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On the Non-Existence of Higher Order Monotone Approximation Schemes for HJB Equations

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Abstract

In this work we present a result on the non-existence of monotone, consistent linear discrete approximation of order higher than 2. This is an essential ingredient, if we want to solve numerically nonlinear and particularly Hamilton-Jacobi-Bellman (HJB) equations.

Keywords: Hamilton-Jacobi-Bellman equation, Monotone numerical schemes, Order of consistency

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1. Introduction

The Hamilton-Jacobi-Bellman (HJB) equation, as well as other nonlinear PDEs may not have solution in the classical sense. Therefore, Crandall, Ishi and Lions [1] introduced in 1992 the concept of viscosity solution, suitable for HJB equations. For a brief introduction to the theory of viscosity solutions we refer to [2]. However, it can be a problem to find even such solution analytically, therefore numerical methods schemes are used [3], [4], [5].

The classical theory proposed by Barles and Souganidis [6], which is widely used for proving the convergence of numerical schemes for HJB equations, is based on the monotonicity of the underlying scheme. Convergent schemes for problems from mathematical finance are often first order-accurate

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in time, and first or second-order accurate in space. Recently, Wang and Forsyth [4] proposed an approach yielding an accuracy close to second order in space. In this work, we prove that no better result can exist.

2. Definitions

Let us first introduce the basic notations. U denotes a suitable function space. Let $FV(x)$, $F: U \to \mathbb{R}$ be any nonlinear differential operator, and

$$GV(x) = G\big(V(x), V(x+b_1h), V(x+b_2h), \ldots, V(x+b_nh)\big) \tag{1}$$

be the corresponding discrete scheme approximating it. $V(x)$ is defined as possibly multidimensional function with suitable properties, b_i, $i = 1, 2, \ldots, n$ is of the same dimension as x, and the uniform step size $h \in \mathbb{R}^+$.

Definition 1 (Monotonicity)**.** *A discrete approximation scheme* (1) *is* monotone*, if the function G is non-increasing in $V(x+b_ih)$ for $b_i \neq 0$, $i = 1, \ldots, n$.*

Definition 2 (Standard Consistency)**.** *The discrete scheme*

$$GV(x) = G\big(V(x), V(x+b_1h), V(x+b_2h), \ldots, V(x+b_nh)\big)$$

is a consistent *approximation of $FV(x)$, if $\lim_{h\to 0} \|FV(x) - GV(x)\|_\infty = 0$, where $V(x)$ is a solution of the equation $FV(x) = 0$. Further, $GV(x)$ is said to be* consistent of order $p > 0$*, if $\|FV(x) - GV(x)\|_\infty = \mathcal{O}(h^p)$, $h \to 0$.*

However, the equation $FV(x) = 0$ may not possess classical solutions, which turns Definition 2 inapplicable. For example, HJB equations often have solutions only in the viscosity sense. Therefore, we use another definition of consistency, that doesn't use a solution of $FV(x) = 0$:

Definition 3 (Consistency in Viscosity-sense)**.** *The discrete scheme*

$$G\phi(x) = G(\phi(x), \phi(x+b_1h), \phi(x+b_2h), \ldots, \phi(x+b_nh))$$

is a consistent *approximation of $FV(x)$ if $\lim_{h\to 0} \|F\phi(x) - G\phi(x)\|_\infty = 0$, for any smooth test function $\phi(x)$. We say it is* consistent of order $p > 0$*, if $\lim_{h\to 0} \|F\phi(x) - G\phi(x)\|_\infty = \mathcal{O}(h^p)$ for any smooth test function $\phi(x)$.*

Let $V(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a locally C^2-function ($x, y \in \mathbb{R}$ are now one-dimensional). We define the differential operator $\mathcal{L} : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$

$$\mathcal{L}V(x,y) = \alpha_1 \frac{\partial^2 V}{\partial x^2} + \alpha_{12} \frac{\partial^2 V}{\partial x \partial y} + \alpha_2 \frac{\partial^2 V}{\partial y^2} + \beta_1 \frac{\partial V}{\partial x} + \beta_2 \frac{\partial V}{\partial y} + \gamma V. \tag{2}$$

We assume $\alpha_1 \neq 0$ and investigate some properties of the linear operator $L : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$ given by

$$\begin{aligned} LV(x,y) = a_0(h)V(x,y) + a_1(h)V(x+b_1h, y+c_1h) \\ + a_2(h)V(x+b_2h, y+c_2h) + \cdots + a_n(h)V(x+b_nh, y+c_nh), \end{aligned} \tag{3}$$

where $b_i \neq 0$, or $c_i \neq 0$, $i = 1, 2, \ldots, n$ and there exist j, k such that $b_j \neq 0$, $c_k \neq 0$. (3) should be an approximation of the differential operator $\mathcal{L}V(x,y)$.

Definition 4 (Positive coefficients approximation)**.** *The linear discrete approximation scheme* (3) *satisfies the positive coefficients condition if* $a_i(h) \geq 0$ *for* $i = 1, 2, \ldots, n$*, for all* $h > 0$*.*

Often a scheme is monotone, if and only if its linear part satisfies positive coefficient condition.

3. Main Results

Theorem 1. *There exist no discrete linear approximation* $LV(x)$ *of* $\mathcal{L}V(x)$ *satisfying the positive coefficients condition which is consistent (in the viscosity sense) of order higher than* 2*.*

Proof. We rewrite $L\phi(x,y)$ in the form of a Taylor expansion up to order m:

$$\begin{aligned} L\phi(x,y) &= a_0(h)\phi(x,y) + a_1(h)\phi(x+b_1h, y+c_1h) \\ &\quad + a_2(h)\phi(x+b_2h, y+c_2h) + \cdots + a_n(h)\phi(x+b_nh, y+c_nh) \\ &= a_0(h)\phi(x,y) + \sum_{i=1}^{n} a_i(h)(\phi(x,y) + \frac{1}{1!}\sum_{j=0}^{1}\binom{1}{j}\frac{\partial^1\phi}{\partial x^{1-j}\partial y^j}(b_ih)^{1-j}(c_ih)^j \\ &\quad + \cdots + \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}\frac{\partial^m\phi}{\partial x^{m-j}\partial y^j}(b_ih)^{m-j}(c_ih)^j) \end{aligned} \tag{4}$$

For an approximation of order p we have $\|\mathcal{L}\phi(x,y) - L\phi(x,y)\|_\infty = \mathcal{O}(h^p)$. Using the expansion (4), this yields the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & b_1h & b_2h & \cdots & b_nh \\ 0 & c_1h & c_2h & \cdots & c_nh \\ 0 & \frac{(b_1h)^2}{2} & \frac{(b_2h)^2}{2} & \cdots & \frac{(b_nh)^2}{2} \\ 0 & b_1c_1h^2 & b_2c_1h^2 & \cdots & b_nc_nh^2 \\ 0 & \frac{(c_1h)^2}{2} & \frac{(c_2h)^2}{2} & \cdots & \frac{(c_nh)^2}{2} \\ 0 & \frac{(b_1h)^3}{3!} & \frac{(b_2h)^3}{3!} & \cdots & \frac{(b_nh)^3}{3!} \\ \vdots & & & & \vdots \\ 0 & \frac{(c_1h)^m}{m!} & \frac{(c_2h)^m}{m!} & \cdots & \frac{(c_nh)^m}{m!} \end{pmatrix} \cdot \begin{pmatrix} a_0(h) \\ a_1(h) \\ a_2(h) \\ \vdots \\ a_n(h) \end{pmatrix} = \begin{pmatrix} \gamma \\ \beta_1 \\ \beta_2 \\ \alpha_1 \\ \alpha_{12} \\ \alpha_2 \\ \mathcal{O}(h^p) \\ \vdots \\ \mathcal{O}(h^p) \end{pmatrix}. \tag{5}$$

We can write (5) shortly as $A(h)a(h) = g(h)$. Let us look at the fourth row of the system $A(h)a(h) = g(h)$:

$$a_1(h)\frac{(b_1h)^2}{2} + a_2(h)\frac{(b_2h)^2}{2} + \cdots + a_n(h)\frac{(b_nh)^2}{2} = \alpha_1. \tag{6}$$

The right-hand side is of order $\mathcal{O}(h^0)$, so should be the left hand side. Therefore, at least one $a_i(h)$ should be of order $\mathcal{O}(h^k)$, $k \leq -2$ such that $b_i \neq 0$. If for all $b_i \neq 0$, $a_i(h) = \mathcal{O}(h^j)$, $j > -2$ holds, then each non-zero term of the left-hand side of (6) is of order $h^2\mathcal{O}(h^j) = \mathcal{O}(h^{2+j})$, where $2 + j > 0$, so the whole left hand side is of order greater than zero.

Now let us assume that we have a solution of $A(h)a(h) = g(h)$ for $p > 2$ satisfying the positive coefficients condition. We consider the 11th row of (5):

$$a_1(h)\frac{(b_1h)^4}{4!} + a_2(h)\frac{(b_2h)^4}{4!} + \cdots + a_n(h)\frac{(b_nh)^4}{4!} = \mathcal{O}(h^p). \tag{7}$$

As we noted, there exists an i such that $b_i \neq 0$ and $a_i(h) = \mathcal{O}(h^k)$, $k \leq -2$. Then, also the term in (7) $a_i(h)\frac{(b_ih)^4}{4!}$ is of order $\mathcal{O}(h^q)$, $q = k + 4 \leq 2$. Due to the positive coefficients condition, each term of (7) is non-negative and hence also the whole left-hand side of (7) will be of order $\mathcal{O}(h^c)$, $c \leq 2$. However, the right hand side should be of order higher than 2, which leads to a contradiction. □

Remark 1. *The proof of the theorem does not take into account the case of schemes without node in x itself. However, this can be seen as a subcase of the above schemes with fixed $a_0(h) = 0$.*

Remark 2. *The proof for higher dimensional function V, with the corresponding second order PDE operator $\mathcal{L}V$ can be done in the same manner.*

Remark 3. *In the case of a linear differential operator with derivatives of order higher than 2 similar results on the non-existence may be feasible, with higher maximal order of consistency (in the viscosity sense).*

Let us define

$$\begin{aligned}&\mathcal{L}_\theta V(x, y)\\&= \alpha_1(\theta)\frac{\partial^2 V}{\partial x^2} + \alpha_{12}(\theta)\frac{\partial^2 V}{\partial x \partial y} + \alpha_2(\theta)\frac{\partial^2 V}{\partial y^2} + \beta_1(\theta)\frac{\partial V}{\partial x} + \beta_2(\theta)\frac{\partial V}{\partial y} + \gamma(\theta)V,\end{aligned} \tag{8}$$

where θ is a parameter, $x, y \in \mathbb{R}$. We now formulate the main result.

Theorem 2. *There exist no monotone discrete approximation*

$$-\sup_{\theta\in\Theta}\big(L_\theta V(x,y)+\delta(\theta)\big) \quad \textit{of} \quad -\sup_{\theta\in\Theta}\big(\mathcal{L}_\theta V(x,y)+\delta(\theta)\big)$$

consistent (in the viscosity sense) of order higher than 2.

Proof. Since the supremum is a non-decreasing function, $L_\theta V(x,y)$ has to satisfy the positive coefficients condition so that $-\sup_{\theta\in\Theta}(L_\theta V(x,y)+\delta(\theta))$ will be monotone. Then,

$$\begin{aligned}-\sup_{\theta\in\Theta}\big(\mathcal{L}_\theta V(x)+\delta(\theta)\big) &= -\sup_{\theta\in\Theta}\big(L_\theta V(x,y)+\mathcal{O}(h^k)+\delta(\theta)\big)\\ &= -\sup_{\theta\in\Theta}\big(L_\theta V(x,y)+\delta(\theta)\big)+\mathcal{O}(h^k),\end{aligned}$$

where according to Theorem 1, $k\le 2$. □

Remark 4. *Non-existence of higher order monotone discrete approximations of $f\big(\mathcal{L}V(x,y)\big)$ for a monotone non-increasing function f can be proven in the same way as in Theorem 2.*

4. Application of the results to HJB equation

Now we apply this result to HJB equation with one space dimension.

Definition 5 (Hamilton-Jacobi-Bellman equation)**.** *The PDE*

$$\frac{\partial V(x,t)}{\partial t}=\sup_{\theta\in\Theta}\Big(\alpha(\theta,x,t)\frac{\partial^2 V}{\partial x^2}+\beta(\theta,x,t)\frac{\partial V}{\partial x}+\gamma(\theta,x,t)V+\delta(\theta,x,t)\Big) \tag{9}$$

is called Hamilton-Jacobi-Bellman (HJB) equation.

The coefficients α, β, γ, δ depend on θ as well as on x and t. However, in each particular time and space, we can treat them as constants with respect to x, t. This allow us to write the HJB equation in the form

$$-\sup_{\theta\in\Theta}\Big(\frac{\partial V}{\partial t}+\alpha(\theta)\frac{\partial^2 V}{\partial x^2}+\beta(\theta)\frac{\partial V}{\partial x}+\gamma(\theta)V+\delta(\theta)\Big)=0 \tag{10}$$

for any particular values of t and x. Now, Theorem 2 applied on the left hand side of (10) with $y:=t$ and

$$\mathcal{L}_\theta V(x,t)=\frac{\partial V}{\partial t}+\alpha(\theta)\frac{\partial^2 V}{\partial x^2}+\beta(\theta)\frac{\partial V}{\partial x}+\gamma(\theta)V$$

states, that we cannot obtain a monotone discrete linear scheme for the HJB equation consistent of order higher than 2 in the viscosity sense.

Remark 5. *As noted in Remark 2, Theorem 2 can be proved also for higher dimensions. Therefore, the same result can be obtained in the case of HJB equations with more space dimensions.*

5. Conclusion

In this work we showed that we cannot apply the convergence theory [6] to prove the convergence of linear discrete schemes which are consistent of order higher than 2 in the viscosity sense, since this theory relies on the monotonicity of the scheme. We used the Definition 3 of the consistency (viscosity sense) because for HJB equations the standard Definition 2 cannot be used. For standard PDEs, also consistency in the standard sense of higher order is feasible. A typical example is a monotone nine-point stencil for the Poisson equation [7], which is $\mathcal{O}(h^4)$-consistent in the standard sense.

It remains the question, if any monotone scheme for the linear part of the HJB equation $-\frac{\partial V(x,t)}{\partial t} + \mathcal{L}_\theta V(x,t)$ being consistent of order higher than 2 in the sense of Definition 2 exists.

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