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Mathematical Modelling of Radicalization Processes: the Example of Right-wing Extremism in Germany

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Abstract

This work deals with an application of mathematical modelling in social sciences. We consider the example of right-wing extremism in Germany and classify the German population into different ideological groups. For modelling attitude dynamics we design a system of ordinary differential equations, and study its mathematical properties like equilibrium points, thresholds and attractors. For the calibration of the system we use biennial studies, called "Mitte-Studien", provided by the German Friedrich-Ebert-Stiftung. Finally, we illustrate our findings presenting numerical simulations and future predictions.

Keywords: epidemiological model, social dynamic model, extreme ideologies, transmission dynamics, attitude dynamics

1. Introduction

Which social and economical factors encourage right-wing extremism? How do cultural values change by the influence of an extreme behaviour? Extreme behaviour is produced by a small group, but affects a large amount of the whole population. Which social principles determine right-wing extremism? On the one hand modelling in social science is an anthropological question: the central assumption is the synthetic theory of evolution [2]. The Human being is part of the genetic process in nature and is seen as a cultural animal. The evolution of cultural values has similarities with the synthetic

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theory of evolution and culture is a result of the same process, which affects all creatures.

On the other hand the spread of right-wing extremism is modelled by an *epidemiological model*. A simple model, which shows the spread of diseases within a population, is known as a susceptible-infected-recovered-model (SIR-model). The population is classified into three subgroups:

- $S(t)$ number of people, who are susceptible to a disease at time t
- $I(t)$ number of people, who are infected with a disease at time t
- $R(t)$ number of people, who have been infected and then removed from the disease, either due to immunization or due to death at time t

A dynamic is described by $S \to I \to R$. The heart of a basic SIR-model is a system of ordinary differential equations (ODEs):

$$\begin{aligned} \frac{S}{dt} &= -\beta SI, \\ \frac{I}{dt} &= \beta SI - \gamma I, \\ \frac{R}{dt} &= \gamma I. \end{aligned}$$

Here β denotes the rate of infection and γ is the mortality rate. The main task of an epidemiological model is to built idealised dynamics, which show social processes, and make them more understandable. Certainly it's only a small part of a much more complex and complicated demographic process.

2. Classification of ideological groups

Manifest and organised right-wing extremism in Germany is integrated in a broad environment of latent right-wing extremism [7]. To describe the ideological landscape of Germany in a mathematical way it is necessary to divide the German population into subgroups. Decker and Brähler [7] undertook in 2002 for the first time a representative study about "right-wing extremism attitudes in Germany". They designed a *questionnaire*, which consists of six topics (so-called "dimensions") with three questions each:

1. Endorsement of a right-wing authoritarian dictatorship
    - In the national interest a dictatorship is under certain circumstances the better form of government.
    - Germany should have a leader, who governs with a firm hand for the good of all.
    - What Germany needs now is one strong party, which embodies the national community.

2. Chauvinism
   - We finally should have the courage for a strong national sentiment.
   - What our country needs now is a hard and strict put through of German interests towards other countries.
   - The main objective of German politics should be greater power and prestige.
3. Xenophobia
   - Foreigners come to Germany just because to exploit the welfare state.
   - If jobs run short, foreigners should be sent back to their home countries.
   - Germany is alienated dangerously by the mass of foreigners.
4. Anti-Semitism
   - Even today the influence of the Jews is still strong.
   - The Jews are using evil tricks to achieve their aims more than other people.
   - The Jews are special and strange, they don't really fit with us.
5. Social Darwinism
   - As in nature also in society the stronger should always prevail.
   - The Germans are superior by nature.
   - There is valuable and worthless life.
6. Trivialisation of National Socialism
   - Without the annihilation of the Jews, Hitler would be seen as a great statesman today.
   - The crimes of national socialism were greatly exaggerated in the historiography.
   - The national socialism had its good sides.

All 18 statements were valued by the survey participants with the help of a Likert scale [9, 1] (strongly disagree, mainly disagree, neither agree nor disagree, mainly agree, strongly agree). This leads to five subgroups for each question. In 2004 the survey was repeated and as from 2006 realised by the "Friedrich-Ebert-Stiftung" every two years. For each survey representative areas in Germany are determined by a sample-point-selection. The following subgroups are thus representative of the German society.

For defining social subgroups related to the right-wing extremism in Germany we calculated the mean value derived from the three questions of each "dimension". Doing so, we determine one specific value for the years 2002 to 2014 for each "dimension" and each subgroup and obtain a classification of the German society related to the right-wing extremism over a time period of 12 years:

- $G(t)$ Number of *opponents*, who disagree with the statements of the respective "dimension".
- $N(t)$ Number of people, who have a *negative* attitude towards the statements of the respective "dimension" and mainly disagree with them.
- $U(t)$ Number of *undecided*, who neither agree nor disagree with the statements of the respective "dimension".
- $S(t)$ Number of *semi fanatics*, who mainly agree with the statements of the respective "dimension".
- $R(t)$ Number of *radicals*, who agree with the statements of the respective "dimension".

Table 1: Averaged results of the "Mitte-Studien" [3] related to xenophobia.

	G(t)	N(t)	U(t)	S(t)	R(t)
2002	25.17	20.77	28.53	18.87	6.63
2004	29.90	19.03	26.67	16.80	7.57
2006	29.30	21.00	25.73	17.73	6.33
2008	31.63	20.37	26.10	16.30	5.60
2010	30.43	20.13	26.70	16.07	6.63
2012	30.53	19.17	24.80	17.50	8.00
2014	40.47	18.20	23.17	12.70	5.57

3. The STV-model

Santonja, Tarazona and Villanueva [10] set up a system of ODEs to describe the spread of an extreme ideology using the example of the ETA in the bask country. In their model (called STV-model in the sequel) the size of the total population is not constant, but depends on intrinsic and extrinsic influencing variables. There is a birth rate and mortality rate, which have some proportional influence on the subgroups. $\Lambda(t)$ is the number of births and $\Phi(t)$ the number of deaths at time t. Extrinsic the number of immigrants $\Gamma(t)$ and the number of emigrants $\Sigma(t)$ influence proportional to their sizes subgroups G and N. In this context the parameters α_1 and α_2 describe the relationship between immigration and emigration due to the social circumstances at time t.

3.1. Transition terms

We model the spread of right-wing extremism in Germany analogously to the STV-model. The dynamics between the subgroups are described by transition terms. It is particularly interesting how the subgroup U, the 'undecided', behaves, because this group is the fragile part of the society, which is susceptible to radicalization. For the incidence rate $B(N, U)$ it is $B(N, 0) = B(0, U) = B(0, 0) = 0$: If there are no people, who have a negative attitude towards right-wing extremism, or people who are undecided or both, then there should be no new cases of radicalization. The parameter c denotes the average number of social contacts per person and per unit time and ρ is the average number of contacts with every single individual, i.e. $c\rho N(t)$ is the total number of contacts per unit time for all individuals in the subgroup N at time t. To simplify we assume that the total population $T(t) = G(t) + N(t) + U(t) + S(t) + R(t)$ is homogeneous, then the number of social contacts per unit time between subgroup N and subgroup U is

$$c\rho N(t) \frac{U(t)}{T(t)}.$$

We denote by $\beta_1 := c\rho$ the transition rate between N and U

$$B(N, U) := \beta_1 N(t) \frac{U(t)}{T(t)},$$

where $\beta_1 > 0$ indicates a transition from N to U and $\beta_1 < 0$ vice versa. Similarly, the transition between G and N reads

$$k\beta_1 G(t) \frac{N(t)}{T(t)}.$$

$k > 0$ is necessary, because the transition between G and N possess the same sign as the transition between N and U, but may have a different weighting. Analogously, a transition between subgroups S and U is described by

$$\beta_2 S(t) \frac{U(t)}{T(t)}$$

Here $\beta_2 > 0$ models the transition from S into U and $\beta_2 < 0$ vice versa. The transition between R and S is described by β_2 and has got the same sign as the transition between S and U, but may also have a different weighting

$$l\beta_2 R(t) \frac{S(t)}{T(t)},$$

where $l > 0$.

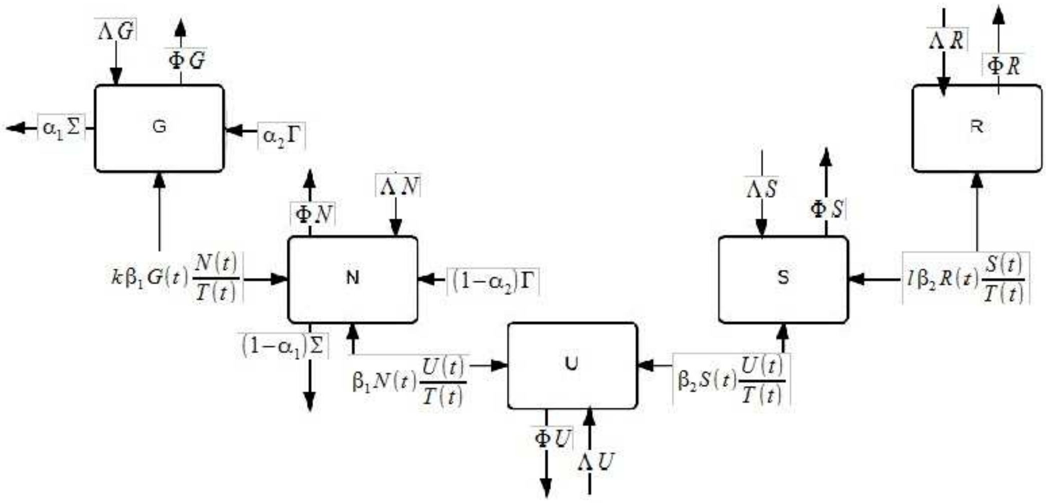

Figure 1: Flow chart of the STV-model defined by (1)–(5)

3.2. Modelling a system of ODEs

The following system of ODEs states a mathematical model describing the temporal evolution of the considered population. The transition terms lead to a system of ODEs, which describes the dynamics of an opinion-forming process respective right-wing extremism in Germany:

$$\frac{dG}{dt} = \Lambda(t)G(t) + \alpha_2\Gamma(t) - k\beta_1 G(t)\frac{N(t)}{T(t)} - \alpha_1\Sigma(t) - \Phi(t)G(t), \tag{1}$$

$$\begin{aligned}\frac{dN}{dt} &= \Lambda(t)N(t) + (1-\alpha_2)\Gamma(t) - \beta_1 N(t)\frac{U(t)}{T(t)} + k\beta_1 G(t)\frac{N(t)}{T(t)} \\ &\quad - (1-\alpha_1)\Sigma(t) - \Phi(t)N(t),\end{aligned} \tag{2}$$

$$\frac{dU}{dt} = \Lambda(t)U(t) + \beta_1 N(t)\frac{U(t)}{T(t)} + \beta_2 S(t)\frac{U(t)}{T(t)} - \Phi(t)U(t), \tag{3}$$

$$\frac{dS}{dt} = \Lambda(t)S(t) - \beta_2 S(t)\frac{U(t)}{T(t)} + l\beta_2 R(t)\frac{S(t)}{T(t)} - \Phi(t)S(t), \tag{4}$$

$$\frac{dR}{dt} = \Lambda(t)R(t) - l\beta_2 R(t)\frac{S(t)}{T(t)} - \Phi(t)R(t), \tag{5}$$

$$T = G + N + U + S + R. \tag{6}$$

3.3. Scaling the model

The first consistency check shows that the study data are percentage, but the ODEs are related to the absolute number of individuals. To correct this inconsistency, we scale the model following [10]. Adding equations (1)–(5) one gets

$$\frac{dT}{dt} = [\Lambda(t) - \Phi(t)]T(t) + \Gamma(t) - \Sigma(t) \tag{7}$$

and dividing both members of (7) by $T(t)$ we have that

$$\frac{\frac{dT}{dt}}{T(t)} = \Lambda(t) - \Phi(t) + \frac{\Gamma(t) - \Sigma(t)}{T(t)}. \tag{8}$$

If we define the rates

$$\begin{aligned} &\qquad\qquad g(t) = \frac{G(t)}{T(t)}, \quad n(t) = \frac{N(t)}{T(t)}, \\ u(t) = \frac{U(t)}{T(t)}, &\quad s(t) = \frac{S(t)}{T(t)}, \quad r(t) = \frac{R(t)}{T(t)}, \\ &\qquad\qquad \gamma(t) = \frac{\Gamma(t)}{T(t)}, \quad \sigma(t) = \frac{\Sigma(t)}{T(t)}, \end{aligned} \tag{9}$$

equation (8) can be transformed to

$$\frac{\frac{dT}{dt}}{T(t)} = \Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t). \tag{10}$$

Using (10) we compute the derivative of g

$$\begin{aligned} \frac{d\frac{G}{T}}{dt} = g'(t) &= \frac{G'(t)T(t) - G(t)T'(t)}{T(t)^2} = \frac{G'(t)}{T(t)} - \frac{G(t)}{T(t)}\frac{T'(t)}{T(t)} \\ &= \frac{G'(t)}{T(t)} - g(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)]. \end{aligned} \tag{11}$$

Analogously we obtain

$$\begin{aligned} n'(t) &= \frac{N'(t)}{T(t)} - n(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)], \\ u'(t) &= \frac{U'(t)}{T(t)} - u(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)], \\ s'(t) &= \frac{S'(t)}{T(t)} - s(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)], \\ r'(t) &= \frac{R'(t)}{T(t)} - r(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)]. \end{aligned} \tag{12}$$

Next, we divide (1) by $T(t)$

$$\frac{\frac{dG}{dt}}{T(t)} = \Lambda(t)\frac{G(t)}{T(t)} + \alpha_2\frac{\Gamma(t)}{T(t)} - k\beta_1\frac{G(t)}{T(t)}\frac{N(t)}{T(t)} - \Phi(t)\frac{G(t)}{T(t)} - \alpha_1\frac{\Sigma(t)}{T(t)}. \tag{13}$$

Using (11) and substituting by the corresponding rates (12) we get

$$g'(t) + g(t)[\Lambda(t) - \Phi(t) + \gamma(t) - \sigma(t)] = \\ \Lambda(t)g(t) + \alpha_2\gamma(t) - k\beta_1 g(t)n(t) - \alpha_1\sigma(t) - \Phi(t)g(t) \quad (14)$$

and obtain finally the scaled ODE system

$$g'(t) = \big(\sigma(t) - \gamma(t)\big)g(t) + \alpha_2\gamma(t) - k\beta_1 g(t)n(t) - \alpha_1\sigma(t), \quad (15)$$

$$n'(t) = \big(\sigma(t) - \gamma(t)\big)n(t) + (1-\alpha_2)\gamma(t) - \beta_1 n(t)u(t) - (1-\alpha_1)\sigma(t), \quad (16)$$

$$u'(t) = \big(\sigma(t) - \gamma(t)\big)u(t) + \beta_1 n(t)u(t) + \beta_2 s(t)u(t), \quad (17)$$

$$s'(t) = \big(\sigma(t) - \gamma(t)\big)s(t) - \beta_2 s(t)u(t) + l\beta_2 r(t)s(t), \quad (18)$$

$$r'(t) = \big(\sigma(t) - \gamma(t)\big)r(t) - l\beta_2 r(t)s(t). \quad (19)$$

Since the transmission rates depend on time, this scaled system of ODEs is a non-autonomous system. Moreover, the transmissions are non-linear, since they contain squared contact-terms.

3.4. Numerical solution

In general solving a system of non-linear ODEs relies on numerical solution methods. A number M of data points and a mathematical model describing the relation between the time and the measured quantity are given to fit the parameter in the ODE system by a calibration procedure. The STV-model gives us a system of ODEs containing six free parameters.

The optimal parameters. In this paragraph we show how to determine the optimal parameters, which approximate the study data best. Therefore we compute the immigration and emigration rate by using data provided by the Federal Statistical Office of Germany for the years t_i=2002, 2004, 2006, 2008, 2010, 2012, 2014:

$$\gamma(t_i) = \frac{\Gamma(t_i)}{T(t_i)}, \quad \sigma(t_i) = \frac{\Sigma(t_i)}{T(t_i)}, \qquad i = 1, \ldots, 7. \quad (20)$$

For providing a continuous function to the ode solver `ode45` we interpolate linearly by the MATLAB built-in function `interp1`.

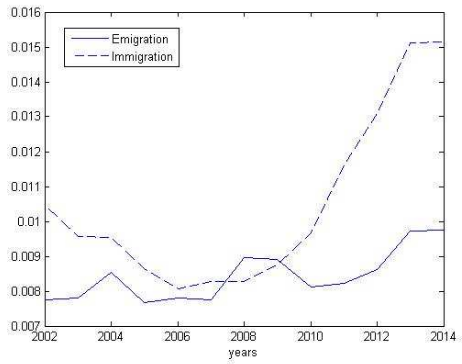

Figure 2: Immigration and emigration rates in Germany. Notice that the missing value for 2014 is replaced by the value of 2013.

In order to compute the optimal parameters, we implemented the error function $\mathcal{E} : \mathbb{R}^6 \to \mathbb{R}$, which determines for every parameter tuple $(\beta_1, \beta_2, k, l, \alpha_1, \alpha_2)$ the mean square error between the study data and the solution of the ODEs. Hence minimizing $\mathcal{E}$ yields the optimal parameters. The following three steps computes the mean square error each parameter tuple.

1. Solve numerically the system of ODEs with initial values from 2002.
2. For the nodal points 2002, 2004, 2006, 2008, 2010, 2012, 2014 the study data should be approximated by the computed solutions of the ODEs.
3. $\mathcal{E}((\beta_1, \beta_2, k, l, \alpha_1, \alpha_2))$ is the mean square error between the values obtained in step 2 and the study data.

The model-parameters β_1 and β_2 are rates, which are either positive or negative and according to amount at the maximum 1. k and l denote weights, which are ≥ 0 and at the maximum 1. α_1 and α_2 model the relation between immigration and emigration, for this reason $\alpha_i \in [0, 1]$, $i = 1, 2$. Therefore the only relevant domain for the model parameter is $D = [-1, 1] \times [-1, 1] \times [0, 1] \times [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^6$. For computing the local minimum of function $\mathcal{E}$, we restrict its domain to D and minimize by the MATLAB routine `fminsearch`. We obtain the following optimal parameters for each "dimension".

Table 2: The optimal parameters of the system (15)–(19)

	Dim. 1	Dim. 2	Dim. 3	Dim. 4	Dim. 5	Dim. 6
β_1	-0.2105	-0.2517	-0.1329	-0.1914	-0.2310	-0.1747
β_2	0.1854	0.2081	0.0767	0.1003	0.1272	0.099
k	0.2836	0.2024	0.0303	0.1633	0.4105	0.2765
l	1	1	0.0467	1	0	0.335
α_1	0	0	0	0	0	0
α_2	1	1	1	1	1	1

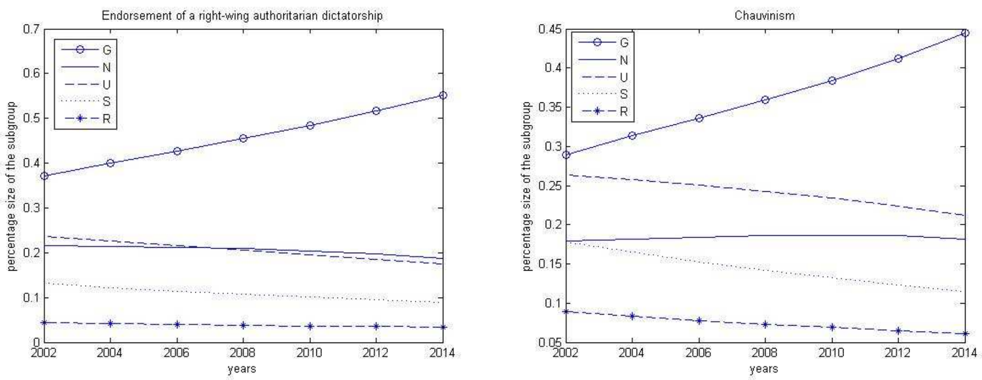

Figure 3: Solutions of the system of ODEs (15)–(19) using the optimal parameter and data from dimensions 1 and 2, 'endorsement of a right-wing authoritarian dictatorship' (left) and 'chauvinism' (right). It indicates a strong increase of the right-wing extremism opponents. Left: The subgroup of people, who are negative, decreases slightly, then decreases more strongly. All other subgroups show a slight decrease. Right: The subgroup of people, who are negative, increases for a short time and afterwards decreases slightly. All other subgroups show a continuous decrease.

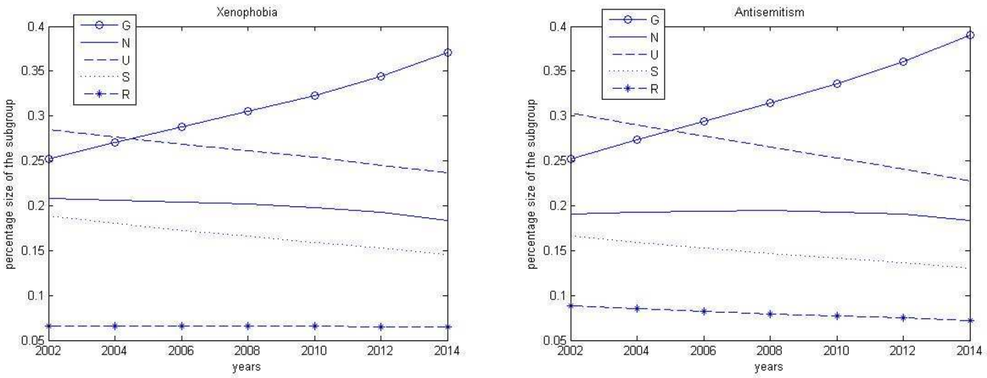

Figure 4: Solutions of the system of ODEs (15)–(19) using the optimal parameter and data from "dimensions 3 and 4" xenophobia and anti-Semitism It indicates a strong increase of the right-wing extremism opponents. Left: The subgroup of people, who are negative, decreases slightly, then decreases more strongly. The radical subgroup decreases slightly, then relatively more strongly. All other subgroups show a continuous decrease. Right: The subgroup of people, who are negative, increases for a short time, then decreases strongly, then decreases more strongly. All other subgroups show a continuous decrease.

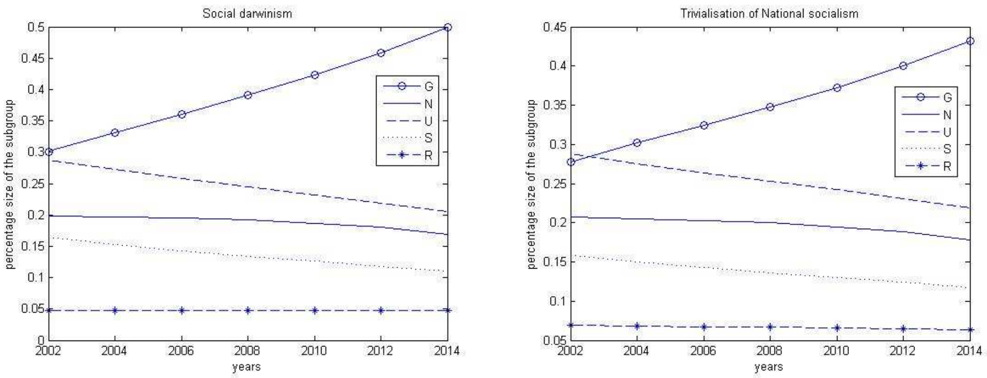

Figure 5: Solutions of the system of ODEs (15)–(19) using the optimal parameter and data from "dimensions 5 and 6" social darwinism and national socialism. It indicates a strong increase of the right-wing extremism opponents. The radicals and the subgroup of people, who are negative, decreases slightly, then decreases more strongly. All other subgroups show a continuous decrease.

3.5. Simulations

In the following simulations we use different values for our model parameters to model fictional and non-fictional scenarios. For prognoses until 2020 we take the size of population, immigration and emigration data from 2013.

The optimal parameters. With the computed optimal parameters we get a first simulation. It represents the "natural" evolution of the German population:

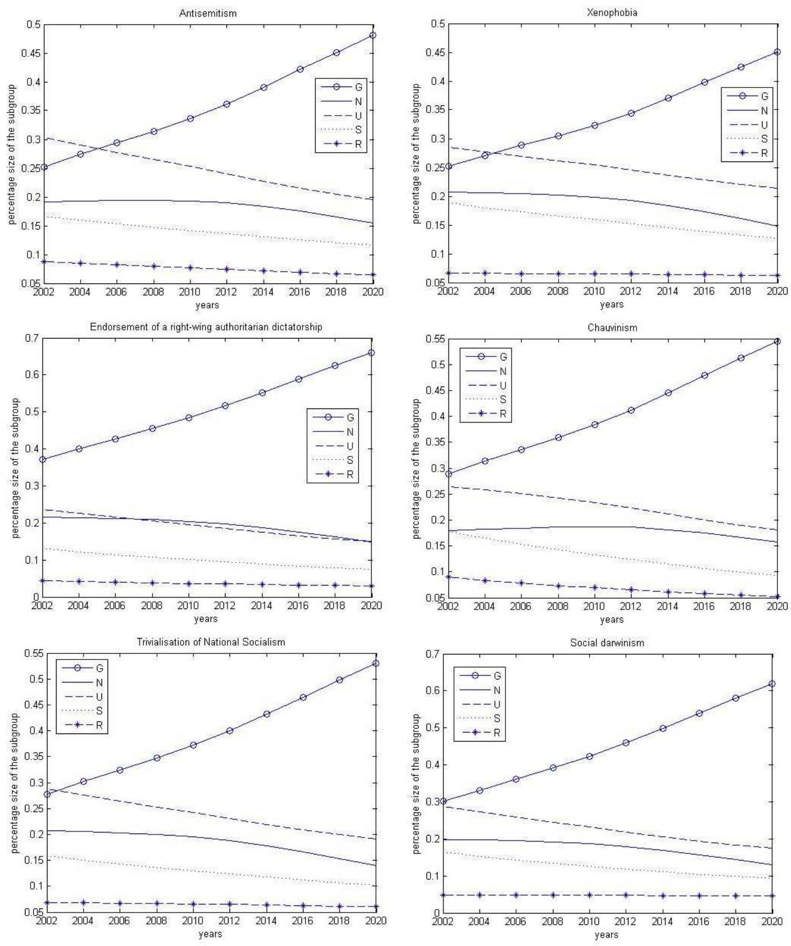

Figure 6: Simulation of STV-model in all six "dimensions" from 2002 to 2020 using the optimal parameters.

A radicalization of society. For modelling a strong recruitment from the subgroup of people, who are undecided, into the subgroup of semi fanatics we set $\beta_2 = -0.2$.

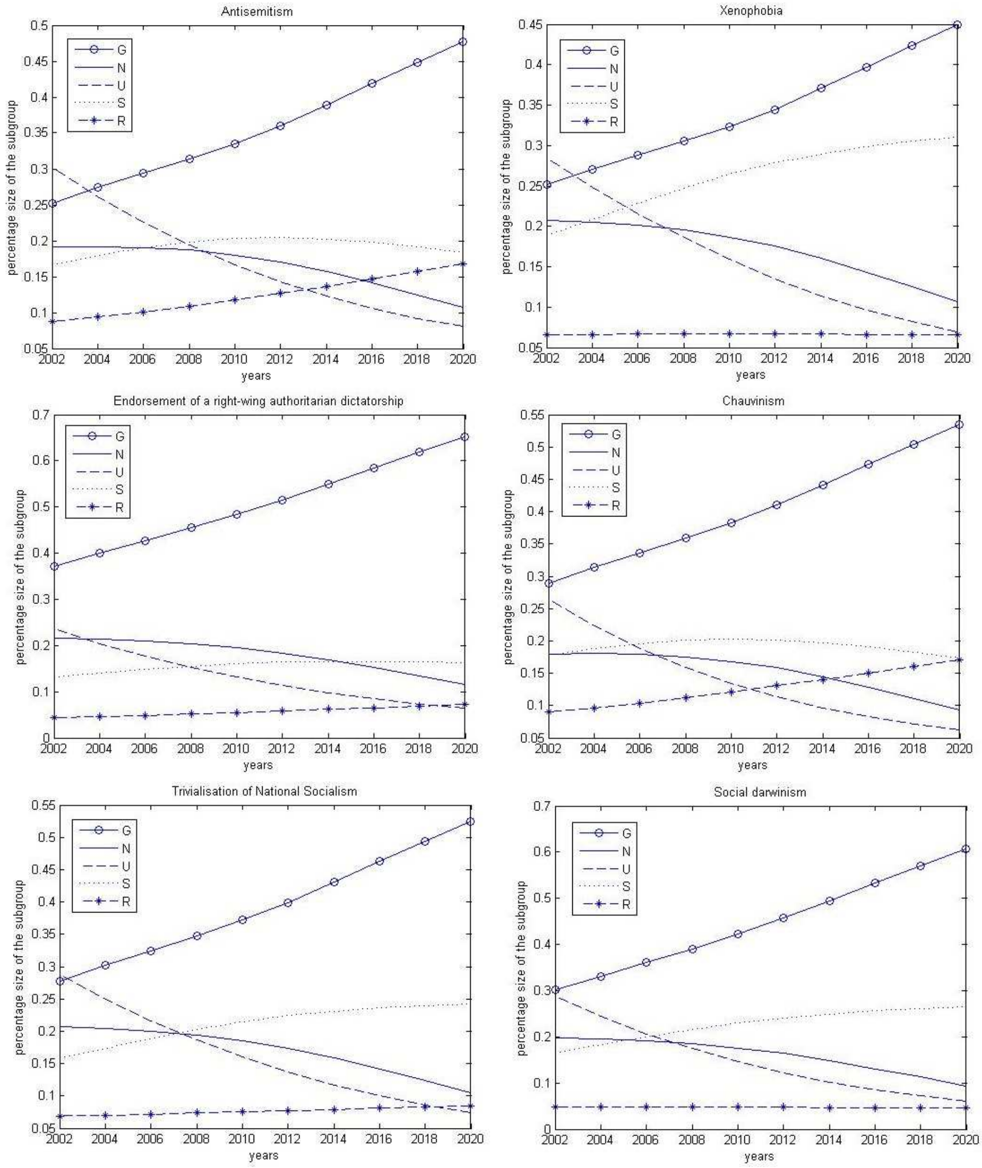

Figure 7: Simulations in all six "dimensions" from 2002 to 2020 using $\beta_2 = -0.2$.

Moderation of society. The next simulation models a strong transition from subgroup of people, who are undecided to the subgroup of people, who are negative. For this purpose we set $\beta_1 = -0.4$.

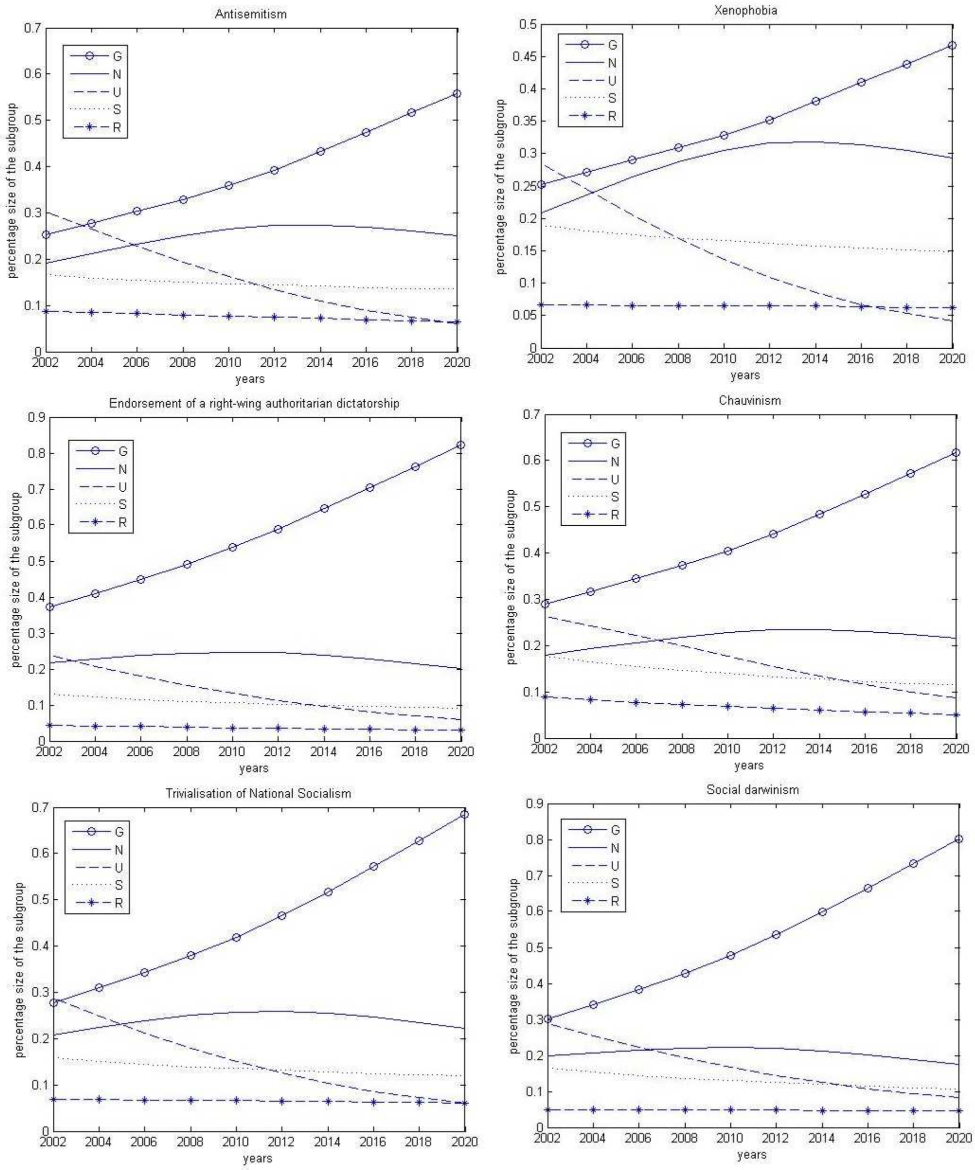

Figure 8: Overview of simulations in all six "dimensions" from 2002 to 2020 using $\beta_1 = -0.4$.

4. The Castillo-Chavez-Song (CCS) model

Castillo-Chavez and Song [5] focus in their model (called CCS-model in the sequel) on the question, which social structures support the existence of fanatic subcultures. Especially the effects of "core" (ultra) fanatics on the spread of extreme ideologies is analysed. Therefore the population is divided in a non-core population $A = G + N$, which is used as a recruitment pool, and a core population $C = U + S + R$, which is the radical part of society. C is relatively small and organized hierarchically with respect to size and radicalization: The subgroup U is the biggest part and the most susceptible one. People in this subgroup are not radical yet, but fragile and susceptible to radicalization. Semi fanatics, who are partially convinced by right-wing extremism, are in subgroup S. The subgroup R is in general the smallest part of society and contains of right-wing extremists. The considered total population is

$$T = A + C = G + N + U + S + R.$$

4.1. Transition terms

Similar to the STV-model we define μ_i as a parameter, which describes the strength of recruitment from one subgroup to another. Opposing γ_i denotes the per-capita recovery rate for each subgroup from the core to A, ($i = 1$, $i = 2$, $i = 3$). Hence $1/\gamma_i$ is the average residence time for each subgroup in the core. Therefore we are able to model dynamics inside the core and dynamics from the core to the moderate part of society:

$$\mu_1 A(t) \, \frac{C(t)}{T(t)}$$

is the transition term from A to C. Inside the core dynamics between U and the two other subgroups are modelled by

$$\mu_2 U(t) \, \frac{S(t) + R(t)}{C(t)}.$$

Analogously we get a transition term for the dynamics inside the core from subgroup S to R by

$$\mu_3 S(t) \, \frac{R(t)}{C(t)}.$$

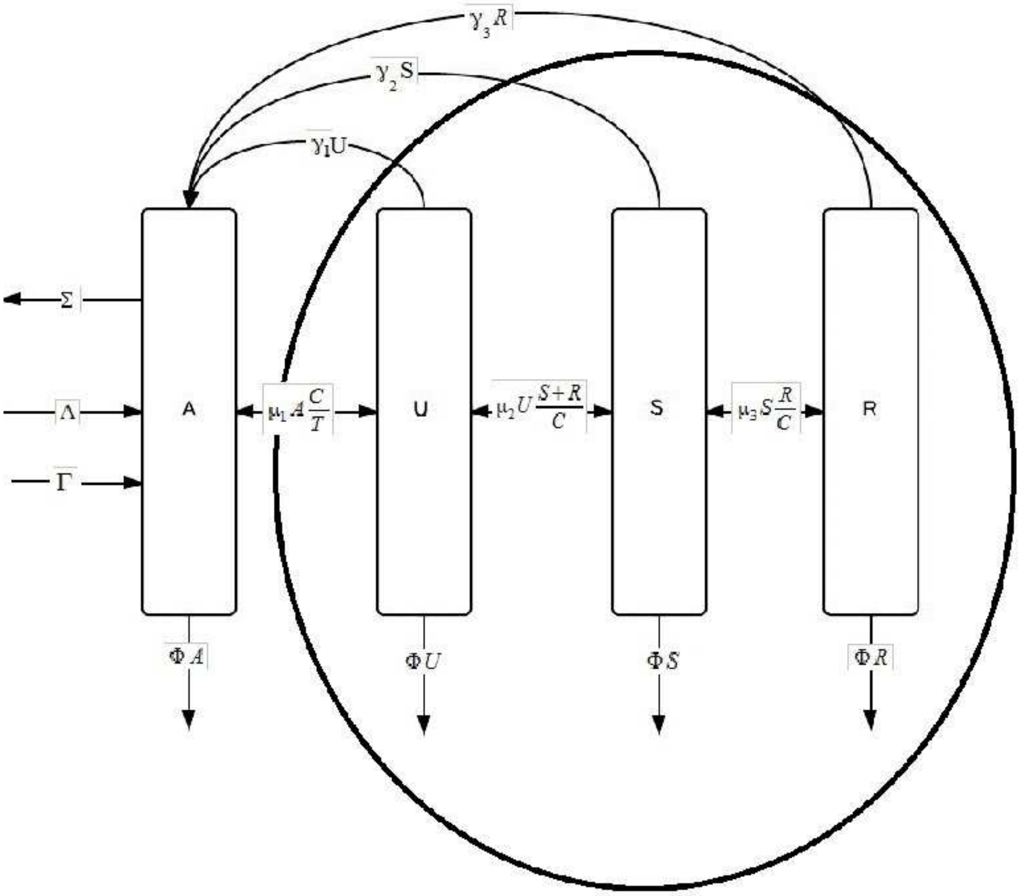

Figure 9: Flow chart of CCS-model (21)–(24)

4.2. Modelling a system of ODEs

In this model the non-core population is increased by a constant birth and immigration rate and decimated by a constant emigration rate and by social contact structures, which lead to a recruitment of the core-population. Every subgroup is decimated by a natural mortality rate Φ that is calculated by the average number of deaths divided by the average size of population.

$$\frac{dR}{dt} = \mu_3 S(t)\frac{R(t)}{C(t)} - \gamma_3 R(t) - \Phi R(t), \tag{21}$$

$$\frac{dS}{dt} = \mu_2 U(t)\frac{S(t)+R(t)}{C(t)} - \mu_3 S(t)\frac{R(t)}{C(t)} - \gamma_2 S(t) - \Phi S(t), \tag{22}$$

$$\frac{dU}{dt} = \mu_1 A(t)\frac{C(t)}{T(t)} - \mu_2 U(t)\frac{S(t)+R(t)}{C(t)} - \gamma_1 U(t) - \Phi U(t), \tag{23}$$

$$\frac{dA}{dt} = \Lambda + \Sigma - \mu_1 A(t)\frac{C(t)}{T(t)} + \gamma_1 U(t) + \gamma_2 S(t) + \gamma_3 R(t) - \Gamma - \Phi A(t). \tag{24}$$

Here it is assumed that

$$A(t_0) > 0, \quad U(t_0) \geq 0, \quad S(t_0) \geq 0 \quad \text{and} \quad R(t_0) \geq 0.$$

4.3. Scaling the Model

In this section our aim is to create a reduced system of ODEs, which is "dynamically" equivalent to the system (21)–(24), i.e. the dynamics (e.g. attractors and equilibria) of the unscaled system and the reduced system are the same [4].

Adding all equations (21)–(24) yields

$$\frac{dT}{dt} = \Lambda + \Sigma - \Gamma - \Phi T(t) \tag{25}$$

as a dynamic for the total German population. By solving the ODE (25) with the initial value $T(t_0) = T_0$ we get

$$T(t) = \frac{\Lambda}{\Phi} + \frac{\Sigma}{\Phi} - \frac{\Gamma}{\Phi} + \left[T_0 - \frac{\Lambda}{\Phi} - \frac{\Sigma}{\Phi} + \frac{\Gamma}{\Phi}\right] e^{-\Phi(t-t_0)}$$

and thus we conclude $\lim_{t\to\infty} T(t) = \frac{\Lambda}{\Phi} + \frac{\Sigma}{\Phi} - \frac{\Gamma}{\Phi}$. We set

$$T(t) = \frac{\Lambda}{\Phi} + \frac{\Sigma}{\Phi} - \frac{\Gamma}{\Phi}, \qquad t \quad \text{large enough}, \tag{26}$$

and get $A(t) = T(t) - C(t) = \frac{\Lambda}{\Phi} + \frac{\Sigma}{\Phi} - \frac{\Gamma}{\Phi} - C(t)$. Hereafter, to shorten notation, we denote: $\gamma_i + \Phi$ with i=1, 2, 3 by γ_i. Because in this model the core population is particularly interesting, scaling the associated ODEs by $\frac{\Lambda}{\Phi} + \frac{\Sigma}{\Phi} - \frac{\Gamma}{\Phi}$ leads to following scaled system:

$$\frac{dU}{dt} = \mu_1\big(1 - C(t)\big)C(t) - \mu_2 U(t)\frac{S(t)+R(t)}{C(t)} - \gamma_1 U(t), \tag{27}$$

$$\frac{dS}{dt} = \mu_2 U(t)\frac{S(t)+R(t)}{C(t)} - \mu_3 S(t)\frac{R(t)}{C(t)} - \gamma_2 S(t), \tag{28}$$

$$\frac{dR}{dt} = \mu_3 S(t)\frac{R(t)}{C(t)} - \gamma_3 R(t). \tag{29}$$

Hence the recruitment into the core population is of logistic form $\mu_1(1 - C(t))C(t)$.

For the model (27)–(29) the domain of interest is the simplex defined by the set

$$\Omega = \{(U, S, R) \in \mathbb{R}^3_+ : 0 \leq U + S + R \leq 1\}.$$

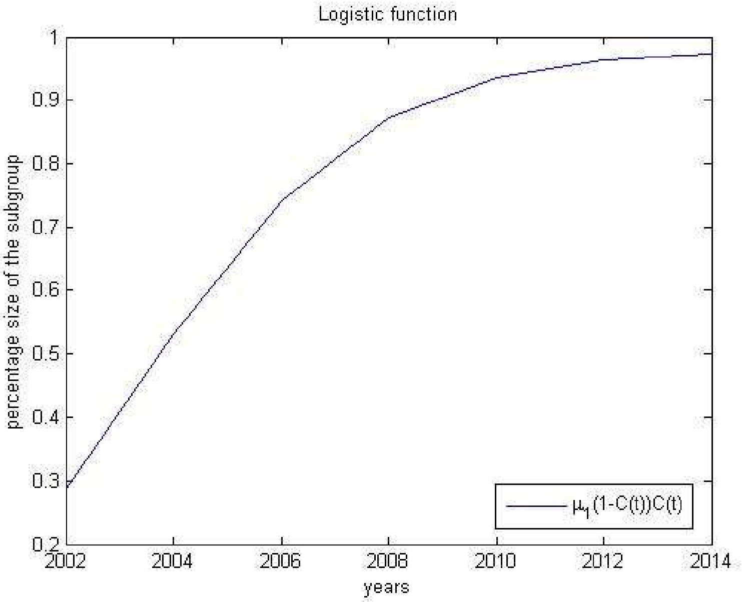

Figure 10: The transition from A into C is of logistic form. Here visualised with initial data from the "dimension" xenophobia and $\mu_1 = 0.5$. The core population increases strongly until it is stopped by the exhaustive recruitment pool.

4.4. Numerical solution

Analogous to the numerical implementation of the STV-model, we implement the CCS-model in Matlab.

The optimal parameters. We get the optimal parameters by finding the local minimum of the function $\mathcal{E} : \mathbb{R}^6 \to \mathbb{R}$ by the built-in function `fminsearch`. In case of the CCS-model the parameter tuple $(\mu_1, \mu_2, \mu_3, \gamma_1, \gamma_2, \gamma_3)$ is displayed in the mean square error between the study data and the solutions of the ODEs. μ_1 is either a positive or negative growth of the logistic function. μ_2 and μ_3 are the model parameters, which describe the transitions between the subgroups and according to amount they are at a maximum 1. The recovery rates γ_1, γ_2, γ_3 describe the repatriation from the core population to the moderate part of society. They are ≥ 0. Therefore the local minimum is determined in

$$D = [-1,1] \times [-1,1] \times [-1,1] \times [0,1] \times [0,1] \times [0,1].$$

.

Table 3: The optimal parameters of the CCS-model (27)–(29) in all six "dimensions".

	Dim. 1	Dim. 2	Dim. 3	Dim. 4	Dim. 5	Dim. 6
μ_1	0.5	-0.0296	0.0088	0.8609	0.9207	0.9837
μ_2	-0.1009	-0.0904	0.0819	-0.0793	-0.0880	0.1051
μ_3	-0.2311	-0.2937	0.1096	-0.1177	0.2742	0.9667
γ_1	0.5985	0.0078	0.0040	0.8744	0.9967	0.9963
γ_2	0.0218	0.0245	0.0444	0	0	0.0003
γ_3	0.0196	0	0.0248	0.0206	0.0515	0.1442

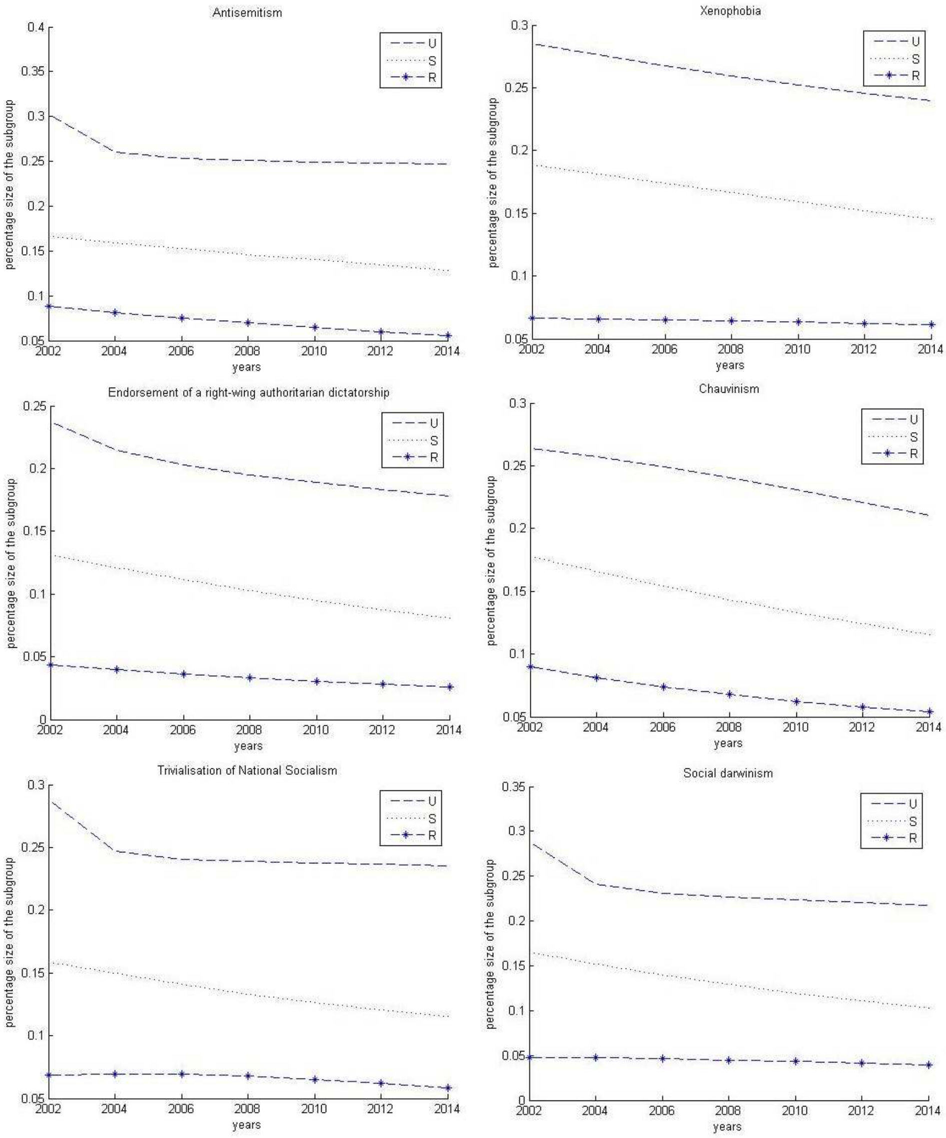

Figure 11: Solutions of the ODEs in all "dimensions" with the optimal parameters. In the "dimension" anti-Semitism, endorsement of a right-wing authoritarian dictatorship and social darwinism the subgroup of undecided people first decreases strongly, then more slightly. The subgroups of semi fanatics and radicals decrease continuously. In the "dimension" xenophobia all subgroups decrease continuously. In the "dimension" chauvinism again all subgroups decrease continuously, whereby the subgroup of radicals tends towards zero. In the "dimension" trivialisation of National Socialism the subgroup of undecided decreases strongly, then slightly. The subgroup of semi fanatics decreases continuously. The subgroup of radicals first increases strongly, then decreases.

4.5. Global Thresholds

The first question is, which are the smallest values for the parameters, such that the dynamical system is characterized by a recognizable change. In the phase space we studied attractors and trajectories for uncovering the time response of the dynamical system.

Establishment of a Core Population. A core population can only be established if either the recruitment rate μ_1 is large enough or the residence time $1/\mu_1$ is long enough, or both [5].

Persistence of a Fanatic Ideology. A second global threshold controls the establishment of the fanatic population and the persistence of the fanatic ideology. From (29) it follows, that if $L_3 = \mu_3/\gamma_3 \leq 1$, then $\lim_{t\to\infty} R(t) = 0$. That is L_3 gives sufficient conditions for the elimination of the fanatic population. The fact that the hyperplane $R = 0$ is a global attractor can be used to reduce the dimension of model (27)–(29). We get the following ODE system with two 'dimensions':

$$\frac{dU}{dt} = \mu_1(1 - C(t))C(t) - \mu_2 U(t)\frac{S(t)}{C(t)} - \gamma_1 U(t), \tag{30}$$

$$\frac{dS}{dt} = \mu_2 U(t)\frac{S(t)}{C(t)} - \gamma_2 S(t), \tag{31}$$

$$C(t) = U(t) + S(t).$$

Theorem 4.1 ([5][Theorem 2). *] Let $\nu = \frac{\gamma_1}{\gamma_2}$ and $L_d = \frac{\nu L_1 L_2}{L_2 - 1 + \nu}$.*

(i) If $L_1 < 1$ and $L_2 < 1$ or $1 < L_2$ and $L_d < 1$, then (0,0) is the global attractor.

(ii) If $1 < L_1$ and $L_2 < 1$, then $(1 - \frac{1}{L_1}, 0)$ is the global attractor.

(iii) If $1 < L_2$ and $L_d > 1$, then the positive equilibrium (U^*, S^*) with

$$U^* = \frac{(L_d - 1)(L_2 - 1 + \nu)}{\nu L_1 L_2^2}, \tag{32}$$

$$S^* = \frac{(L_2 - 1)(L_d - 1)(L_2 - 1 + \nu)}{\nu L_1 L_2^2} \tag{33}$$

is the global attractor.

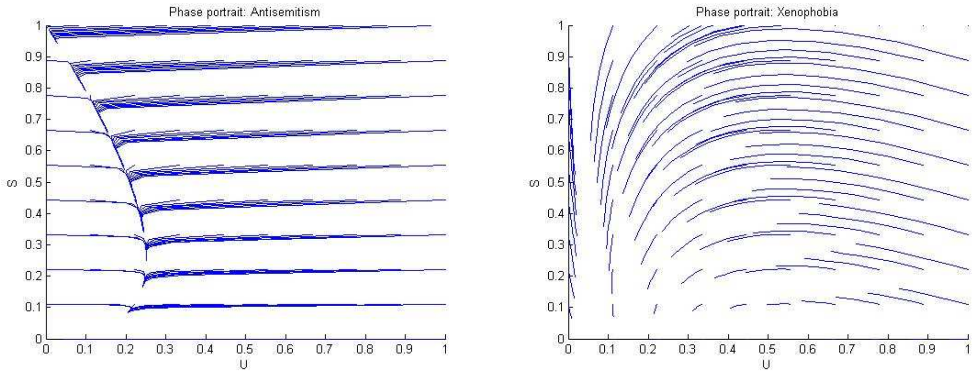

Figure 12: Phase portrait for system (30)–(31) using the optimal parameters. In "dimensions" anti-Semitism and xenophobia (0,0) is the global attractor.

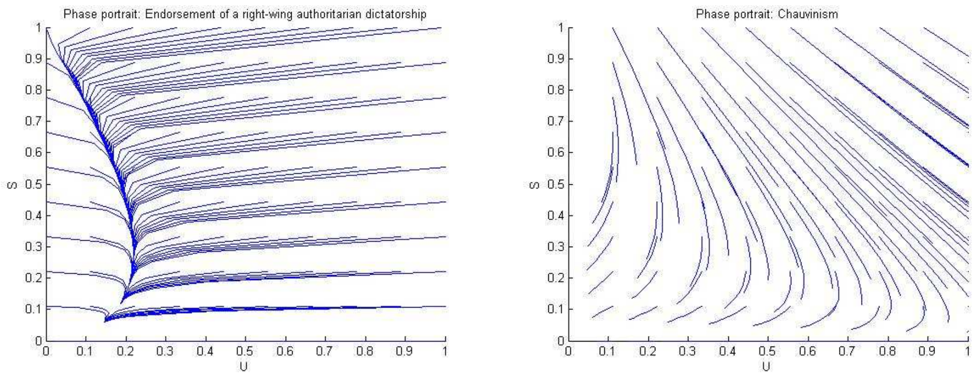

Figure 13: Phase portrait for system (30)–(31) using the optimal parameters. In "dimensions" endorsement of a right-wing authoritarian dictatorship and chauvinism (0,0) is the global attractor.

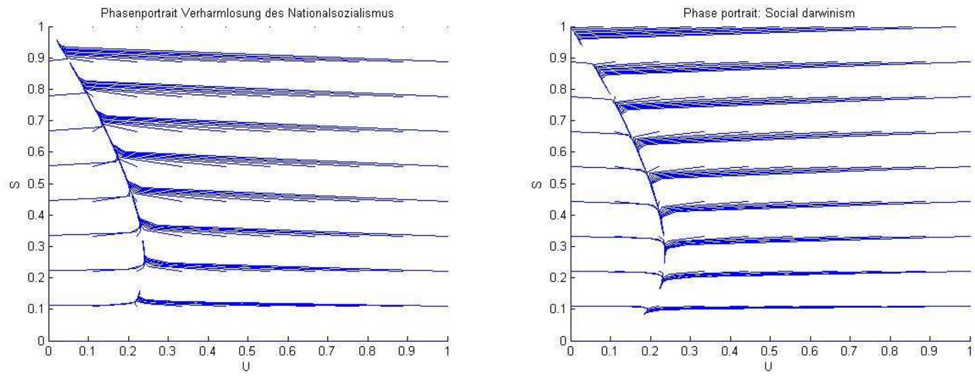

Figure 14: Phase portrait for system (30)–(31) using the optimal parameters. In "dimensions" trivialisation of National Socialism and social darwinism (0,0) is the global attractor.

4.6. Simulations

The optimal Parameters. The "natural" evolution of the German population from 2002 to 2020 with respect to right-wing extremism is shown by a simulation using the optimal parameters.

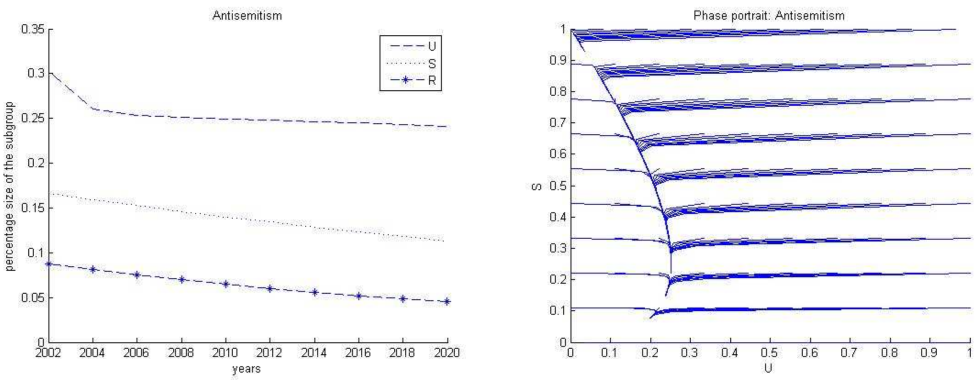

Figure 15: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" anti-Semitism. The subgroup of undecided people decreases strongly, then slightly to a value less then 0.25. The subgroups of semi fanatics and radicals decrease continuously to a value less then 0.1 and less then 0.05. The global attractor is (0,0).

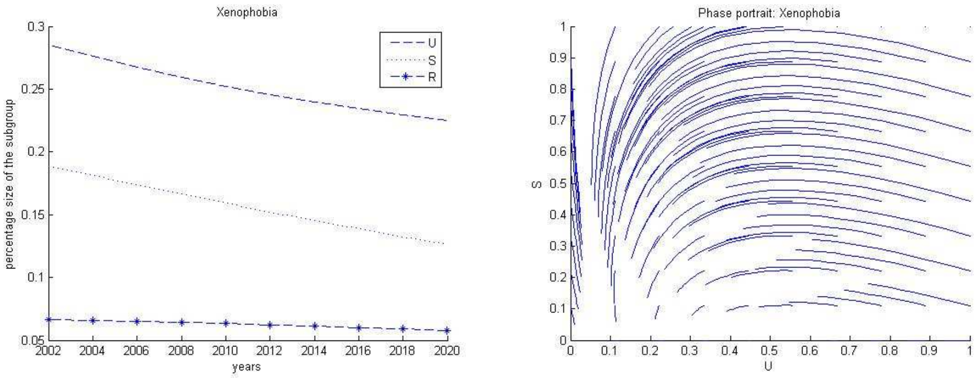

Figure 16: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" xenophobia. All subgroups decrease continuously. The subgroup of undecided people decreases to 0.225. The subgroup of semi fanatics decrease to 0.125 and the subgroup of radicals decrease to 0.05. The global attractor is (0,0).

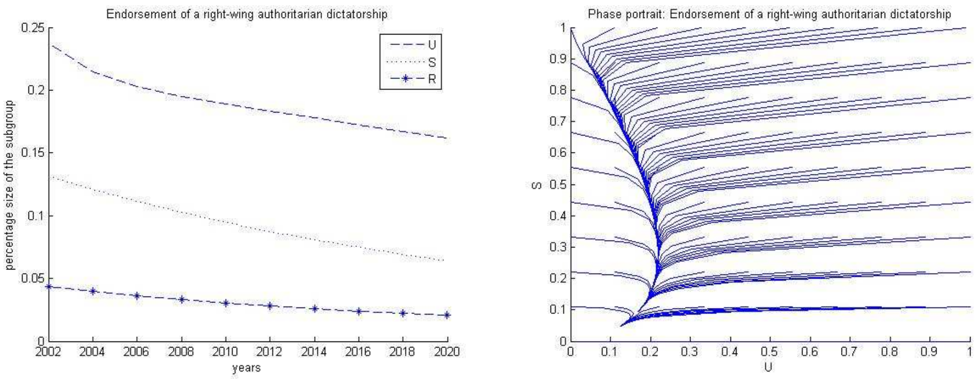

Figure 17: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" endorsement of a right-wing authoritarian dictatorship. The subgroup of undecided people decreases strongly, then slightly to 0.15. The subgroups of semi fanatics and radicals decrease continuously to 0.05 and 0.025. The global attractor is (0,0).

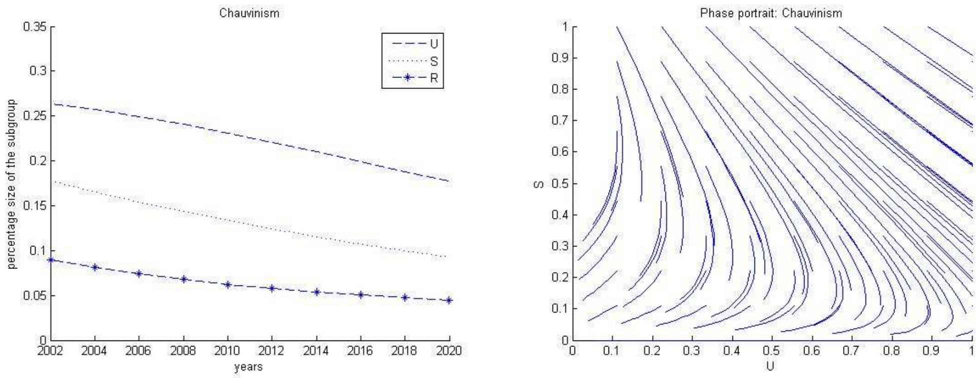

Figure 18: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" chauvinism. All subgroups decrease continuously. The subgroup of undecided people decreases to 0.175. The subgroups of semi fanatics and radicals decrease to a value less then 0.1 and less then 0.05. The global attractor is (0,0).

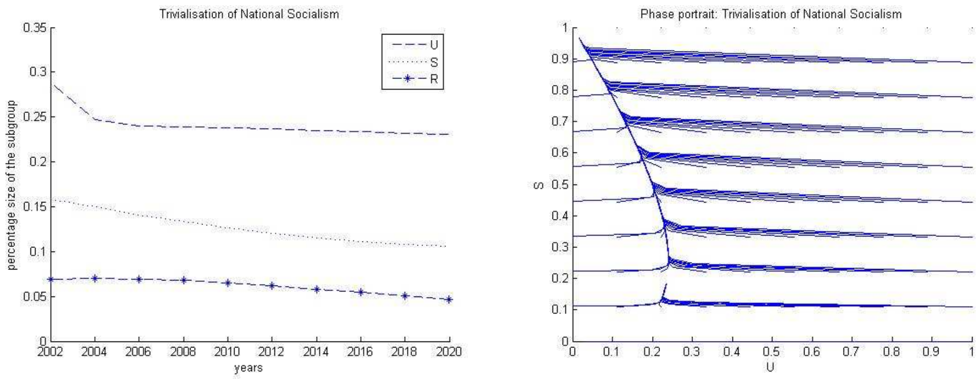

Figure 19: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" trivialisation of National Socialism. The subgroup of undecided people decreases strongly, then slightly to 0.225. The subgroup of semi fanatics decreases continuously to 0.1. The subgroup of radicals increases slightly, then decreases to a value less then 0.05. The global attractor is (0,0).

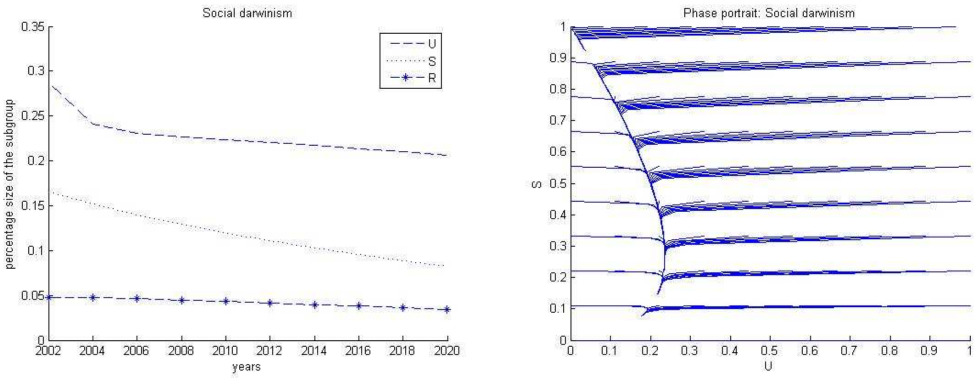

Figure 20: Solutions and phase portrait of the simulation with the optimal parameters and data from the "dimension" social darwinism. The subgroup of undecided people decreases strongly, then slightly to 0.2. The subgroups of semi fanatics and radicals decrease continuously to a value less then 0.1 and less then 0.05. The global attractor is (0,0).

Recruitment parameters. In the following simulations some parameters are changed and unless otherwise noted we use the optimal parameters. The aim of this paragraph is to show the functioning of the model.

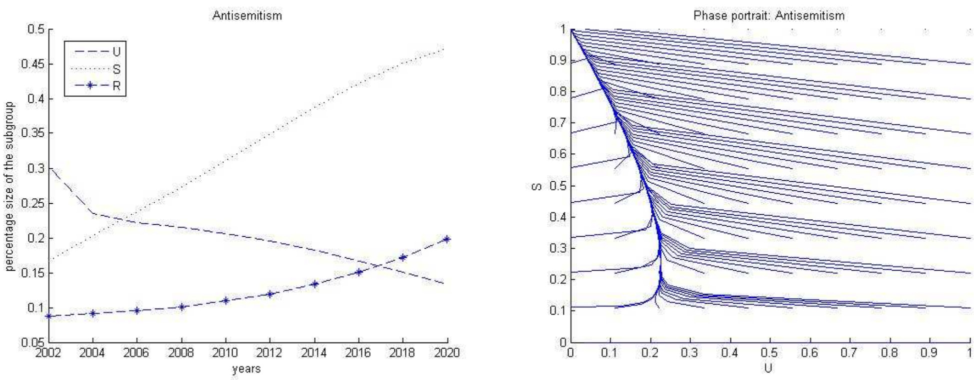

Figure 21: Solutions and phase portrait of the simulation with $\mu_2 = 0.3$ and $\mu_3 = 0.2$ in the "dimension" anti-Semitism. The subgroup of undecided people decreases very strongly, then strongly. The subgroup of semi fanatics increases continuously. The subgroup of radicals increases more and more. The global attractor is (0,1).

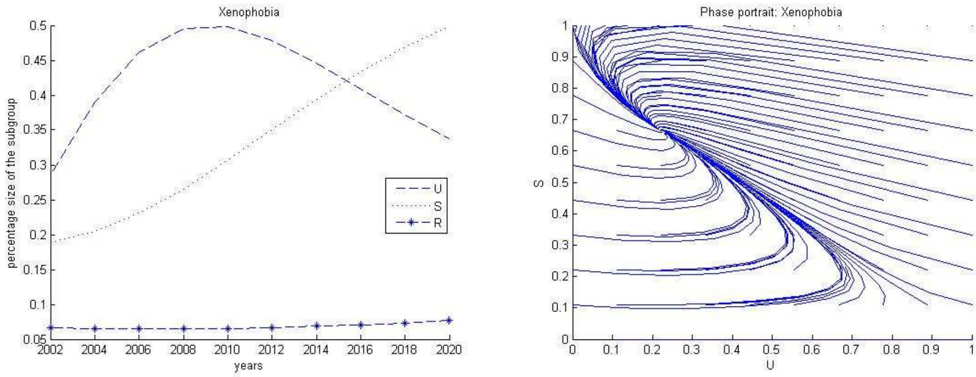

Figure 22: Solutions and phase portrait of the simulation with $\mu_1 = 0.3$, $\mu_2 = 0.2$ and $\mu_3 = 0.1$ in the "dimension" xenophobia. The subgroup of undecided people increases strongly, then decreases strongly. The subgroup of semi fanatics increases continuously. The subgroup of radicals increases slightly. The global attractor is (0.19578, 0.686).

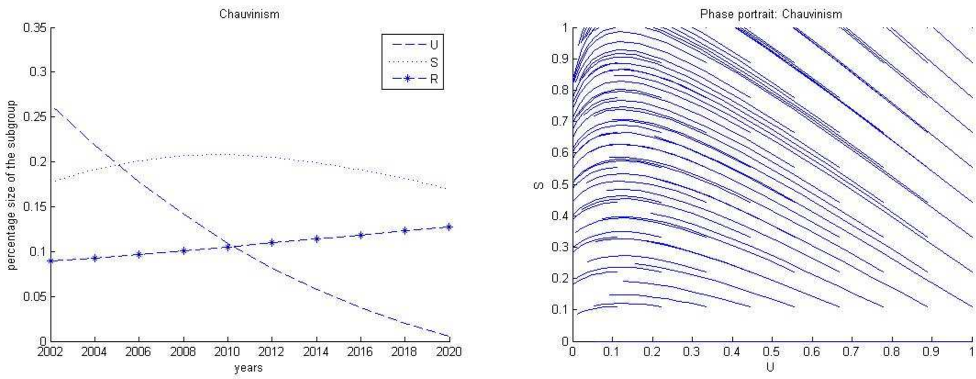

Figure 23: Solutions and phase portrait of the simulation with $\mu_2 = 0.2$ and $\mu_3 = 0.1$ in the "dimension" chauvinism. The subgroup of undecided people decreases strongly. The subgroup of semi fanatics increases slightly, then decreases slightly. The subgroup of radicals increases strongly. The global attractor is (0,0).

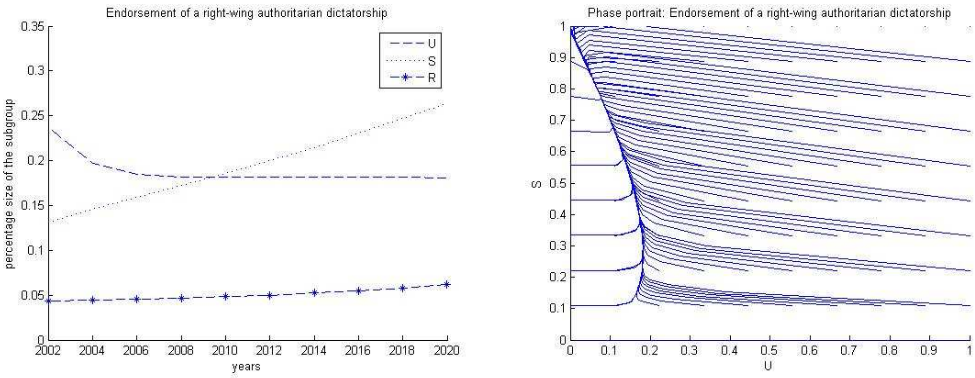

Figure 24: Solutions and phase portrait of the simulation with$\mu_2 = 0.3$ and $\mu_3 = 0.2$ in the "dimension" endorsement of a right-wing authoritarian dictatorship. The subgroup of undecided people decreases strongly, then decreases slightly. The subgroup of semi fanatics increases linear strongly. The subgroup of radicals increases more and more. The global attractor is (0.034, 0.8).

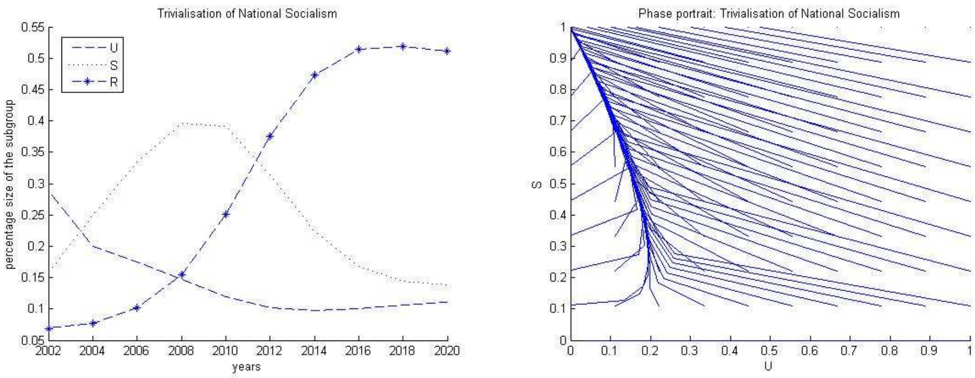

Figure 25: Solutions and phase portrait of the simulation with $\mu_2 = 0.95$ in the "dimension" trivialisation of national socialism. The subgroup of undecided people decreases strongly, then decreases slightly and finally increases slightly. The subgroup of semi fanatics increases strongly, then is more or less constant, decreases strongly and finally decreases slightly. The subgroup of radicals increases more and more, then decreases slightly. The global attractor is (0,1).

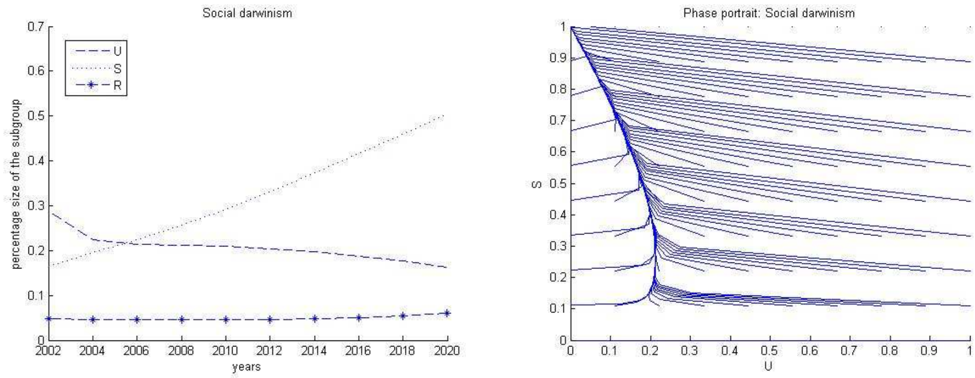

Figure 26: Solutions and phase portrait of the simulation with $\mu_2 = 0.3$ and $\mu_3 = 0.2$ in the "dimension" social darwinism. The subgroup of undecided people decreases strongly, then decreases slightly. The subgroup of semi fanatics increases linear strongly. The subgroup of radicals increases more and more. The global attractor is (0,1).

Recovery parameters. For modelling more scenarios we change the recovery parameters γ_i with $i = 1, 2, 3$ in addition to the recruitment parameters μ_i.

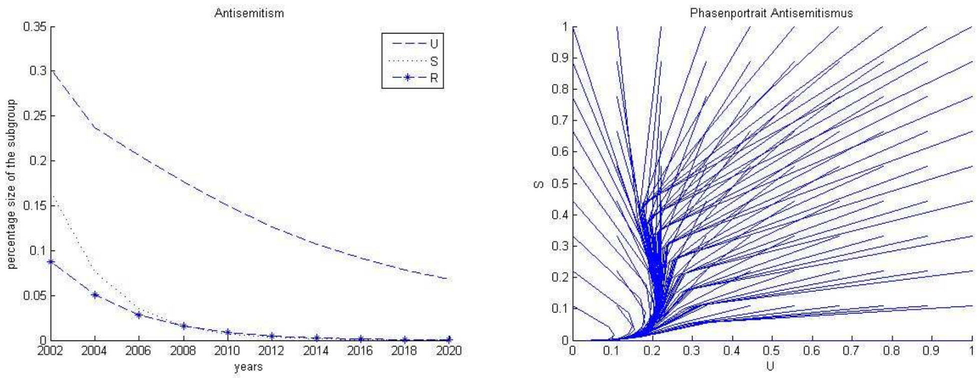

Figure 27: Solutions and phase portrait of the simulation with $\mu_2 = 0.3$, $\mu_3 = 0.2$ and $\gamma_2 = 0.5$, $\gamma_3 = 0.3$ in the "dimension" anti-Semitism. All subgroups decrease strongly, then decrease more and more slightly. The global attractor is (0,0).

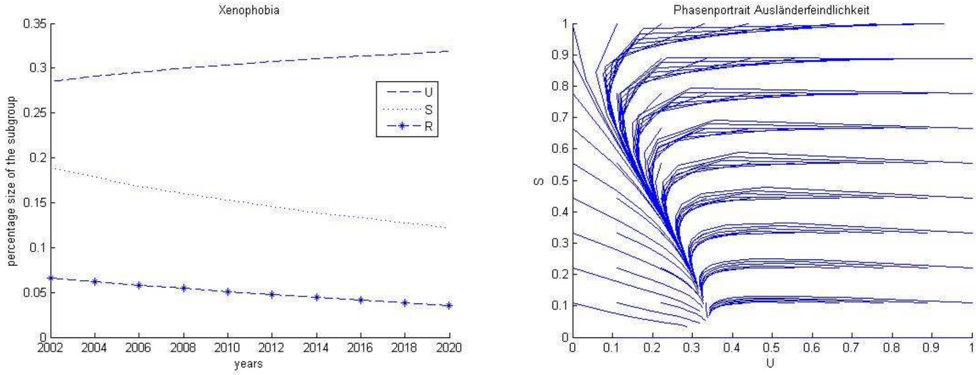

Figure 28: Solutions and phase portrait of the simulation with $\mu_1 = 0.3$, $\mu_2 = 0.2$, $\mu_3 = 0.1$ and $\gamma_1 = 0.2$, $\gamma_2 = 0.1$, $\gamma_3 = 0.05$ in the "dimension" xenophobia. The subgroup of undecided people increases slightly. The subgroups of semi fanatics and radicals decrease slightly. The global attractor is (0.25,0.25).

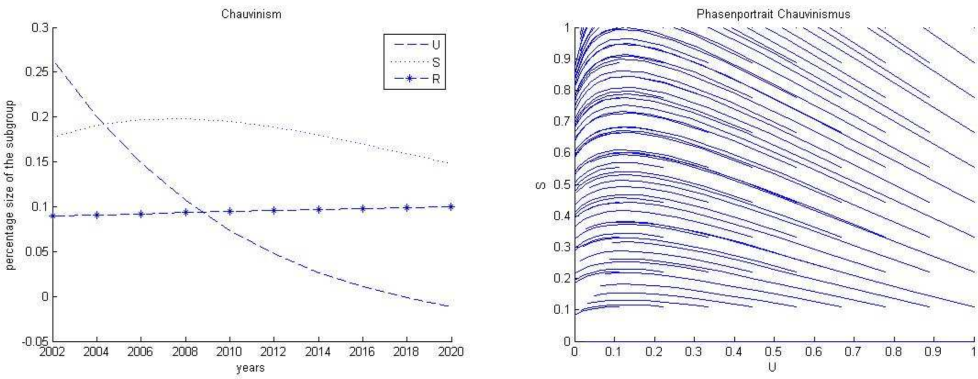

Figure 29: Solutions and phase portrait of the simulation with $\mu_2 = 0.2$, $\mu_3 = 0.1$ and $\gamma_1 = 0.05$, $\gamma_2 = 0.025$, $\gamma_3 = 0.0125$ in the "dimension" chauvinism. The subgroup of undecided people decreases less and less strongly. The subgroup of semi fanatics increases slightly, then decreases slightly. The subgroup of radicals increases linear slightly. The global attractor is (0,0).

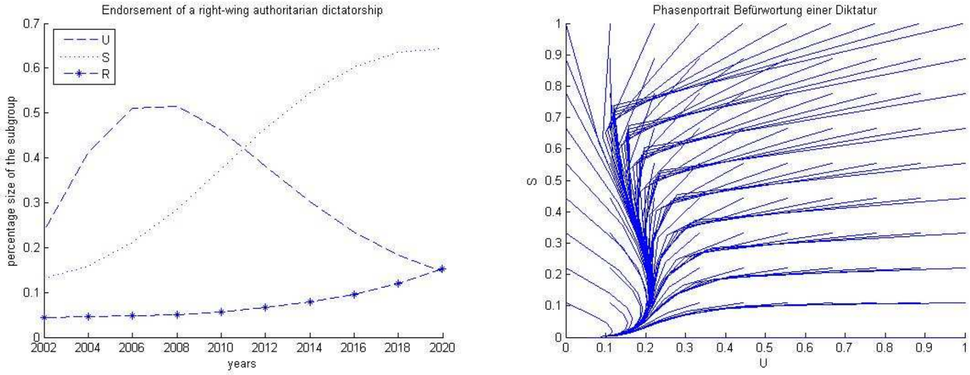

Figure 30: Solutions and phase portrait of the simulation with $\mu_2 = 0.3$, $\mu_3 = 0.2$ and $\gamma_1 = 0.5$, $\gamma_2 = 0.25$, $\gamma_3 = 0.125$ in the "dimension" endorsement of a right-wing authoritarian dictatorship. The subgroup of undecided people decreases continuously. The subgroups of semi fanatics and radicals decrease strongly, then decrease less and less. The global attractor is (0.069, 0.0139).

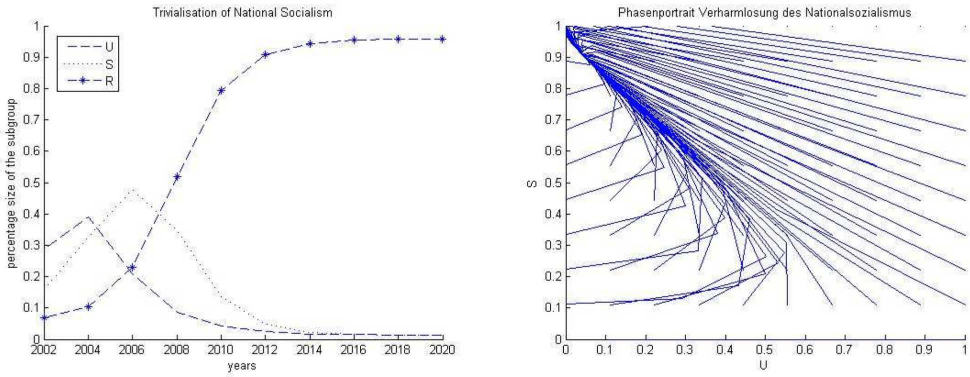

Figure 31: Solutions and phase portrait of the simulation with $\mu_2 = 0.95$ and $\gamma_1 = 0.1$, $\gamma_2 = 0.05$, $\gamma_3 = 0.0125$ in the "dimension" trivialisation of National Socialism. The subgroups of undecided people and semi fanatics increase strongly, then decrease strongly. The subgroup of radicals increases more and more, then increases slightly. The global attractor is (0.05, 0.89).

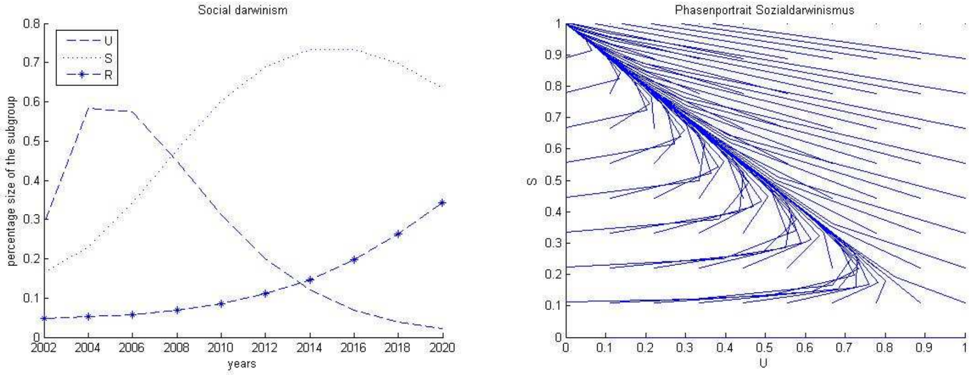

Figure 32: Solutions and phase portrait of simulation with $\mu_2 = 0.3$, $\mu_3 = 0.2$ and $\gamma_1 = \gamma_2 = \gamma_3 = 0$ in the "dimension" social darwinism. The subgroup of undecided people increases strongly, then decreases slightly and finally decreases strongly. The subgroup of semi fanatics increases more and more, then decreases slightly. The subgroup of radicals increases more and more. The global attractor is (0,1).

5. Conclusion

The six different "dimensions" of right-wing extremism have complex analysis facilities. The classification of society respective right-wing extremism leads to two different models each "dimension", which at their hearts consist of a system of ODEs. Contrary to a SIR-model both models built in this paper do not consider the possibility of immunity. Comparing the models with each other, it is clear that, contrary to the STV-model, the CCS-model includes a circulation (see Fig. 9). Individuals, which come from a core subgroup and go via "recovery" γ_i, i=1, 2, 3, back to the moderate part of society, become part of the recruitment pool again. γ_1 in the most "dimensions" of right-wing extremism is the highest one, whereby a remarkable small value ($\gamma_1 = 0.004$) occurs in "dimension" xenophobia. This implies that the decimation of subgroup U is significantly due to a transition to A. In "dimension" xenophobia subgroup U could be more susceptible to radicalization than to recovery. This conclusion is further suggested by $\mu_2 = 0.0819$, which is a positive value. When γ_1 is the highest parameter, it can be derived that a radicalization is stopped after the transition from A to U by an early movement back to the moderate part of society. Just a few individuals become more radical. Besides xenophobia, the "dimension" chauvinism makes an exception. In this "dimensions" μ_3 is according to amount the highest parameter. Because it is $\gamma_3 = 0$ at the same time, the decimation of subgroup R is explained by a transition from R to S.

In the STV-model the dynamics occurring are linear $G \leftrightarrow N \leftrightarrow U \leftrightarrow$

$S \leftrightarrow R$, which implies a step-by-step radicalization or moderation. Explicitly a separate immigration and emigration rate for subgroups G and N are considered. For each ”dimension” of right-wing extremism we get $\alpha_1 = 0$ and $\alpha_2 = 1$. The model gives the impression that the social climate in Germany causes an immigration to subgroup G and an emigration from subgroup N. The parameter l in ”dimension” xenophobia is relatively small, which implies a slight dynamic $R \to S$. In the ”dimensions” endorsement of a right-wing authoriterian dictatorship, chauvinism and anti-Semitism is $l = 1$, which leads to equipollent transitions $U \leftrightarrow S$ and $S \leftrightarrow R$. In all ”dimensions” is $\beta_1 < 0$ and at the same time $\beta_2 > 0$, whereby according to amount β_1 is higher than β_2. We interpret this as the central dynamical process

$$G \leftarrow N \leftarrow U \leftarrow S \leftarrow R.$$

Especially in the ”dimension” social darwinism k is relatively high, which implies a strong transition from N to G.
By calculating thresholds and attractors Castillo-Chavez and Song [5] showed that the most effective approach for the eradication of the right-wing extremism comes from sufficient effort to limit recruitment into the radical core group C: The control threshold is $L_1 = \mu_1/\gamma_1$. One way to bring this quantity below 1 is to reduce the value of μ_1. This reduction corresponds to an increase in the resistance of the general population A to ”advances” from the core. The elimination of the fanatic population R is critically important. $L_3 = \mu_3/\gamma_3 < 1$ implies that the fanatic population will crash regardless of its size. Since the value of γ_3 is actually a tiny number or, equivalently, since the residence time $1/\gamma_3$ is long, then it is quite unlikely that L_3 could be made less than 1, see [5, Sec. 7.5]. This scenario leads to an increase of R.

Definitions	Threshold Values	Attractors
$L_1 = \mu_1/\gamma_1$	$L_1 < 1$	$(0, 0, 0)$
$L_2 = \mu_2/\gamma_2$	$L_2 < 1 < L_1$	$(1 - 1/L_1, 0, 0)$
$L_d = \frac{\nu L_1 L_2}{L_2 - 1 + \nu}$	$L_d < 1$	$(0, 0, 0)$
$L_3 = \mu_3/\gamma_3$	$L_3 < 1$, $L_2 < 1$ and $L_1 < 1$	$(0, 0, 0)$
	$L_3 < 1$ and $L_2 < 1 < L_1$	$(1 - 1/L_1, 0, 0)$
	$L_3 < 1 < L_2$ and $L_d > 1$	$(U^*, S^*, 0)$

Table 4: Summary of Analytical results of the CCS-model cf. [5, Table 7.1]

Future work will consist in including stochastic terms and probabilistic prediction technique, following [6], to improve the modelling of the stochastic fluctuations between the different subgroups.

Acknowledgement

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