmatrix}, \tag{37}$$

$$P^m = \begin{pmatrix} 0.0205\text{e}{-3} & 0.0243\text{e}{-3} & 0.0534\text{e}{-3} \\ 0.0243\text{e}{-3} & 0.0828\text{e}{-3} & 0.0725\text{e}{-3} \\ 0.0534\text{e}{-3} & 0.0725\text{e}{-3} & 0.1539\text{e}{-3} \end{pmatrix}, \tag{38}$$

$$P^* = \begin{pmatrix} 0.0132\text{e}{-3} & 0.0177\text{e}{-3} & 0.0435\text{e}{-3} \\ 0.0177\text{e}{-3} & 0.0770\text{e}{-3} & 0.0705\text{e}{-3} \\ 0.0435\text{e}{-3} & 0.0705\text{e}{-3} & 0.1670\text{e}{-3} \end{pmatrix} \tag{39}$$

which correspond to the correlations in (27), (28) and (29) and have same singular values to those in $\overline{S}$. Furthermore, we also report the estimated standard deviations

$$\hat{\sigma}_0 = \begin{pmatrix} 0.0069 \\ 0.0102 \\ 0.0103 \end{pmatrix}, \ \hat{\sigma}^m = \begin{pmatrix} 0.0045 \\ 0.0091 \\ 0.0124 \end{pmatrix} \text{ and } \hat{\sigma}^* = \begin{pmatrix} 0.0036 \\ 0.0088 \\ 0.0129 \end{pmatrix}, \tag{40}$$

### 4.3 Construction of covariance and correlation flows

We start with the SVDs of (37), (38) and (39),

$$P_0 = Q_0^\top S_0 Q_0, \ P^m = Q^{m\top} S^m Q^m \text{ and } P^* = Q^{*\top} S^* Q^*, \tag{41}$$

note that $S_0 = S^m = S^* = \overline{S}$. Based on our model (10) and (11) in the non-commutative case, the covariance flows are given by $P(t) = Q^\top(t) P_0 Q(t) = \left(\mathrm{e}^{C+\int_0^t k(s)\,ds}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\int_0^t k(s)\,ds}\mathrm{e}^{-C}\right)$.

Similar to (15) and (16), we need to find suitable models for $k(t)$ such that the covariance flows $P(t),\ t \in [0,1]$ pass through $P^m$ at $t = 0.5$ and $P^*$ at $t = 1$. For this we use (22), namely

$$k(t) = \left(a + b\,\mathrm{sech}^2(bt)\right) \mathrm{sech}^2\left(at + \tanh(bt)\right). \tag{42}$$

and thus

$$\tilde{B}(t) = C + \tanh\left(at + \tanh(bt)\right). \tag{43}$$

Then, the unknown constant matrices $a, b$ and $C$ in (43) can be obtained by solving

$$\begin{cases} P(0.5) = Q^\top(0.5) P_0 Q(0.5) = P^m, \\ P(1) = Q^\top(1) P_0 Q(1) = P^*, \end{cases} \tag{44}$$

i.e.,

$$\begin{cases} \left(\mathrm{e}^{C+\tanh(0.5a+\tanh(0.5b))}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\tanh(0.5a+\tanh(0.5b))}\mathrm{e}^{-C}\right) = P^m, \\ \left(e^{C+\tanh(a+\tanh(b))}\mathrm{e}^{-C}\right)^\top P_0 \left(e^{C+\tanh(a+\tanh(b))}\mathrm{e}^{-C}\right) = P^*, \end{cases} \tag{45}$$

where $P_0, P^m$ and $P^*$ have been already given in (37), (38) and (39). Our numerical results read

$$a = \begin{pmatrix} 0.2290 & -0.1756 & -0.3114 \\ 0.3222 & -0.2390 & 0.0051 \\ 0.4569 & 0.4342 & 0.1211 \end{pmatrix}, \tag{46}$$

$$b = \begin{pmatrix} -0.2165 & 0.2053 & 1.2017 \\ -0.3834 & 0.2298 & 0.0052 \\ -0.9240 & -0.9742 & -0.1243 \end{pmatrix} \tag{47}$$

and

$$C = \begin{pmatrix} -0.6217 & 1.2092 & 0.4554 \\ 0.8537 & 0.0985 & -0.6926 \\ -0.1354 & 0.4918 & 0.4374 \end{pmatrix} \tag{48}$$

which can be used to compute $\tilde{B}(t)$, namely $Q(t)$. For example, we obtain

$$\tilde{B}(0.5) := \tilde{B}^m = \begin{pmatrix} -0.5476 & 0.8407 & 0.6844 \\ 0.6780 & 0.0931 & -0.5962 \\ -0.3262 & 0.2515 & 0.4102 \end{pmatrix} \tag{49}$$

$$\tilde{B}(1) := \tilde{B}^* = \begin{pmatrix} -0.5412 & 0.8443 & 0.7523 \\ 0.6697 & 0.0851 & -0.5930 \\ -0.3853 & 0.1737 & 0.4094 \end{pmatrix} \tag{50}$$

and

$$Q(0.5) := Q^m = \begin{pmatrix} 0.9668 & -0.0739 & 0.2447 \\ 0.0467 & 0.9922 & 0.1153 \\ -0.2513 & -0.1000 & 0.9627 \end{pmatrix} \tag{51}$$

$$Q(1) := Q^* = \begin{pmatrix} 0.9423 & -0.0994 & 0.3197 \\ 0.0514 & 0.9866 & 0.1550 \\ -0.3308 & -0.1297 & 0.9347 \end{pmatrix} \tag{52}$$

which are both rotation. Finally, the covariance flows can be generated by

$$P(t) = Q(t)^\top P_0 Q(t) = \left(\mathrm{e}^{C+\tanh(at+\tanh(bt))}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\tanh(at+\tanh(bt))}\mathrm{e}^{-C}\right), \tag{53}$$

where the matrices $a, b$ and $C$ are given in (46), (47) and (49), respectively. By converting (53) the corresponding correlation flows can be immediately obtained, which are valid, i.e., all correlation matrices satisfy the constraints 1) and 2) at each time point. In Figure 3 we compare the correlation flows generated by (53) to the benchmark. We observe,

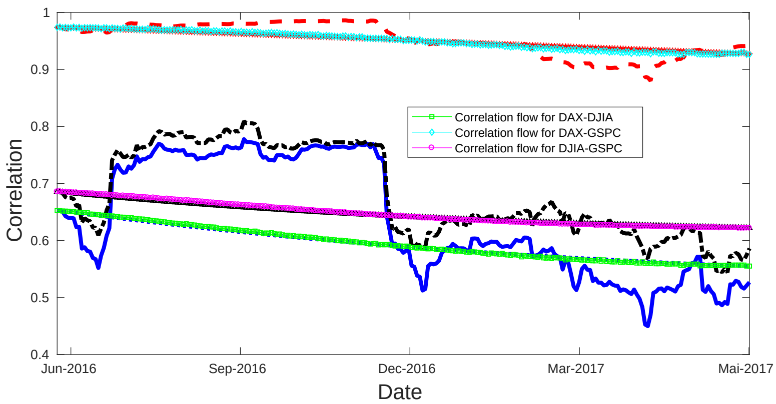


Figure 3: The generated correlation flows between GSPC, DAX and DJIA with (53).

although we only have used two covariance matrices $P^m$ and $P^*$ to construct $B(t)$, namely $Q(t)$ which can control the tendency of matrix flows, the generated correlation flows approach the benchmark quite well. However, we can actually imagine that there should be infinite many different ways of correlation flows moving from the initial matrix to the target matrices. Therefore, this phenomenon shows that, given the initial and target matrices our methodology have generated meaningful correlation flows. To confirm our observations we do exactly the procedure as above but for another historical data, which are GSPC, DAX and the exchanges rates between US Dollar and Euro. Instead of 100-day moving correlations we consider in this experiment 50-day moving correlations. Furthermore, 1 year historical correlations are analyzed, we thus use the historical prices from March 16, 2016 to May 26,2017. We plot all the results in Figure 4, from which the same conclusion as those of the previous example can be drawn.

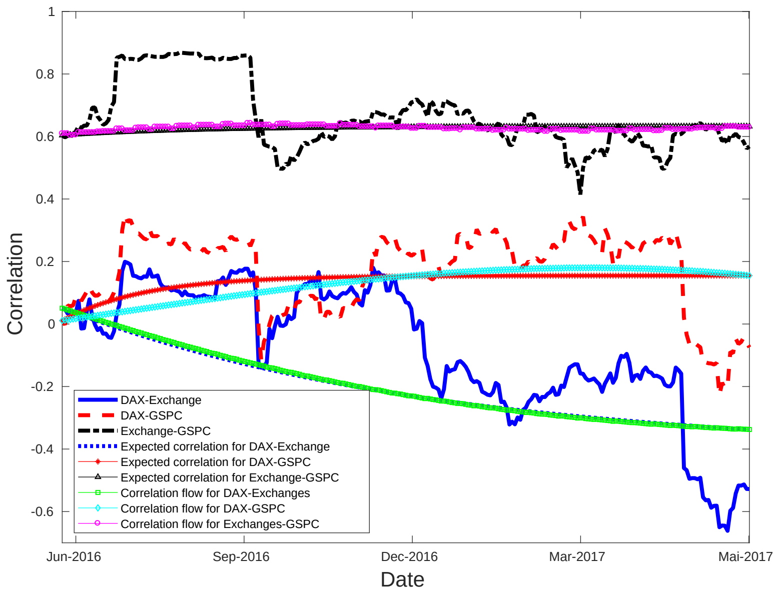


Figure 4: The 50-day historical correlations between GSPC, DAX and the exchange rates (US Dollar/Euro) from 27 May, 2016 to 26 May, 2017, the computed expected correlations and the generated correlation flows, source of data: www.yahoo.com

Note that, in the experiments above we have only considered two target matrices. For a better approximation to the benchmark one can choose more target matrices, similar to (43), e.g.,

$$B(t) := \tanh(at + \tanh(bt) + \tanh(ct) + \tanh(dt) + \cdots). \tag{54}$$

Then, one needs to solve a larger equation system than (44).

## 5 Conclusion

We have proposed a new methodology to create valid time-dependent covariance and correlation matrices (covariance and correlation flows) based on isospectral flows. Given an initial correlation matrix, the tendency of the correlation flows can be controlled by the rotation matrices in the model. For example, one can require that correlation flows should give some correlation matrices which are equal to some prespecified target matrices at some time points or as $t \to \infty$.

As an application, we model correlation as a stochastic process and calibrate the correlation process with the historical data. Then we compute the expected values of correlation processes at each time instant, namely obtain time-dependent expected correlation matrices based on the historical data, which are taken as benchmark. From the benchmark we choose the initial and two target matrices, from which we determine the rotation matrices which are used to generate the flows. By comparing the generated correlation flows to the benchmark, we find that the correlation flows are meaningful in the sense of expected correlation values at each time instant. Many more applications are expected to show the ability of our model, which is regarded as future work.

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