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Abstract. In this work we combine high-order-compact (HOC) and alternating-direction-implicit (ADI) schemes for pricing basket options in a sparse grid setting. HOC schemes exploit the structure of the underlying partial differential equation to obtain a high order of consistency while employing a compact stencil. As time discretisation we propose an efficient ADI splitting to derive a stable scheme. The combination technique is used to construct the so called sparse grid solution, which leads to a significant reduction of necessary grid points and thus to a lower computational effort.

1. Introduction.

We consider the d-dimensional Black-Scholes partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i} - rV = 0$$

in $\Omega \times \Omega_t$ with $\Omega = [0, S_1^{\max}] \times \ldots \times [0, S_d^{\max}]$ and $\Omega_t = [0,T]$. The volatility of the single assets S_i is denoted by $\sigma_i > 0$, their correlation is given by ρ_{ij} for $i,j = 1,\ldots,d$. The risk-free interest rate is given by r. At maturity $t = T$ the option value is given by its payoff

$$g(S_1,\ldots,S_d) = (K - S_1 - \ldots - S_d)^+ \quad \text{(Put)}, \qquad g(S_1,\ldots,S_d) = (S_1 + \ldots + S_d - K)^+ \quad \text{(Call)}$$

with the strike price $K > 0$. We apply the transformations $x_i = \log(S_i)$ for $i = 1,\ldots,d$, $\tau = T - t$ and $u = e^{r\tau}V$, which leads to the transformed PDE

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \left(r - \tfrac{1}{2}\sigma_i^2\right)\frac{\partial u}{\partial x_i} = 0, \tag{1}$$

with space-independent coefficients. In order to solve the PDE (1) numerically we use the method-of-lines approach and end up with a semi-discretisation in space

$$\frac{\partial u}{\partial \tau} = F(u(\tau)), \qquad 0 \le \tau \le T, \qquad u(0) = g.$$

We consider ADI splitting schemes in the time domain with the decomposition of the spatial discretisation

$$F(u) = F_0(u) + F_1(u) + \ldots + F_d(u),$$

where F_0 stems from all mixed derivatives and F_i for $i = 1,\ldots,d$ belongs to the unidirectional contribution of the i-th coordinate in the PDE (1). Within the ADI framework the F_0 part will always be treated explicitly. We propose a HOC finite difference discretisation of the F_i-terms to compute a highly accurate solution while employing a compact stencil. To reduce the number of grid points we use the combination technique to compute the so called sparse grid solution. Compared to a tensor-based full grid with $\mathcal{O}(h^{-d})$ points in space, the sparse grid consists of only $\mathcal{O}(h^{-1}\log(h^{-1})^{d-1})$ nodes. Under suitable regularity assumptions the pointwise rate of convergence is $\mathcal{O}(h^4\log(h^{-1})^{d-1})$ if a fourth order scheme is used to compute the sub solutions.

2. HOC Finite Differences.

We now derive a HOC approximation of the single F_i arising in the decomposition of F. Throughout this article we use standard finite difference operators to approximate the derivatives. A central discretisation to the first and second derivative of order two is given by

$$\delta_{x_i}^2 u_k = \tfrac{1}{h_i^2}\left(u_{k+1} - 2u_k + u_{k-1}\right) = \frac{\partial^2 u}{\partial x_i^2} + \mathcal{O}(h_i^2), \qquad \delta_{x_i}^0 u_k = \tfrac{1}{2h_i}\left(u_{k+1} - u_{k-1}\right) = \frac{\partial u}{\partial x_i} + \mathcal{O}(h_i^2).$$

The mixed derivative term F_0 is approximated with the help of fourth order stencils

$$\tilde{\delta}_{x_i}^0 u_k = \tfrac{1}{12h_i}\left(-u_{k+2} + 8u_{k+1} - 8u_{k-1} + u_{k-2}\right) = \frac{\partial u}{\partial x_i} + \mathcal{O}(h_i^4).$$

Thus we can approximate F_0 via

$$F_0(u) = \sum_{\substack{i,j=1\\ i\neq j}}^{d} \tfrac{1}{2}\rho_{ij}\sigma_i\sigma_j\tilde{\delta}_{x_i}^0\tilde{\delta}_{x_j}^0 u_{kl} + \sum_{i,j}\mathcal{O}(h_i^4 h_j^4).$$

As F_0 is always treated explicitly we do not expect any significant adverse effects incorporating these large stencils regarding the computational effort. The unidirectional contributions are given by

$$F_i(u) = \tfrac{1}{2}\sigma_i^2\frac{\partial^2 u}{\partial x_i^2} + \left(r - \tfrac{1}{2}\sigma_i^2\right)\frac{\partial u}{\partial x_i} = f, \tag{2}$$

for $i = 1,...,d$ and some arbitrary right hand side f. Inserting the finite difference operators we obtain

$$F_i(u_k) = \tfrac{1}{2}\sigma_i^2\delta_{x_i}^2 u_k - \tfrac{1}{2}\sigma_i^2\tfrac{h_i^2}{12}\frac{\partial^4 u}{\partial x_i^4} + \left(r - \tfrac{1}{2}\sigma_i^2\right)\delta_{x_i}^0 u_k - \left(r - \tfrac{1}{2}\sigma_i^2\right)\tfrac{h_i^2}{6}\frac{\partial^3 u}{\partial x_i^3} + \mathcal{O}(h_i^4) = f_k. \tag{3}$$

Since the truncation error in (3) is of order two, we can derive a fourth order approximation if the third and fourth derivative are approximated with second order accuracy. In order to derive these approximations, we differentiate equation (2) once with respect to x_i and get

$$\frac{\partial^3 u}{\partial x_i^3} = \frac{2}{\sigma_i^2}\frac{\partial f}{\partial x_i} - \left(\frac{2r}{\sigma_i^2} - 1\right)\frac{\partial^2 u}{\partial x_i^2}. \tag{4}$$

Differentiating (2) twice with respect to x_i gives

$$\frac{\partial^4 u}{\partial x_i^4} = \frac{2}{\sigma_i^2}\frac{\partial f}{\partial x_i} - \left(\frac{2r}{\sigma_i^2} - 1\right)\left(\frac{2}{\sigma_i^2}\frac{\partial f}{\partial x_i} - \left(\frac{2r}{\sigma_i^2} - 1\right)\frac{\partial^2 u}{\partial x_i^2}\right). \tag{5}$$

The derivatives (4) and (5) can be approximated via central discretisation on the compact stencil with second order. Hence using (4) and (5) in (3) leads to a fourth order accurate approximation

$$\left(\frac{h_i^2\left(r - \frac{\sigma_i^2}{2}\right)^2}{6\sigma_i^2} + \frac{\sigma_i^2}{2}\right)\delta_{x_i}^2 u_k + \left(r - \frac{\sigma_i^2}{2}\right)\delta_{x_i}^0 u_k = f_k + \frac{h_i^2}{12}\delta_{x_i}^2 f_k + \frac{h_i^2\left(r - \frac{\sigma_i^2}{2}\right)}{6\sigma_i^2}\delta_{x_i}^0 f_k. \tag{6}$$

Rewriting this scheme in terms of matrices or symbolic operators gives

$$A_{x_i}U = B_{x_i}F$$

for vectors U and F, where A_{x_i} corresponds to the left hand side of (6) and B_{x_i} to its right hand side. The semi-discrete scheme can thus be written as

$$\frac{\partial u}{\partial t} = F_0(u) + B_{x_1}^{-1}A_{x_1}u + ... + B_{x_d}^{-1}A_{x_d}u + \mathcal{O}(h_1^4) + ... + \mathcal{O}(h_d^4) + \sum_{i,j}\mathcal{O}(h_i^4 h_j^4).$$

3. HOC-ADI schemes. We now apply three well known ADI schemes to the spatial discretisation given in the previous section, namely

HOC Douglas scheme:

$$\begin{cases} Z_0 & = \prod_{j=1}^d B_{x_j} u_n + \Delta_t \left(\prod_{j=1}^d B_{x_j} F_0(u_n) + \sum_{i=1}^d \prod_{\substack{j=1 \\ j \neq i}}^d B_{x_j} A_{x_i} u_n \right) \\ (B_{x_i} - \theta \Delta_t A_{x_i})\, Z_i & = Z_{i-1} - \theta \Delta_t \prod_{j=i+1}^d B_{x_j} A_{x_i} u_n \text{ for } i = 1, ..., d \\ u_{n+1} & = Z_d, \end{cases} \tag{7}$$

HOC Craig-Sneyd scheme:

$$\begin{cases} Z_0 & = \prod_{j=1}^d B_{x_j} u_n + \Delta_t \left(\prod_{j=1}^d B_{x_j} F_0(u_n) + \sum_{i=1}^d \prod_{\substack{j=1 \\ j \neq i}}^d B_{x_j} A_{x_i} u_n \right) \\ (B_{x_i} - \theta \Delta_t A_{x_i})\, Z_i & = Z_{i-1} - \theta \Delta_t \prod_{j=i+1}^d B_{x_j} A_{x_i} u_n \text{ for } i = 1, ..., d \\ \tilde{Z}_0 & = Z_0 + \frac{1}{2} \Delta_t \left(\prod_{j=1}^d B_{x_j} F_0(Z_d) - \prod_{j=1}^d B_{x_j} F_0(u_n) \right) \\ (B_{x_i} - \theta \Delta_t A_{x_i})\, \tilde{Z}_i & = \tilde{Z}_{i-1} - \theta \Delta_t \prod_{j=i+1} B_{x_j} A_{x_i} u_n \text{ for } i = 1, ..., d \\ u_{n+1} & = \tilde{Z}_d. \end{cases} \tag{8}$$

HOC Modified Craig-Sneyd scheme:

$$\begin{cases} Z_0 & = \prod_{j=1}^d B_{x_j} u_n + \Delta_t \left(\prod_{j=1}^d B_{x_j} F_0(u_n) + \sum_{i=1}^d \prod_{\substack{j=1 \\ j \neq i}}^d B_{x_j} A_{x_i} u_n \right) \\ (B_{x_i} - \theta \Delta_t A_{x_i})\, Z_i & = Z_{i-1} - \theta \Delta_t \prod_{j=i+1}^d B_{x_j} A_{x_i} u_n \text{ for } i = 1, ..., d \\ \hat{Z}_0 & = Z_0 + \theta \Delta_t \left(\prod_{j=1}^d B_{x_j} F_0(Z_d) - \prod_{j=1}^d B_{x_j} F_0(u_n) \right) \\ \tilde{Z}_0 & = \hat{Z}_0 + (\frac{1}{2} - \theta) \Delta_t \left(\prod_{j=1}^d B_{x_j} F(Z_d) - \prod_{j=1}^d B_{x_j} F(u_n) \right) \\ (B_{x_i} - \theta \Delta_t A_{x_i})\, \tilde{Z}_i & = \tilde{Z}_{i-1} - \theta \Delta_t \prod_{j=i+1} B_{x_j} A_{x_i} u_n \text{ for } i = 1, ..., d \\ u_{n+1} & = \tilde{Z}_d. \end{cases} \tag{9}$$

The Douglas scheme, see [2], exhibits a consistency order 2 in time if $\theta = \frac{1}{2}$ and $F_0 = 0$, order 1 otherwise. The consistency order in time of the Craig-Sneyd scheme, see [1], is given by 2 if and only if $\theta = \frac{1}{2}$. The modified Craig-Sneyd scheme, see [4], exhibits consistency order 2 in time for any θ. The Craig-Sneyd and the Modified Craig-Sneyd scheme can be seen as an extension of the Douglas scheme.

4. Combination technique. In order to construct the solution on the sparse grid we use the combination technique, which exploits the error splitting structure to linearly combine an anisotropic sequence of solutions in such a way that low order error terms cancel out. We assume

$$u(\mathbf{x_h}) - u_\mathbf{l} = \sum_{k=1}^{d} \sum_{\substack{\{j_1,...,j_k\} \\ \subseteq \{1,...,d\}}} w_{j_1,...j_k}(.; h_{j_1}, ..., h_{j_k}) h_{j_1}^4 \cdots h_{j_k}^4,$$

as error with bounded coefficient functions w. The analytical solution on the discrete grid $\mathbf{x_h}$ is denoted by $u(\mathbf{x_h})$. Note that such an splitting structure can be shown for a wide class of PDEs and linear finite difference schemes [7]. Combining the solutions according to

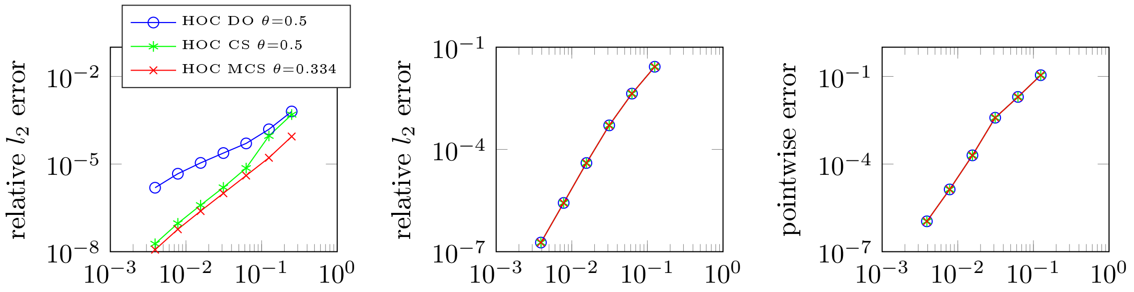
$$u_n^s = \sum_{q=0}^{d-1} (-1)^q \binom{d-1}{q} \sum_{|\mathbf{l}|_1 = n-q} u_\mathbf{l},$$

we can expect a pointwise rate of convergence of $\mathcal{O}(h^4 \log(h^{-1})^{d-1})$. Here u_n^s denotes the sparse grid solution on level n. The numerical sub solutions $u_\mathbf{l}$ are computed on a grid with step sizes $(h_1, h_2, ..., h_d) = \left(2^{-l_1} \cdot c_1, 2^{-l_2} \cdot c_2, ..., 2^{-l_d} \cdot c_d\right)$ with multi-index $\mathbf{l} = (l_1, l_2, ..., l_d)$ and grid length c_i in coordinate direction i for $i = 1, ..., d$.

5. Numerical experiments. In this section we apply our numerical schemes to a European basket put option with two underlyings with parameters

$$T = 1, \quad K = 20, \quad \sigma_1 = 0.4, \quad \sigma_2 = 0.3, \quad \rho_{12} = 0.5, \quad x_i^{\min} = -5 \quad \text{and} \quad x_i^{\max} = \log(5K)$$

for $i = 1, 2$. Figure 1 shows the results of our numerical tests. In the time domain we use the lowest θ



(a) Error in time, $\Delta_t \to 0$ (b) Spatial error, $h \to 0$ (c) Error on sparse grid for $h \to 0$

Figure 1: Numerical convergence plots

value ensuring unconditionally stability in the case of standard second order finite differences [5, 3]. All three schemes show a stable behaviour, see 1(a) and lead to their expected convergence order. Figure 1(b) and 1(c) show the evolution of the error on the full and sparse grid. We compute the sparse grid error at the central grid node, which is the only point that belongs to all sub grids and is therefore not influenced by the interpolation technique used to combine the solutions. The convergence in both plots is in line with the theoretical findings. Please note that the initial value has been smoothed according to Kreiss et. al. [6] in order to overcome the deteriorations from the non-smooth option's payoff.

6. Conclusion and further research. In this work we introduced HOC-ADI schemes to price basket options. The number of grid points could be reduced significantly using sparse grids and the combination technique. In a forthcoming paper we generalise these schemes to problems settings with space-dependent coefficients. Furthermore we analyse the stability in the von Neumann framework.

References

[1] I.J.D. Craig and A.D. Sneyd. An alternating-direction implicit scheme for parabolic equations with mixed derivatives. *Comput. & Math. Appl.*, 16(4):341–350, 1988.

[2] J. Douglas and H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. *Trans. Amer. Math. Soc.*, 82:421–439, 1956.

[3] K. J. in't Hout and C. Mishra. Stability of the Modified Craig-Sneyd scheme for two-dimensional convection-diffusion equations with mixed derivative terms. *Math. Comp. Simul.*, 81:2540–2548, 2011.

[4] K. J. in't Hout and B. Welfert. Unconditional stability of second-order ADI schemes applied to multi-dimensional diffusion equations with mixed derivative terms. *Appl. Numer. Math.*, 59(3-4):677–692, 2009.

[5] K. J. in't Hout and B. D. Welfert. Stability of ADI schemes applied to convection-diffusion equations with mixed derivative terms. *Appl. Numer. Math.*, 57(1):19–35, Jan. 2007.

[6] H. O. Kreiss, V. Thomée, and O. Widlund. Smoothing of initial data and rates of convergence for parabolic difference equations. *Commun. Pure and Appl. Math.*, 23(2):241–259, 1970.

[7] C. Reisinger. Analysis of linear difference schemes in sparse grid combination technique. *IMA J. Numer. Anal.*, 33(2):544–581, 2013.