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Essentially high-order compact schemes with application to stochastic volatility models on non-uniform grids

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Abstract

We present high-order compact schemes for a linear second-order parabolic partial differential equation (PDE) with mixed second-order derivative terms in two spatial dimensions. The schemes are applied to option pricing PDE for a family of stochastic volatility models. We use a non-uniform grid with more grid-points around the strike price. The schemes are fourth-order accurate in space and second-order accurate in time for vanishing correlation. In our numerical convergence study we achieve fourth-order accuracy also for non-zero correlation. A combination of Crank-Nicolson and BDF-4 discretisation is applied in time. Numerical examples confirm that a standard, second-order finite difference scheme is significantly outperformed.

1 Introduction

We consider the following parabolic partial differential equation for $u = u(x_1, x_2, t)$ in two spatial dimensions and time,

$$du_\tau + a_1 u_{x_1x_1} + a_2 u_{x_2x_2} + b_{12} u_{x_1x_2} + c_1 u_{x_1} + c_2 u_{x_2} = 0 \quad \text{in } \Omega\times]0,T] =: Q_T, \tag{1}$$

subject to suitable boundary conditions and initial condition $u(x_1, x_2, 0) = u_0(x_1, x_2)$ with $T > 0$ and $\Omega = \left[x_{\min}^{(1)}, x_{\max}^{(1)}\right] \times \left[x_{\min}^{(2)}, x_{\max}^{(2)}\right] \subset \mathbb{R}^2$ with $x_{\min}^{(i)} < x_{\max}^{(i)}$ for $i = 1, 2$. The functions $a_i = a_i(x_1, x_2, \tau) < 0$, $b_{12} = b_{12}(x_1, x_2, \tau)$, $c_i = c(x_1, x_2, \tau)$, $d = d(x_1, x_2, \tau)$ map Q_T to $\mathbb{R}$, and $a_i(\cdot, \tau)$, $b(\cdot, \tau)$, $c_i(\cdot, \tau)$, and $d(\cdot, \tau)$ are assumed to be in $C^2(\Omega)$ and $u(\cdot, t) \in C^6(\Omega)$ for all $\tau \in]0, T]$. We define a uniform spatial grid G with step size Δx_k in x_k direction for $k = 1, 2$. Setting $f = -du_\tau$ and applying a standard, second-order central difference approximation leads to the elliptic problem

$$\begin{aligned} f =& A_0 - \frac{a_1(\Delta x_1)^2}{12}\frac{\partial^4 u}{\partial x_1^4} - \frac{a_2(\Delta x_2)^2}{12}\frac{\partial^4 u}{\partial x_2^4} - \frac{b_{12}(\Delta x_1)^2}{6}\frac{\partial^4 u}{\partial x_1^3\partial x_2} \\ & - \frac{b_{12}(\Delta x_2)^2}{6}\frac{\partial^4 u}{\partial x_1\partial x_2^3} - \frac{c_1(\Delta x_1)^2}{6}\frac{\partial^3 u}{\partial x_1^3} - \frac{c_2(\Delta x_2)^2}{6}\frac{\partial^3 u}{\partial x_2^3} + \varepsilon, \end{aligned} \tag{2}$$

with $A_0 := a_1 D_1^c D_1^c U_{i_1,i_2} + a_2 D_2^c D_2^c U_{i_1,i_2} + b_{12} D_1^c D_2^c U_{i_1,i_2} + c_1 D_1^c U_{i_1,i_2} + c_2 D_2^c U_{i_1,i_2}$, where D_k^c denotes the central difference operator in x_k direction, and $\varepsilon \in \mathcal{O}\left(h^4\right)$ if $\Delta x_k \in \mathcal{O}(h)$ for $h > 0$. We call a finite difference scheme high-order compact (HOC) if its consistency error is of order $\mathcal{O}\left(h^4\right)$ for $\Delta x_1, \Delta x_2 \in \mathcal{O}(h)$ for $h > 0$, and it uses only points on the compact stencil, $U_{k,p}$ with $k \in \{i_1 - 1, i_1, i_1 + 1\}$ and $p \in \{i_2 - 1, i_2, i_2 + 1\}$, to approximate the solution at $(x_{i_1}, x_{i_2}) \in \mathring{G}$.

2 Auxiliary relations for higher derivatives

Our aim is to replace the third- and fourth-order derivatives in (2) which are multiplied by second-order terms by equivalent expressions which can be approximated with second order on the compact

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stencil. Indeed, if we differentiate (1) (using $f = -du_\tau$) once with respect to x_k ($k = 1, 2$), we obtain relations

$$\frac{\partial^3 u}{\partial x_1^3} = A_1, \qquad \frac{\partial^3 u}{\partial x_2^3} = A_2, \tag{3}$$

where we can discretise A_i with second order on the compact stencil using the central difference operator. Analogously, we obtain

$$\begin{aligned}
\frac{\partial^4 u}{\partial x_1^4} &= B_1 - \frac{b_{12}}{a_1}\frac{\partial^4 u}{\partial x_1^3 \partial x_2} &\iff&& \frac{\partial^4 u}{\partial x_1^3 \partial x_2} &= \frac{a_1}{b_{12}} B_1 - \frac{a_1}{b_{12}}\frac{\partial^4 u}{\partial x_1^4},\\
\frac{\partial^4 u}{\partial x_2^4} &= B_2 - \frac{b_{12}}{a_2}\frac{\partial^4 u}{\partial x_1 \partial x_2^3} &\iff&& \frac{\partial^4 u}{\partial x_1 \partial x_2^3} &= \frac{a_2}{b_{12}} B_2 - \frac{a_2}{b_{12}}\frac{\partial^4 u}{\partial x_2^4},\\
\frac{\partial^4 u}{\partial x_1^3 \partial x_2} &= C_1 - \frac{a_2}{a_1}\frac{\partial^4 u}{\partial x_1 \partial x_2^3} &\iff&& \frac{\partial^4 u}{\partial x_1 \partial x_2^3} &= C_2 - \frac{a_1}{a_2}\frac{\partial^4 u}{\partial x_1^3 \partial x_2},
\end{aligned} \tag{4}$$

where we can approximate B_k and C_k with second order on the compact stencil using the central difference operator. A detailed derivation can be found in [4].

3 Derivation of high-order compact schemes

In general it is not possible to obtain a HOC scheme for (1), since there are four fourth-order derivatives in (2), but only three auxiliary equations for these in (4). Hence, we propose four different versions of the numerical schemes, where only one of the fourth-order derivatives in (2) is left as a second-order remainder term. Using (3) and (4) in (2) we obtain as *Version 1* scheme

$$\begin{aligned}
f =& A_0 - \frac{c_1(\Delta x_1)^2}{6} A_1 - \frac{c_2(\Delta x_2)^2}{6} A_2 - \frac{a_2(\Delta x_2)^2}{12} B_2 - \frac{b_{12}(\Delta x_2)^2}{12} C_2 \\
&- \frac{a_1\left(2a_2(\Delta x_1)^2 - a_1(\Delta x_2)^2\right)}{12a_2} B_1 + \frac{a_1\left(a_2(\Delta x_1)^2 - a_1(\Delta x_2)^2\right)}{12a_2}\frac{\partial^4 u}{\partial x_1^4} + \varepsilon,
\end{aligned} \tag{5}$$

as *Version 2* scheme

$$\begin{aligned}
f =& A_0 - \frac{c_1(\Delta x_1)^2}{6} A_1 - \frac{c_2(\Delta x_2)^2}{6} A_2 - \frac{a_1(\Delta x_1)^2}{12} B_1 - \frac{b_{12}(\Delta x_1)^2}{12} C_1 \\
&- \frac{a_2\left(2a_1(\Delta x_2)^2 - a_2(\Delta x_1)^2\right)}{12a_1} B_2 + \frac{a_2\left(a_1(\Delta x_2)^2 - a_2(\Delta x_1)^2\right)}{12a_1}\frac{\partial^4 u}{\partial x_2^4} + \varepsilon,
\end{aligned} \tag{6}$$

as *Version 3* scheme

$$\begin{aligned}
f =& A_0 - \frac{c_1(\Delta x_1)^2}{6} A_1 - \frac{c_2(\Delta x_2)^2}{6} A_2 - \frac{a_1(\Delta x_1)^2}{12} B_1 - \frac{a_2(\Delta x_2)^2}{12} B_2 \\
&- \frac{b_{12}(\Delta x_2)^2}{12} C_2 + \frac{b_{12}\left(a_1(\Delta x_2)^2 - a_2(\Delta x_1)^2\right)}{12a_2}\frac{\partial^4 u}{\partial x_1^3 \partial x_2} + \varepsilon,
\end{aligned} \tag{7}$$

and, finally, as *Version 4* scheme

$$\begin{aligned}
f =& A_0 - \frac{c_1(\Delta x_1)^2}{6} A_1 - \frac{c_2(\Delta x_2)^2}{6} A_2 - \frac{a_1(\Delta x_1)^2}{12} B_1 - \frac{a_2(\Delta x_2)^2}{12} B_2 \\
&- \frac{b_{12}(\Delta x_1)^2}{12} C_1 + \frac{b_{12}\left(a_2(\Delta x_1)^2 - a_1(\Delta x_2)^2\right)}{12a_1}\frac{\partial^4 u}{\partial x_1 \partial x_2^3} + \varepsilon.
\end{aligned} \tag{8}$$

Employing the central difference operator with $\Delta x = \Delta y = h$ for $h > 0$ to discretise A_i, B_i, C_i, in (5)–(8) and neglecting the remaining lower-order term leads to four semi-discrete (in space) schemes. A more detailed description of this approach can be found in [4]. When $a_1 \equiv a_2$ or $b_{12} \equiv 0$ these schemes are fourth-order consistent in space, otherwise second-order.

In time, we apply the implicit BDF4 method on an equidistant time grid with stepsize $k \in \mathcal{O}(h)$. The necessary starting values are obtained using a Crank-Nicolson time discretisation, where we subdivide the first timesteps with a step size $k' \in \mathcal{O}\left(h^2\right)$ to ensure the fourth-order time discretisation in terms of h.

With additional information on the solution of (1) even better results are possible. If the specific combination of pre-factors in (1) and the higher derivatives in the second-order terms is sufficiently small, the second-order term dominates the computational error only for very small step-sizes h. Before this error term becomes dominant one can observe a fourth-order numerical convergence. In this case we call the scheme essentially high-order compact (EHOC).

4 Application to option pricing

In this section we apply our numerical schemes to an option pricing PDE in a family of stochastic volatility models, with a generalised square root process for the variance with nonlinear drift term,

$$dS_t = \mu S_t \mathrm{d}t + \sqrt{v_t} S_t \mathrm{d}W_t^{(1)}, \qquad dv_t = \kappa v_t^{\alpha} \left(\theta - v_t\right) \mathrm{d}t + \sigma \sqrt{v_t} \mathrm{d}W_t^{(2)},$$

with $\alpha \geq 0$, a correlated, two-dimensional standard Brownian motion, $dW_t^{(1)} dW_t^{(2)} = \rho dt$, as well as drift $\mu \in \mathbb{R}$ of the stock price S, long run mean $\theta > 0$, mean reversion speed $\kappa > 0$, and volatility of volatility $\sigma > 0$. For $\alpha = 0$ one obtains the standard Heston model, for $\alpha = 1$ the SQRN model, see [1]. Using Itô's lemma and standard arbitrage arguments, the option price $V = V(S, v, t)$ solves

$$\frac{\partial V}{\partial t} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma v S \frac{\partial^2 V}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} + \kappa v^{\alpha}\left(\theta - v\right)\frac{\partial V}{\partial v} - rV = 0, \tag{9}$$

where $S, \sigma > 0$ and $t \in [0, T[$ with $T > 0$. For a European Put with exercise price K we have the final condition $V(S, T) = \max\left(K - S, 0\right)$. The transformations $\tau = T - t$, $u = e^{r\tau} V/K$, $\hat{S} = \ln(S/K)$, $y = v/\sigma$ as well as $\hat{S} = \varphi\left(x\right)$ [2], lead to

$$\varphi_x^3 u_\tau + \frac{\sigma y}{2}\left[\varphi_x u_{xx} + \varphi_x^3 u_{yy}\right] - \rho\sigma y \varphi_x^2 u_{xy} + \left[\frac{\sigma y \varphi_{xx}}{2} + \left(\frac{\sigma y}{2} - r\right)\varphi_x^2\right] u_x - \kappa\sigma^{\alpha} y^{\alpha}\frac{\theta - \sigma y}{\sigma}\varphi_x^3 u_y = 0,$$

with initial condition $u(x, y, 0) = \max\left(1 - e^{\varphi(x)}, 0\right)$. The function φ is considered to be four times differentiable and strictly monotone. It is chosen in such a way that grid points are concentrated around the exercise price K in the S–v plane when using a uniform grid in the x–y plane.

Dirichlet boundary conditions are imposed at $x = x_{\min}$ and $x = x_{\max}$ similarly as in [2],

$$u(x_{\min}, y, \tau) = u(x_{\min}, y, 0), \quad u(x_{\max}, y, \tau) = u(x_{\max}, y, 0) \quad \forall\, \tau \in [0, \tau_{\max}] \quad \forall\, y \in [y_{\min}, y_{\max}].$$

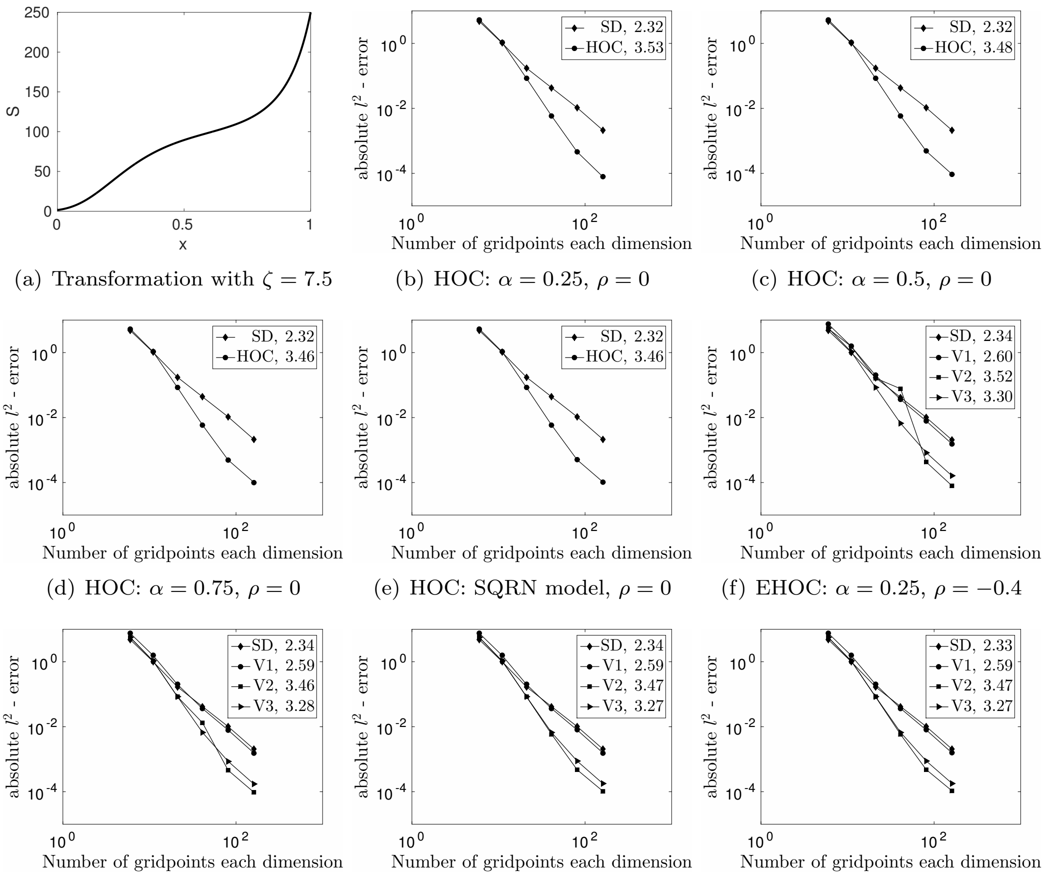
At the boundaries $y = y_{\min}$ and $y = y_{\max}$ we employ the discretisation of the interior spatial domain and extrapolate the resulting ghost-points using

$$U_{i,-1} = 3U_{i,0} - 3U_{i,1} + U_{i,2} + \mathcal{O}\left(h^3\right), \quad U_{i,M+1} = 4U_{i,M} - 6U_{i,M-1} + 4U_{i,M-2} - U_{i,M-3} + \mathcal{O}\left(h^4\right),$$

for $i = 0, \ldots, N$. Third-order extrapolation is sufficient here to ensure overall fourth-order convergence [3].

5 Numerical experiments

We employ the function $\varphi(x) = \sinh(c_2 x + c_1(1-x))/\zeta$, where $c_1 = \operatorname{asinh}(\zeta \hat{S}_{\min})$, $c_2 = \operatorname{asinh}(\zeta \hat{S}_{\max})$ and $\zeta > 0$. We use $\kappa = 1.1$, $\theta = 0.2$, $v = 0.3$, $r = 0.05$, $K = 100$, $T = 0.25$, $v_{\min} = 0.1$, $v_{\max} = 0.3$, $S_{\min} = 1.5$, $S_{\max} = 250$, $\rho = 0, -0.4$ and $\zeta = 7.5$. Hence, $x_{\max} - x_{\min} = y_{\max} - y_{\min} = 1$. For the Crank-Nicolson method we use $k'/h^2 = 0.4$, for the BDF4 method $k/h = 0.1$. We smooth the initial condition according to [5], so that the smoothed initial condition tends towards the original initial condition for $h \to 0$. We neglect the case $\alpha = 0$ (Heston model), since a numerical study of that case has been performed in [2]. In the numerical convergence plots we use a reference solution U_{ref} on a fine grid ($h = 1/320$) and report the absolute l^2-error compared to U_{ref}. The numerical convergence order is computed from the slope of the linear least square fit of the points in the log-log plot.



(a) Transformation with $\zeta = 7.5$ (b) HOC: $\alpha = 0.25$, $\rho = 0$ (c) HOC: $\alpha = 0.5$, $\rho = 0$

(d) HOC: $\alpha = 0.75$, $\rho = 0$ (e) HOC: SQRN model, $\rho = 0$ (f) EHOC: $\alpha = 0.25$, $\rho = -0.4$

(g) EHOC: $\alpha = 0.5$, $\rho = -0.4$ (h) EHOC: $\alpha = 0.75$, $\rho = -0.4$ (i) EHOC: SQRN model, $\rho = -0.4$

Figure 1: Transformation of the spatial grid and numerical convergence plots.

Figure 1(a) shows the transformation from x to S. The transformation focuses on the region around the strike price. Figures 1(b), 1(c), 1(d) and 1(e) show that the HOC schemes lead to a numerical convergence order of about 3.5, whereas the standard, second-order central difference discretisation (SD) leads to convergence orders of about 2.3, in the case of vanishing correlation. In all cases with non-vanishing correlation ($\rho \neq 0$) we observe only slightly improved convergence for Version 1 (V1) when comparing it to the standard discretisation. Version 2 (V2) and Version 3 (V3), however, lead to similar convergence orders as the HOC scheme, even for non-vanishing correlation. Results of Version 4 are not shown as this scheme shows instable behaviour in this example.

In summary, we obtain high-order compact schemes for vanishing correlation and achieve high-order convergence also for non-vanishing correlation for the family (9) of stochastic volatility model. A standard, second-order discretisation is significantly outperformed in all cases.

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