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Optimal Actuator and Observation Location for Time-Varying Systems on a Finite-Time Horizon *

Xueran Wu^{†‡}, Birgit Jacob[‡] and Hendrik Elbern^{†§}

Abstract

The choice of the location of controllers and observations is of great importance for designing control systems and improving the estimations in various practical problems. For time-varying systems in Hilbert spaces, the existence and convergence of the optimal location based on linear-quadratic control on a finite-time horizon is studied. The optimal location of observations for improving the estimation of the state at the final time, based on Kalman filter, is considered as the dual problem to the LQ optimal problem of the control locations. Further, the existence and convergence of optimal locations of observations for improving the estimation at the initial time, based on Kalman smoother is discussed. The obtained results are applied to a linear advection-diffusion model.

Keywords: Approximation, Kalman filter, Kalman smoother, linear-quadratic control, optimal observation location

1 Introduction

The choice of the locations of control hardware, such as sensors and actuators, plays an important role in the designs of control systems for many physical and engineering problems. Proper locations of sensors and actuators is essential to improve the performance of the controlled system. Many researchers have focused on the study of finding the optimal locations of control hardware and different criteria of optimising control locations were established, such as maximization of observability and controllability [17], [24], minimizing the linear quadratic regulator cost [23]. Geromel [14] successfully reformulated the LQ cost function into a convex optimization problem by mapping the locations of controller into zero-one vectors and expressed the solution of classic LQ problem in terms of a Riccati equation. Morris [22] optimized controller locations of time-invariant systems on an infinite-time

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horizon in Hilbert spaces by solving an algebraic Riccati equation and showed the convergence of optimal controller locations of a sequence of approximated finite-dimensional systems. Further, an algorithm [10] for the linear quadratic optimal problem of controller locations based on the convexity shown in [14] are introduced.

The issue of observations is also of great importance of many estimation problems for stochastic systems, such as weather forecasting and data assimilation in meteorology. For this kind of problems, observations always have low temporal and spatial density. The lack of observations is a major barrier of preventing the improvement of estimations and leading to the accuracy of predictions. On one hand, the insufficient observations become the main reason that many works are introduced to improve approaches of estimations in in the recent years. On the other hand, one possibility to improve the predictive or estimation skill for specific problems is to target the locations of observations which can potentially result in the largest forecast improvement in order to make observations more efficient. The better choice of locations of observations can help making more progress of the predictive or estimation skills. In contrast, improper observations probably make no sense to the accuracy of predictions and lead to the waste of resources by optimizing the improper parameters. There are several papers focusing on this problem from the perspective of applications. For finite-dimensional systems in practice, approaches based on singular value decomposition ([3], [4]) always help determining the direction with the strong influence of observations. However, it cannot solve the optimal problem of observation locations. Motivated by problems of data assimilation in meteorology, we estimate unknown random variables by Kalman filter and smoother, which has been theoretical foundation of one of the most popular data assimilation approaches in last decade. In fact, since 1960's, besides of applications in meteorology, the Kalman filter and smoother [19] were widely applied in many other fields to produce optimal linear estimations of states and parameters through a series of observations over time. It provides us an opportunity to define and search for optimal locations of observations by minimizing the covariance based on Kalman filter and smoother.

In this paper, we will start from the infinite-dimensional state space to consider the optimal location problem of controllers and observations for time-varying systems on a finite-time horizon. First, we study the linear-quadratic optimal location control problem for both deterministic and stochastic systems and develop conditions guaranteeing the existence of optimal locations of linear quadratic control problems in Section 2. Associated with practical applications, since optimal control problems cannot be solved directly in infinite-dimensional spaces, a sequence of approximations of the original time-varying system have to be considered. Thus, in Section 3, analogical to the approximation theory of time-invariant systems, we introduce the similar approximation conditions of evolution operators so as to ensure that the approximated control problems converge to the optimal control problem of the original infinite-dimensional time-varying system. Further, we show the convergence of minimal costs and optimal locations of the sequence of approximations. In Section 4 and Section 5, we derive the Kalman filter and smoother of time-varying systems in the integral form on Hilbert spaces. Then, by duality between Kalman filter and linear-quadratic optimal control, under certain conditions, the nuclearity of the covariance can be guaranteed. In Section 6 based on Kalman filter and smoother, the existence and convergence of optimal location observations of the estimation of the model state for stochastic

systems is shown. Finally, we apply the obtained results to a three-dimensional advection-diffusion model with the special construction of the emission rate in Section 7. In this example, the operator splitting technique with spatial and temporal discretization is applied to simulate the practical application in meteorology.

2 Existence of optimal actuator locations

Throughout this paper, we will always assume that the state space of the time-varying system is a real separable Hilbert space X , and the input and output space are Hilbert spaces denoted by U and Y , respectively. First, we introduce the notion of mild evolution operators for the time-varying system.

Definition 2.1. Denote $\Gamma_a^b := \{(t, s) \mid -\infty < a \leq s \leq t \leq b < \infty\}$. We call $T(\cdot, \cdot) : \Gamma_a^b \rightarrow \mathcal{L}(X)$ a mild evolution operator if

1. $T(t, t) = I$,
2. $T(t, r)T(r, s) = T(t, s)$, $a \leq s \leq r \leq t \leq b$,
3. $T(\cdot, s) : [s, b] \rightarrow \mathcal{L}(X)$ and $T(t, \cdot) : [a, t] \rightarrow \mathcal{L}(X)$ are strongly continuous.
4. $\lambda := \sup_{(t,s) \in \Gamma_a^b} \|T(t, s)\| < \infty$.

In the following we assume that $T(\cdot, \cdot) : \Gamma_a^b \rightarrow \mathcal{L}(X)$ is a mild evolution operator, and $B \in L_s^\infty(a, b; U, X)$ with $B^* \in L_s^\infty(a, b; X, U)$. Here

$$L_s^\infty(a, b; X, Y) := \{F : [a, b] \rightarrow \mathcal{L}(X, Y) \mid F \text{ is strongly measurable and } \|F\|_\infty := \operatorname{esssup}_{t \in [a, b]} \|F(t)\| < \infty\}.$$

For an initial time $t_0 \in [a, b]$, we consider the time-varying system described by

$$x(t) = T(t, t_0)x_0 + \int_{t_0}^t T(t, s)B(s)u(s)ds, \quad t \in [t_0, b], \quad (1)$$

where $x_0 \in X$ and $u \in L^2(t_0, b; U)$. We are interested in the following linear-quadratic optimal control problem.

Linear-Quadratic Optimal Control Problem: Find for $x_0 \in X$ a control $u_0 \in L^2(t_0, b; U)$ which minimizes the cost functional

$$J(t_0, x_0, u) = \langle x(b), Gx(b) \rangle + \int_{t_0}^b \|C(s)x(s)\|^2 + \langle u(s), F(s)u(s) \rangle ds, \quad (2)$$

where the function x is given by (1). Here $C \in L_s^\infty(a, b; X, Y)$, $G \in \mathcal{L}(X)$ and $F \in L_s^\infty(a, b; U, U)$, $F(t)$ is self-adjoint and nonnegative for fixed t , and $F^{-1} \in L_s^\infty(a, b; U, U)$.

It is well known, see [15], that the linear-quadratic optimal control problem possesses for $x_0 \in X$ a unique solution u_0 , which is given by $u_0(t) = -L(t)x(t)$, $t \in [t_0, b]$, $L(t) = F^{-1}(t)B^*(t)\Pi(t)$, such that the minimum of the cost functional is given by

$$\min_{u \in L^2(t_0, b; U)} J(t_0, x_0, u) = J(t_0, x_0, u_0) = \langle x_0, \Pi(t_0)x_0 \rangle,$$

where the self-adjoint nonnegative operator $\Pi(t)$ is the unique solution of the first integral Riccati equation (IRE)

$$\begin{aligned} \Pi(t)x &= T^*(b, t)GT(b, t)x \\ &+ \int_t^b T^*(s, t)[C^*(s)C(s) - \Pi(s)B(s)F^{-1}(s)B^*(s)\Pi(s)]T(s, t)x ds \end{aligned} \quad (3)$$

and the second IRE

$$\begin{aligned} \Pi(t)x &= T_{\Pi}^*(b, t)GT_{\Pi}(b, t)x \\ &+ \int_t^b T_{\Pi}^*(s, t)[C^*(s)C(s) + \Pi(s)B(s)F^{-1}(s)B^*(s)\Pi(s)]T_{\Pi}(s, t)x ds, \end{aligned} \quad (4)$$

where

$$T_{\Pi}(t, \tau)x = T(t, \tau)x - \int_{\tau}^t T(t, s)B(s)F^{-1}(s)B^*(s)\Pi(s)T_{\Pi}(s, \tau)x ds, \quad (t, \tau) \in \Gamma_a^b.$$

Now we consider the situation having the opportunity to choose m locations to control and each location varies over a compact set $\Omega \subset \mathbb{R}^l$. We indicate these m locations by the parameter $r \in \Omega^m$, and denote the location-dependent input operator $B(\cdot)$ by $B_r(\cdot)$. Throughout the rest of the paper, by a time-varying system with location-dependent input operator the time-varying system (1) and the cost functional (2) with B_r instead of B is meant. The corresponding solution of the IRE and the Riccati operator L are denoted by Π_r and L_r , respectively.

In most cases, the initial state x_0 is not fixed. This indicates several different ways to define the optimal actuator location problem. We take two possible ways into account here. The first one is to minimize the cost with the worst choice of initial value, which is

$$\max_{\|x_0\|=1} \min_{u \in L^2(t_0, b; U)} J_r(x_0, u) = \max_{\|x_0\|=1} \langle x_0, \Pi_r(t_0)x_0 \rangle = \|\Pi_r(t_0)\|.$$

Let $\ell^r(t_0) := \|\Pi_r(t_0)\|$, the optimal performance of r is $\hat{\ell}(t_0) = \inf_{r \in \Omega^m} \|\Pi_r(t_0)\|$.

The second one is to assume that the system is stochastic. Thus, we need to consider the trace of $\Pi_r(t_0)$ instead, since the trace indicates the sum of the deviation of the state vector in each coordinate. Thus the evaluation of the particular performance of r is given by the nuclear norm of $\Pi_r(t_0)$, which is $\ell_1^r(t_0) = \|\Pi_r(t_0)\|_1$. Further, the optimal performance is

$$\hat{\ell}_1(t_0) = \inf_{r \in \Omega^m} \|\Pi_r(t_0)\|_1.$$

For time-invariant problems on an infinite time horizon this problem was studied in [22]. In this section we prove the existence of optimal control locations for deterministic as well as stochastic time varying systems on a finite-time horizon.

Theorem 2.2. Let $\{B_r\}_r$, $r \in \Omega^m$, be a family of compact operator valued functions with the property that $\lim_{r \rightarrow r_0} \|B_r - B_{r_0}\|_\infty = 0$ for some $r_0 \in \Omega^m$. Then the solutions of the corresponding integral Riccati equations Π_r satisfy

$$\lim_{r \rightarrow r_0} \|\Pi_r(t) - \Pi_{r_0}(t)\| = 0, \quad t \in [a, b],$$

and there exists an optimal location \hat{r} such that for any initial time $t_0 \in [a, b]$,

$$\hat{\ell}(t_0) = \|\Pi_{\hat{r}}(t_0)\| = \inf_{r \in \Omega^m} \|\Pi_r(t_0)\|.$$

Proof. Thanks to the assumptions on B_r , there exists $\delta > 0$ such that $\lambda_B := \sup\{\|B_r(t)\| \mid t \in [a, b], \|r - r_0\| \leq \delta\} < \infty$. We denote by $\mathbf{B}(r_0, \delta)$ the set $\mathbf{B}(r_0, \delta) := \{r \in \Omega^m : \|r - r_0\| \leq \delta\}$. Thus, [15, Theorem 5.1] implies for every $x \in X$

$$\Pi_r(t)x \rightarrow \Pi_{r_0}(t)x, \quad r \rightarrow r_0.$$

For any feedback control $\tilde{u}(t) = \tilde{L}(t)x(t)$, $\tilde{L} \in L_s^\infty(a, b; X, U)$,

$$\begin{aligned} \langle x(t), \Pi_r(t)x(t) \rangle &\leq J(t, x(t), \tilde{u}) \\ &= \langle x(b), Gx(b) \rangle + \int_t^b \|C(s)x(s)\|^2 + \langle \tilde{L}(s)x(s), F(s)\tilde{L}(s)x(s) \rangle ds \\ &= \|G^{\frac{1}{2}}T_{\tilde{L}, r}(b, t)x(t)\|^2 \\ &\quad + \int_t^b \|C(s)T_{\tilde{L}, r}(s, t)x(t)\|^2 + \|F^{\frac{1}{2}}(s)\tilde{L}(s)T_{\tilde{L}, r}(s, t)x(t)\|^2 ds, \end{aligned}$$

where $T_{\tilde{L}, r}(t, \tau)x = T(t, \tau)x + \int_\tau^t T(t, s)B_r(s)\tilde{L}(s)T_{\tilde{L}, r}(s, \tau)x ds$, $(t, \tau) \in \Gamma_a^b$.

Since the family B_r is uniformly bounded by λ_B on $\mathbf{B}(r_0, \delta)$, [15, Theorem 2.1] implies for all $r \in \mathbf{B}(r_0, \delta)$, $(t, \tau) \in \Gamma_a^b$, $\|T_L(t, \tau)\| \leq \lambda \exp(\lambda \lambda_B \|\tilde{L}\|_\infty (t - \tau))$. Further, because $C \in L_s^\infty(a, b; X, Y)$, $F \in L_s^\infty(a, b; U, U)$, there exists a constant λ_Π , independent of t and $r \in \mathbf{B}(r_0, \delta)$, such that $\|\Pi_r\|_\infty \leq \lambda_\Pi$.

For $S_r = C^*C - L_r^*FL_r$, where $L_r = F^{-1}B_r^*\Pi_r$, we obtain

$$\Pi_r(t)x - \Pi_{r_0}(t)x = \int_t^b T^*(s, t)(S_r(s) - S_{r_0}(s))T(s, t)x ds, \quad x \in X.$$

Since $F^{-1} \in L_s^\infty(a, b; U, U)$ and the operator $B_{r_0}(t)$ is compact for any $t \in [a, b]$, we have

$$\begin{aligned} \|L_r^*(t) - L_{r_0}^*(t)\| &\leq \|F^{-1}\|_\infty (\|\Pi_r(t)\| \|B_r(t) - B_{r_0}(t)\| \\ &\quad + \|(\Pi_r(t) - \Pi_{r_0}(t))B_{r_0}(t)\|) \longrightarrow 0, \quad r \rightarrow r_0, \end{aligned}$$

which shows

$$\begin{aligned} \|S_r(t) - S_{r_0}(t)\| &\leq \|L_{r_0}^*(t) - L_r^*(t)\| \|F(t)L_{r_0}(t)\| \\ &\quad + \|L_r^*(t)F(t)\| \|L_{r_0}(t) - L_r(t)\| \longrightarrow 0, \quad r \rightarrow r_0. \end{aligned}$$

From the uniform boundedness of F , B_r and Π_r on $\mathbf{B}(r_0, \delta)$, L_r and further S_r are uniformly bounded for all $t \in [a, b]$ and $\mathbf{B}(r_0, \delta)$. According, thanks to the dominated convergence theorem, we obtain $\|\Pi_r(t) - \Pi_{r_0}(t)\| \rightarrow 0$, $r \rightarrow r_0$.

Additionally, since $r \in \Omega^m$, Ω^m is a compact set, there exists an optimal location \hat{r} such that $\|\Pi_{\hat{r}}(t_0)\| = \inf_{r \in \Omega^m} \|\Pi_r(t_0)\|$. \square

Theorem 2.2 shows the continuity of optimal actuator locations and existence of the optimal location in the operator norm. For stochastic systems, the above problem leads to the nuclear norm. Thus, first we develop conditions which guarantee that the Riccati operator is a nuclear operator. Similar to [8, Theorem 3.1], we have

Theorem 2.3. *Let $T(\cdot, \cdot)$ be a mild evolution operator on X , $B \in L_s^\infty(a, b; \mathbb{C}^p, X)$, and $C \in L_s^\infty(a, b; X, \mathbb{C}^q)$. Then for any $t_0 \in [a, b]$ we have:*

1. *The observability operator $\mathcal{C}_{t_0} : X \rightarrow L^2(t_0, b; \mathbb{C}^q)$ defined by*

$$(\mathcal{C}_{t_0}x_0)(\cdot) = C(\cdot)T(\cdot, t_0)x_0, \quad x_0 \in X,$$

is a Hilbert-Schmidt operator;

2. *The controllability operator $\mathcal{B}_{t_0} : L^2(t_0, b; \mathbb{C}^p) \rightarrow X$ defined by*

$$\mathcal{B}_{t_0}u = \int_{t_0}^b T(b, s)B(s)u(s)ds$$

is a Hilbert-Schmidt operator;

3. *$\mathcal{C}_{t_0}^* \mathcal{C}_{t_0}$ and $\mathcal{B}_{t_0} \mathcal{B}_{t_0}^*$ are nuclear operators.*

Proof. 1. Defining $\mathcal{C}_{t_0, i} : X \rightarrow L^2(t_0, b)$, $i \in \{1, \dots, q\}$

$$(\mathcal{C}_{t_0, i}x_0)(s) = \langle C(s)T(s, t_0)x_0, e_i \rangle, \quad s \geq t_0,$$

where $\{e_i\}$ is the standard orthogonal basis of \mathbb{C}^q . We have

$$\begin{aligned} |(\mathcal{C}_{t_0, i}x_0)(s)| &= |\langle C(s)T(s, t_0)x_0, e_i \rangle| \leq \|C(s)T(s, t_0)x_0\| \|e_i\| \\ &\leq \|C(s)\| \|T(s, t_0)\| \|x_0\| < \infty. \end{aligned}$$

[26, Theorem 6.12] implies that $\mathcal{C}_{t_0, i}$ is Hilbert-Schmidt, that is, for any orthogonal basis $\{\bar{e}_i\}$ of X , we have $\sum_{i=1}^q \sum_{j=1}^\infty \|\mathcal{C}_{t_0, i} \bar{e}_j\|_{L^2(t_0, b)}^2 < \infty$. Since

$$\|\mathcal{C}_{t_0} \bar{e}_j\|_{L^2(t_0, b)}^2 = \sum_{i=1}^q \|\mathcal{C}_{t_0, i} \bar{e}_j\|_{L^2(t_0, b)}^2,$$

we have

$$\sum_{j=1}^\infty \|\mathcal{C}_{t_0} \bar{e}_j\|_{L^2(t_0, b; \mathbb{C}^q)}^2 = \sum_{i=1}^q \sum_{j=1}^\infty \|\mathcal{C}_{t_0, i} \bar{e}_j\|_{L^2(t_0, b)}^2 < \infty,$$

which shows that \mathcal{C}_{t_0} is a Hilbert-Schmidt operator.

2. According to [26, Theorem 6.9], \mathcal{B}_{t_0} is Hilbert-Schmidt if and only if $\mathcal{B}_{t_0}^*$ is Hilbert-Schmidt. An easy calculation shows $\mathcal{B}_{t_0}^* : X \rightarrow L^2(t_0, b; U)$,

$$(\mathcal{B}_{t_0}^* x)(\cdot) = B_{t_0}^*(\cdot)T^*(b, \cdot)x$$

From part 1, $\mathcal{B}_{t_0}^*$ is Hilbert-Schmidt, and so is \mathcal{B}_{t_0} .

3. Since $\|\mathcal{C}_{t_0}^* \mathcal{C}_{t_0}\|_1 \leq \|\mathcal{C}_{t_0}^*\|_{HS} \|\mathcal{C}_{t_0}\|_{HS} < \infty$ and $\|\mathcal{B}_{t_0}^* \mathcal{B}_{t_0}\|_1 \leq \|\mathcal{B}_{t_0}^*\|_{HS} \|\mathcal{B}_{t_0}\|_{HS} < \infty$, $\mathcal{C}_{t_0}^* \mathcal{C}_{t_0}$ and $\mathcal{B}_{t_0}^* \mathcal{B}_{t_0}$ are nuclear operator. \square

Corollary 2.4. *Assume that the input space U and the output space Y are finite-dimensional and G that is a nuclear operator, then the unique nonnegative self-adjoint solution $\Pi(t_0)$ of the integral Riccati equation is a nuclear operator.*

Proof. Defining the bounded operator $\mathcal{C}_{t_0} : X \rightarrow L^2(t_0, b; U \times Y)$ by

$$(\mathcal{C}_{t_0} x_0)(\cdot) = \begin{pmatrix} C(\cdot) \\ F^{\frac{1}{2}}(\cdot)L(\cdot) \end{pmatrix} T_L(\cdot, t_0)x_0, \quad L = F^{-1}B^*\Pi.$$

\mathcal{C}_{t_0} is Hilbert-Schmidt by Theorem 2.3.1 The second IRE (4) can be rewritten as

$$\Pi(t_0)x = T_L^*(b, t_0)GT_L(b, t_0)x + \mathcal{C}_{t_0}^* \mathcal{C}_{t_0} x, \quad x \in X.$$

Form Theorem 2.3.3 and the nuclearity of G , $\Pi(t)$ is a nuclear operator. \square

Lemma 2.5. *Assume $T(\cdot, \cdot)$ and $T_i(\cdot, \cdot)$, $i \in \mathbb{N}$, are mild evolution operators which are uniformly bounded by λ_T , $D_i, D \in L_s^\infty(a, b; X, X)$ satisfy $\|D_i(t)x - D(t)x\| \rightarrow 0$ as $i \rightarrow \infty$ for every $x \in X$ and $\sup_i \{\|D_i\|_\infty, \|D\|_\infty\} \leq \lambda_D$. $T_{D_i}(\cdot, \cdot)$, $T_D(\cdot, \cdot)$ denote the perturbed evolution operators corresponding to the perturbation of $T_i(\cdot, \cdot)$ by D_i and $T(\cdot, \cdot)$ by D . If $\|T_i(t, \tau)x - T(t, \tau)x\| \rightarrow 0$ as $i \rightarrow \infty$ for $x \in X$, then for any $(t, \tau) \in \Gamma_a^b$ and $x \in X$,*

$$\|T_{D_i}(t, \tau)x - T_D(t, \tau)x\| \rightarrow 0, \quad i \rightarrow \infty.$$

Proof. As in [6], we construct $T_{D_i}(t, \tau)$ as $T_{D_i}(t, \tau) = \sum_{n=0}^{\infty} T_{D_i, n}(t, \tau)$, where

$$T_{D_i, 0}(t, \tau) = T_i(t, \tau), \quad T_{D_i, n}(t, \tau)x = \int_{\tau}^t T_i(t, s)D_i(s)T_{D_i, n-1}(s, \tau)x ds, \quad x \in X.$$

By induction we obtain $\|T_{D_i, n}(t, \tau)\| \leq \lambda_T(\lambda_T \lambda_D)^n \frac{(t-\tau)^n}{n!}$. $T_D(t, \tau)$ can be constructed in a similar manner with the same upper bound.

Defining $d_{i, n}(t, \tau) = T_{D_i, n}(t, \tau) - T_{D, n}(t, \tau)$, we have $d_{i, 0}(t, \tau) = T_i(t, \tau) - T(t, \tau)$,

$$\begin{aligned} d_{i, n}(t, \tau) &= \int_{\tau}^t T_i(t, s)D_i(s)d_{i, n-1}(s, \tau)ds + \int_{\tau}^t T_i(t, s)[D_i(s) - D(s)]T_{D, n-1}(s, \tau)ds \\ &\quad + \int_{\tau}^t [T_i(t, s) - T(t, s)]D(s)T_{D, n-1}(s, \tau)ds. \end{aligned}$$

The uniform boundedness of $\{T_{D_i}(t, \tau)\}_{i \in \mathbb{N}}$ and $T_D(t, \tau)$ implies

$$\left\| \sum_{n=0}^{\infty} d_{i,n}(t, \tau) \right\| \leq \|T_{D_i}(t, \tau)\| + \|T_D(t, \tau)\| < \infty.$$

Due to $\sup_i \{\|D_i\|_{\infty}, \|D\|_{\infty}\} \leq \lambda_D$ and $T(\cdot, \cdot)$, $T_i(\cdot, \cdot)$ are uniformly bounded, the mild evolution operators $T_D(\cdot, \cdot)$, $T_{D_i}(\cdot, \cdot)$, are uniformly bounded, and further for any $n \in \mathbb{N}$, $\sup_i \sup_{(t, \tau) \in \Gamma_{t_0}^b} \|d_{i,n}(t, \tau)\| < \infty$. Meanwhile, since $\|D_i(t)x - D(t)x\| \rightarrow 0$, $\|d_{i,0}(t, \tau)x\| = \|T_i(t, \tau)x - T(t, \tau)x\| \rightarrow 0$, $i \rightarrow \infty$. Hence,

$$\begin{aligned} \|d_{i,n}(t, \tau)x\| &\leq \int_{\tau}^t \|T_i(t, s)\| \|D_i(s)\| \|d_{i,n-1}(t, \tau)x\| ds \\ &\quad + \int_{\tau}^t \|T_i(t, s)\| \|[D_i(s) - D(s)]T_{D,n-1}(s, \tau)x\| ds \\ &\quad + \int_{\tau}^t \|(T_i(t, s) - T(t, s))(s)T_{D,n-1}(s, \tau)x\| ds \rightarrow 0, \quad i \rightarrow \infty. \end{aligned} \quad (5)$$

By dominated convergence theorem,

$$\|T_{D_i}(t, \tau)x - T_D(t, \tau)x\| \leq \sum_{n=0}^{\infty} \|d_{i,n}(t, \tau)x\| \rightarrow 0, \quad i \rightarrow \infty.$$

□

Corollary 2.6. *For any mild evolution operator $T(\cdot, \cdot)$ with uniform bound λ_T and $D_i, D \in L_s^{\infty}(t_0, b; X, X)$ with $\sup_i \{\|D_i\|_{\infty}, \|D\|_{\infty}\} \leq \lambda_D$, if $\|D_i(t) - D(t)\| \rightarrow 0$, then for $T_D(\cdot, \cdot)$ which is the perturbation of $T(\cdot, \cdot)$ by D_i and $T_{D_i}(\cdot, \cdot)$ which is the perturbation evolution operator of $T(\cdot, \cdot)$ by D , we have*

$$\|T_{D_i}(t, \tau) - T_D(t, \tau)\| \rightarrow 0, \quad i \rightarrow \infty.$$

Proof. From the assumptions, let $T_i = T$ in Lemma 2.5, replace (5) by

$$\begin{aligned} \|d_{i,n}(t, \tau)\| &\leq \int_{\tau}^t \|T(t, s)\| \|D_i(s)\| \|d_{i,n-1}(t, \tau)\| ds \\ &\quad + \int_{\tau}^t \|T(t, s)\| \|[D_i(s) - D(s)]T_{D,n}(s, \tau)\| ds \rightarrow 0, \quad i \rightarrow \infty. \end{aligned}$$

Then, we can prove the uniform convergence of $T_{D_i}(t, \tau)$ by the dominated convergence theorem in the similar way with Lemma 2.5. □

Theorem 2.7. *We consider the time-varying system (1) with the location-dependent input operators and the cost functional (2). Assume $\{B_r\}_{r \in \Omega^m}$ satisfies $\lim_{r \rightarrow r_0} \|B_r - B_{r_0}\|_{\infty} = 0$, for some $r_0 \in \Omega^m$, U and Y are finite-dimensional and G is a nuclear operator, then*

$$\lim_{r \rightarrow r_0} \|\Pi_r(t) - \Pi_{r_0}(t)\|_1 = 0, \quad t \in [t_0, b]$$

and there exists an optimal location \hat{r} such that

$$\hat{\ell}_1(t) = \|\Pi_{\hat{r}}(t_0)\|_1 = \inf_{r \in \Omega^m} \|\Pi_r(t_0)\|_1.$$

Proof. Similar to Theorem 2.2, there exists $\delta > 0$ such that $\sup_{r \in \mathbf{B}(r_0, \delta)} \|B_r\| < \infty$, $r_0 \in \Omega^m$ and for every $x \in X$ and $t \in [t_0, b]$,

$$\Pi_r(t)x \rightarrow \Pi_{r_0}(t)x, \quad r \rightarrow r_0.$$

Further, from (5), we have Π_r are uniformly bounded with λ_Π for any $t \in [t_0, b]$ and $r \in \mathbf{B}(r_0, \delta)$.

Defining the operator $\mathcal{C}_{t,r} : X \rightarrow L^2(t, b; U \times Y)$, $t \in [t_0, b]$,

$$(\mathcal{C}_{t,r}x(t))(\cdot) = \begin{pmatrix} C(\cdot) \\ -F^{\frac{1}{2}}(\cdot)B_r^*(\cdot)\Pi_r(\cdot) \end{pmatrix} T_{L,r}(\cdot, t)x(t). \quad (6)$$

Corollary 2.4 has shown $\mathcal{C}_{t,r}$ is a Hilbert-Schmidt operator and

$$\Pi_r(t) = T_{L,r}^*(b, t)GT_{L,r}(b, t) + \mathcal{C}_{t,r}^*\mathcal{C}_{t,r}.$$

is nuclear if G is nuclear.

Now let us show that $\mathcal{C}_{t,r}$ uniformly converges to \mathcal{C}_{t,r_0} in Hilbert-Schmidt norm. Let $\{e_i\}_{i=1}^{p+q}$ and $\{\bar{e}_i\}_{i=1}^\infty$ be respectively the orthogonal basis of $U \times Y$ and X , then

$$\begin{aligned} & \|\mathcal{C}_{t,r} - \mathcal{C}_{t,r_0}\|_{HS} \\ &= \sum_{i=1}^{\infty} \int_t^b \sum_{j=1}^{p+q} \langle (\mathcal{C}_{t,r}\bar{e}_i)(s) - (\mathcal{C}_{t,r_0}\bar{e}_i)(s), e_j \rangle_{U \times Y} ds \\ &= \sum_{i=1}^{\infty} \int_t^b \sum_{j=1}^{p+q} |\langle \bar{e}_i, T_{L,r}^*(s, t)[C^*(s), L_r^*(s)F^{\frac{1}{2}}(s)]e_j \\ & \quad - T_{L,r_0}^*(s, t)[C^*(s), L_{r_0}^*(s)F^{\frac{1}{2}}(s)]e_j \rangle_X|^2 ds \\ &= \int_t^b \sum_{i=1}^{\infty} \sum_{j=1}^{p+q} |\langle \bar{e}_i, T_{L,r}^*(s, t)[C^*(s), L_r^*(s)F^{\frac{1}{2}}(s)]e_j \\ & \quad - T_{L,r_0}^*(s, t)[C^*(s), L_{r_0}^*(s)F^{\frac{1}{2}}(s)]e_j \rangle_X|^2 ds \\ &= \sum_{j=1}^{p+q} \int_t^b \|T_{L,r}^*(s, t)[C^*(s), L_r^*(s)F^{\frac{1}{2}}(s)]e_j - T_{L,r_0}^*(s, t)[C^*(s), L_{r_0}^*(s)F^{\frac{1}{2}}(s)]e_j\|_X^2 ds, \end{aligned}$$

where $L_r = F^{-1}B_r^*\Pi_r$.

From Theorem 2.2, we have $\lim_{r \rightarrow r_0} \|L_r(t) - L_{r_0}(t)\| = 0$ and $\|L_r\|_\infty < \infty$. Then,

$$\lim_{r \rightarrow r_0} \|B_r(t)L_r(t) - B_{r_0}(t)L_{r_0}(t)\| = 0$$

and $\|B_r L_r\|_\infty < \infty$. Hence, from Corollary 2.6, for any $(s, t) \in \Gamma_{t_0}^b$, $T_{L,r}(s, t)$ uniformly converges to $T_{L,r_0}(s, t)$. Therefore,

$$\begin{aligned} & \|T_{L,r}^*(s, t)[C^*(s), L_r^*(s)F^{\frac{1}{2}}(s)]e_j - T_{L,r_0}^*(s, t)[C^*(s), L_{r_0}^*(s)F^{\frac{1}{2}}(s)]e_j\|_X \\ \leq & \|(T_{L,r}^*(s, t) - T_{L,r_0}^*(s, t)^*)[C^*(s), L_r^*(s)F^{\frac{1}{2}}(s)]e_j\| \\ & + \|T_{L,r_0}^*(s, t)[0, (L_r^*(s) - L_{r_0}^*(s))F^{\frac{1}{2}}(s)]e_j\| \longrightarrow 0, \quad r \rightarrow r_0. \end{aligned} \quad (7)$$

By dominated convergence theorem, $\|\mathcal{C}_{t,r} - \mathcal{C}_{t,r_0}\|_{HS} \rightarrow 0$, $r \rightarrow r_0$. Further, if G is a nuclear operator,

$$\begin{aligned} & \|\Pi_r(t) - \Pi_{r_0}(t)\|_1 \\ \leq & \|T_{L,r}^*(b, t) - T_{L,r_0}^*(b, t)\| \|GT_{L,r}(b, t)\|_1 + \|T_{L,r_0}^*(b, t)G\|_1 \|T_{L,r}(b, t) - T_{L,r_0}(b, t)\| \\ & + \|\mathcal{C}_{t,r}^* - \mathcal{C}_{t,r_0}^*\|_{HS} \|\mathcal{C}_{t,r}\|_{HS} + \|\mathcal{C}_{t,r_0}^*\|_{HS} \|\mathcal{C}_{t,r} - \mathcal{C}_{t,r_0}\|_{HS} \rightarrow 0, \quad r \rightarrow r_0. \end{aligned}$$

By the compactness of Ω^m , the optimal location \hat{r} exists in nuclear norm. \square

3 Convergence of optimal control locations

In practice, the integral Riccati equation in an infinite-dimensional space cannot be solved directly. Usually, we approximate and solve it in finite-dimensional space by a sequence of approximations from various numerical methods. Let X_n be a family of finite-dimensional subspaces of X and \mathbf{P}_n be the corresponding orthogonal projection of X onto X_n . The finite-dimensional spaces $\{X_n\}$ inherit the norm from X . For every $n \in \mathbb{N}$, let $T_n(\cdot, \cdot)$ be a mild evolution operator on X_n , $B_n(t) \in L_s^\infty(t_0, b; U, X_n)$ and $C_n(t) = C(t)\mathbf{P}_n$, $G_n \in \mathcal{L}(X_n)$. This defines a sequence of approximations

$$x(t) = T_n(t, t_0)x(t_0) + \int_{t_0}^t T_n(t, s)B_n(s)u(s)ds, \quad t \in [t_0, b]$$

with the cost functional

$$J_n(t, x, u) = \langle x(b), G_n x(b) \rangle + \int_t^b \langle C_n(s)x(s), C_n(s)x(s) \rangle + \langle u(s), F(s)u(s) \rangle ds.$$

We denote the optimal control of the approximation by $u_n(t) = -L_n(t)\mathbf{P}_n x(t)$, $t \in [t_0, b]$, where $L_n(t) = F^{-1}B_n^*\Pi_n$, the perturbed evolution operator of $T_n(\cdot, \cdot)$ by $-B_n L_n$ by $T_{L_n}(\cdot, \cdot)$ and the Riccati operator of the approximation by Π_n .

In order to guarantee that $\Pi_n(t)$ converges to $\Pi(t)$, the following assumptions are needed in the approximation of control problem for partial differential equations [15]. For each $x \in X$, $u \in U$, $y \in Y$, when $n \rightarrow \infty$,

$$(a1) \quad (i) \quad T_n(t, s)\mathbf{P}_n x \rightarrow T(t, s)x; \quad (ii) \quad T_n^*(t, s)\mathbf{P}_n x \rightarrow T^*(t, s)x$$

and $\sup_n \|T_n(t, s)\| < \infty$, $(t, s) \in \Gamma_{t_0}^b$.

$$(a2) \quad (i) \quad B_n(t)u \rightarrow B(t)u; \quad (ii) \quad B_n^*(t)\mathbf{P}_n x \rightarrow B^*(t)x, \quad a.e..$$

- (a3) (i) $C_n(t)\mathbf{P}_n x \rightarrow C(t)x$; (ii) $C_n^*(t)y \rightarrow C^*(t)y$, a.e..
(a4) $\sup_n \|G_n\| < \infty$ and $G_n\mathbf{P}_n x \rightarrow Gx$.

Before we study the uniform convergence from $\Pi_n(t)$ to $\Pi(t)$, we study under which condition the compactness of $\Pi(t)$ can be guaranteed. The following lemma shows this.

Lemma 3.1. *We consider the time-varying system (1) with the cost functional (2). If $B(t)$, $C(t)$, $t \in [t_0, b]$ and G are compact operators, then the unique solution $\Pi(t)$ of the integral Riccati equation (4) is compact.*

Proof. Denote $S = C^*C + \Pi BF^{-1}B^*\Pi$,

$$\Pi(t) = T_L^*(b, t)GT_L(b, t) + \int_t^b T_L^*(s, t)S(s)T_L(s, t)ds.$$

Since $B(t)$, $C(t)$ and G are compact, $T_L^*(b, t)GT_L(b, t)$ and $T_L^*(s, t)S(s)T_L(s, t)$, $(s, t) \in \Gamma_{t_0}^b$ are compact. Let us only consider the integral part of $\Pi(t)$ firstly. It is clear that there exists a set of orthogonal projections $\{\mathbf{P}_n\}$ to some finite-dimensional spaces X_n , $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \|\mathbf{P}_n T_L^*(s, t)S(s)T_L(s, t) - T_L^*(s, t)S(s)T_L(s, t)\| = 0$. Then, since $T_L(\cdot, \cdot)$ and $S(\cdot)$ are uniformly bounded, it is easy to obtain $\mathbf{P}_n T_L^* S T_L$ is also uniformly bounded in any time and n . By the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \left\| \int_t^b \mathbf{P}_n T_L^*(s, t)S(s)T_L(s, t)ds - \int_t^b T_L^*(s, t)S(s)T_L(s, t)ds \right\| = 0.$$

Obviously, $\int_t^b \mathbf{P}_n T_L^*(s, t)S(s)T_L(s, t)ds$ is still finite-rank operator and bounded, so it is compact.

Therefore, $\int_t^b T_L^*(s, t)S(s)T_L(s, t)ds$ is compact. Further, $\Pi(t)$ is compact. \square

The following theorem shows the uniform convergence of $\Pi_n(t)$.

Theorem 3.2. *For the sequence of approximations under the assumptions (a1) – (a4), if $B(t)$, $C(t)$, $t \in [t_0, b]$ and G are compact operators and $\lim_{n \rightarrow \infty} \|B_n - \mathbf{P}_n B\|_\infty = 0$, then*

$$\lim_{n \rightarrow \infty} \|\Pi_n(t)\mathbf{P}_n - \Pi(t)\| = 0, \quad t \in [t_0, b].$$

Proof. From $\lim_{n \rightarrow \infty} \|B_n - \mathbf{P}_n B\|_\infty = 0$ and $\sup_{t \in [t_0, b]} \|B(t)\| < \infty$, we have

$$\sup_{n \in \mathbb{N}, t \in [t_0, b]} \|B_n(t)\| < \infty.$$

Moreover, because $B(t)$ is compact and \mathbf{P}_n is strongly convergent to the identity operator I , $\lim_{n \rightarrow \infty} \|\mathbf{P}_n B(t) - B(t)\| = 0$, $t \in [t_0, b]$. Further,

$$\|B_n(t) - B(t)\| \leq \|B_n(t) - \mathbf{P}_n B(t)\| + \|\mathbf{P}_n B(t) - B(t)\| \rightarrow 0, \quad t \in [t_0, b], \quad n \rightarrow \infty.$$

Meanwhile, by the uniform boundedness of $\|T_n(\cdot, \cdot)\|$, $\|C_n\|_\infty$ and $\|G_n\|$ and [15, Theorem 5.1], for any $x \in X$,

$$\lim_{n \rightarrow \infty} \|\Pi_n(t)x - \Pi(t)x\| = 0, \quad t \in [t_0, b].$$

Similar to the proof of the uniform boundedness of Π_r in Theorem 2.2, for the approximations with arbitrary feedback control $\tilde{u}_n(t) = \tilde{L}_n(t)x(t) = \tilde{L}(t)\mathbf{P}_n x(t)$, $\tilde{L} \in L_s^\infty(t_0, b; X, U)$, there exists $\lambda_\Pi > 0$, such that $\sup_n \|\Pi_n\|_\infty < \lambda_\Pi$.

To proof the uniform convergence of $\Pi_n(t)$, we define $S_n = C_n^* C_n + \Pi_n B_n F^{-1} B_n^* \Pi_n$ and S with the similar way, then

$$\begin{aligned} & \|\Pi_n(t)\mathbf{P}_n - \Pi(t)\| \leq \|(T_{L_n}^*(b, t) - T_L^*(b, t))G_n\mathbf{P}_n\| \|T_{L_n}(b, t)\mathbf{P}_n\| \\ & + \|T_L^*(b, t)\| \|(G_n\mathbf{P}_n - G)T_{L_n}(b, t)\mathbf{P}_n\| + \|T_L^*(b, t)\| \|G(T_{L_n}(b, t)\mathbf{P}_n - T_L(b, t))\| \\ & + \int_t^b \|(T_{L_n}^*(s, t) - T_L^*(s, t))S_n(s)\mathbf{P}_n\| \|T_{L_n}(s, t)\mathbf{P}_n\| ds \\ & + \int_t^b \|T_L^*(s, t)\| \|S_n(s)\mathbf{P}_n - S(s)\| \|T_{L_n}(s, t)\mathbf{P}_n\| ds \\ & + \int_t^b \|T_L(s, t)\| \|S(s)(T_{L_n}(s, t)\mathbf{P}_n - T_L(s, t))\| ds. \end{aligned}$$

As a result of the uniform boundedness of $\|T_n(\cdot, \cdot)\|$, $\|\Pi_n\|_\infty$ and $\|B_n\|_\infty$ in n , $\|L_n\|_\infty$ is uniform bounded and

$$\begin{aligned} \|L_n^*(t) - \mathbf{P}_n L^*(t)\| & \leq \|F^{-1}\|_\infty (\|\Pi_n(t)\| \|B_n(t) - \mathbf{P}_n B(t)\| \\ & + \|\Pi_n(t) - \mathbf{P}_n \Pi(t)\| \|B(t)\|) \rightarrow 0, \quad r \rightarrow r_0. \end{aligned}$$

so $\lim_{n \rightarrow \infty} \|L_n(t)\mathbf{P}_n - L(t)\| = 0$ and $\lim_{n \rightarrow \infty} \|B_n(t)L_n(t)\mathbf{P}_n - B(t)L(t)\| = 0$.

According to Lemma 2.5 and assumption (a1),

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T_{L_n}(t, s)\mathbf{P}_n x - T_L(t, s)x\| & = 0, \\ \lim_{n \rightarrow \infty} \|T_{L_n}^*(t, s)\mathbf{P}_n x - T_L^*(t, s)x\| & = 0, \quad x \in X. \end{aligned}$$

Finally, because of the compactness of the self-adjoint operator G_n and G , we have

$$\lim_{n \rightarrow \infty} \|(T_{L_n}(t, s) - T_L(t, s))G_n\mathbf{P}_n\| = 0$$

and $\lim_{n \rightarrow \infty} \|G(T_{L_n}(t, s)\mathbf{P}_n - T_L(t, s))\| = 0$. Meanwhile, $\|S_n\|_\infty \leq \|C_n^* C_n\|_\infty + \|\Pi_n B_n F^{-1} B_n^* \Pi_n\|_\infty < \infty$, $n \in \mathbb{N}$. Since $C_n = C\mathbf{P}_n$ is compact,

$$\begin{aligned} & \|S_n(t)\mathbf{P}_n - S(t)\| \\ & \leq \|C_n^*(t)C_n(t) - C^*(t)C(t)\| + \|L_n^*(t)F(t)L_n(t)\mathbf{P}_n - L(t)F(t)L(t)\| \\ & \leq \|C^*(t)\mathbf{P}_n - C^*(t)\| \|C_n\|_\infty + \|C^*\|_\infty \|C(t)\mathbf{P}_n - C(t)\| \\ & \quad + \|L_n^*(t) - L^*(t)\| \|F\|_\infty \|L_n\|_\infty + \|L^*\|_\infty \|F\|_\infty \|L_n(t)\mathbf{P}_n - L(t)\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

By dominated convergence theorem, $\|\Pi_n(t)\mathbf{P}_n - \Pi(t)\| \rightarrow 0$, $n \rightarrow \infty$. \square

Next we show that the optimal control locations of approximations converge to the optimal control location of the original system.

Theorem 3.3. *Under the assumptions (a1) – (a4) and further assume $B_{r,n} = \mathbf{P}_n B_r$, $r \in \Omega^m$, if $B_r(t)$, $C(t)$ and G , $t \in [t_0, b]$ are compact operators and $\lim_{r \rightarrow r_0} \|B_r - B_{r_0}\| = 0$, then*

$$\hat{\ell}_n(t) \rightarrow \hat{\ell}(t), \quad \hat{r}_n \rightarrow \hat{r}, \quad n \rightarrow \infty.$$

Proof. From Theorem 3.2, $\lim_{n \rightarrow \infty} \|\Pi_{r,n}(t)\mathbf{P}_n - \Pi_r(t)\| = 0$, $r \in \Omega^m$.

Since $\lim_{r \rightarrow r_0} \|B_r - B_{r_0}\|_\infty = 0$,

$$\|B_{r,n} - B_{r_0,n}\|_\infty \leq \|\mathbf{P}_n\| \|B_r - B_{r_0}\|_\infty \rightarrow 0, \quad r \rightarrow r_0.$$

From Theorem 2.2, for any $n \in \mathbb{N}$, there exists $\hat{\ell}_n(t) = \inf_{r \in \Omega^m} \|\Pi_{r,n}(t)\|$.

On one hand,

$$\begin{aligned} \hat{\ell}_n(t) &= \inf_{r \in \Omega^m} \|\Pi_{r,n}(t)\| \leq \|\Pi_{\hat{r},n}(t)\| \leq \|\Pi_{\hat{r},n}(t) - \Pi_{\hat{r}}(t)\| + \|\Pi_{\hat{r}}(t)\| \\ &\rightarrow \|\Pi_{\hat{r}}(t)\| = \hat{\ell}(t), \quad n \rightarrow \infty, \end{aligned}$$

so $\lim_{n \rightarrow \infty} \sup_n \hat{\ell}_n(t) \leq \hat{\ell}$.

On the other hand, there exists a subsequence $\{\hat{\ell}_{n_k}(t)\}$ such that $\lim_{k \rightarrow \infty} \hat{\ell}_{n_k}(t) = \lim_{n \rightarrow \infty} \inf_n \hat{\ell}_n(t)$, where $\hat{\ell}_{n_k}(t) = \inf_{r \in \Omega^m} \|\Pi_{r,n_k}(t)\| = \|\Pi_{r_{n_k},n_k}(t)\|$. Due to the compactness of Ω^m , without loss of the generality, we assume $\lim_{k \rightarrow \infty} \hat{r}_{n_k} = \bar{r}$,

$$\|B_{\hat{r}_{n_k},n_k} - B_{\bar{r}}\|_\infty \leq \|\mathbf{P}_{n_k}\| \|B_{\hat{r}_{n_k}} - B_{\bar{r}}\|_\infty + \|\mathbf{P}_{n_k} B_{\bar{r}} - B_{\bar{r}}\|_\infty \rightarrow 0, \quad k \rightarrow \infty$$

and

$$\|\Pi_{\hat{r}_{n_k},n_k}(t) - \Pi_{\bar{r}}(t)\| \leq \|\Pi_{\hat{r}_{n_k},n_k}(t) - \Pi_{r_{n_k}}(t)\| + \|\Pi_{r_{n_k}}(t) - \Pi_{\bar{r}}(t)\| \rightarrow 0, \quad k \rightarrow \infty. \quad (8)$$

Hence,

$$\lim_{n \rightarrow \infty} \inf_n \hat{\ell}_n(t) = \lim_{k \rightarrow \infty} \hat{\ell}_{n_k}(t) = \lim_{k \rightarrow \infty} \|\Pi_{\hat{r}_{n_k},n_k}(t)\| = \|\Pi_{\bar{r}}(t)\| \geq \|\Pi_{\hat{r}}(t)\| = \hat{\ell}(t),$$

so $\lim_{n \rightarrow \infty} \hat{\ell}_n(t) = \hat{\ell}(t)$. Further, $\lim_{n \rightarrow \infty} \hat{\ell}_n(t) = \lim_{n \rightarrow \infty} \inf_n \hat{\ell}_n(t) = \hat{\ell}(t)$, so

$$\lim_{k \rightarrow \infty} \|\Pi_{\hat{r}_{n_k},n_k}(t)\| = \|\Pi_{\bar{r}}(t)\| = \|\Pi_{\hat{r}}(t)\|.$$

By the continuity with respect to r_{n_k} in (8), $\lim_{k \rightarrow \infty} \hat{r}_{n_k} = \hat{r}$. \square

For the proof of the uniform convergence of the Riccati operators of the approximations in nuclear norm for stochastic systems, we need the following lemma.

Lemma 3.4. *Let G be a nonnegative nuclear operator in a separable Hilbert space X and assume that T_n strongly converges to T , $T_n, T \in \mathcal{L}(X)$ are uniformly bounded by λ_T . Then*

$$\lim_{n \rightarrow \infty} \|(T_n - T)G\|_1 = 0.$$

Proof. Assume $\{e_i\}$ is the orthogonal basis in X and there exist a partial isometry V such that $G = V|G|$, where $|G| = (G^*G)^{\frac{1}{2}}$, then,

$$\|(T_n - T)V|G|^{\frac{1}{2}}e_i\| \leq \|T_n - T\| \|V|G|^{\frac{1}{2}}e_i\| \leq 2\lambda_T \|V|G|^{\frac{1}{2}}e_i\|.$$

Because of the strong convergence of T_n , $\lim_{n \rightarrow \infty} \|(T_n - T)V|G|^{\frac{1}{2}}e_i\| = 0$.

Since G is a nuclear operator, then $|G|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, so

$$\sum_{i=1}^{\infty} \|(T_n - T)V|G|^{\frac{1}{2}}e_i\| = 2\lambda_T \sum_{i=1}^{\infty} \|V|G|^{\frac{1}{2}}e_i\| < \infty.$$

By the dominated convergence theorem,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|(T_n - T)V|G|^{\frac{1}{2}}\|_{HS} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|(T_n - T)V|G|^{\frac{1}{2}}e_i\| = \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} \|(T_n - T)V|G|^{\frac{1}{2}}e_i\| = 0. \end{aligned}$$

Then, $\|(T_n - T)G\|_1 \leq \|(T_n - T)V|G|^{\frac{1}{2}}\|_{HS} \| |G|^{\frac{1}{2}} \|_{HS} \rightarrow 0$, $n \rightarrow \infty$. \square

Associated with Corollary 2.4, the following theorem guarantees the uniform convergence of the Riccati operators of approximations to the Riccati operator of the original system in nuclear norm.

Theorem 3.5. *For the sequence of approximations under the assumptions (a1) – (a4), if U and Y are finite dimensional, $\lim_{n \rightarrow \infty} \|B_n - \mathbf{P}_n B\|_{\infty} = 0$, G is nuclear operator and $\lim_{n \rightarrow \infty} \|G_n \mathbf{P}_n - G\|_1 = 0$, then*

$$\lim_{n \rightarrow \infty} \|\Pi_n(t) \mathbf{P}_n - \Pi(t)\|_1 = 0.$$

Proof. Defining \mathcal{C}_t in the same way with Corollary 2.4 and define $\mathcal{C}_{t,n}$ by substituting n into r in (6), from Theorem 2.3.1, $\Pi_n(t) = T_{L_n}^*(b, t)G_n T_{L_n}(b, t) + \mathcal{C}_{t,n}^* \mathcal{C}_{t,n}$ is nuclear. The same with Theorem 3.2, we also have the uniform boundedness of $\|T_n(\cdot, \cdot)\|$, $\|\Pi_n\|_{\infty} \|B_n\|_{\infty}$, $\|L_n\|_{\infty}$ in n and $\lim_{n \rightarrow \infty} \|L_n(t) \mathbf{P}_n - L(t)\| = 0$, $\lim_{n \rightarrow \infty} \|T_{L_n}(t, s) \mathbf{P}_n x - T_L(t, s)x\| = 0$. Hence, similar to Theorem 2.7,

$$\begin{aligned} & \|\mathcal{C}_{t,n} - \mathcal{C}_t\|_{HS} \leq \sum_{j=1}^{p+q} \int_t^b \|(T_{L_n}^*(s, t) - T_L(s, t)^*)[C_n^*(s), L_n^*(s)F^{\frac{1}{2}}(s)]e_j\| \\ & + \|T_L^*(s, t)[C_n(s) - C(s), (L_n^*(s) - L^*(s))F^{\frac{1}{2}}(s)]e_j\| ds \rightarrow 0, \quad r \rightarrow r_0, \quad (s, t) \in \Gamma_{t_0}^b. \end{aligned}$$

Then, since G is nuclear operator with $\lim_{n \rightarrow \infty} \|G_n \mathbf{P}_n - G\|_1 = 0$,

$$\begin{aligned} & \|\Pi_n(t) \mathbf{P}_n - \Pi(t)\|_1 \\ & \leq \|T_{L_n}^*(b, t)\| \|G_n(T_{L_n}(b, t) \mathbf{P}_n - T_L(b, t))\|_1 + \|(T_{L_n}^*(b, t) - T_L^*(b, t))G_n\|_1 \|T_L(b, t)\| \\ & + \|T_L^*(b, t)\| \|G_n \mathbf{P}_n - G\|_1 \|T_L(b, t)\| + \|\mathcal{C}_{t,n}^* - \mathcal{C}_t^*\|_{HS} \|\mathcal{C}_{t,n}\|_{HS} \\ & + \|\mathcal{C}_t^*\|_{HS} \|\mathcal{C}_{t,n} - \mathcal{C}_t\|_{HS} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

\square

Theorem 3.6. *Under the assumptions (a1) – (a4) and further assume $B_{r,n} = \mathbf{P}_n B_r$, $r \in \Omega^m$, if the input space U and the output space Y are finite dimensional, $\lim_{r \rightarrow r_0} \|B_r - B_{r_0}\| = 0$, G is nuclear operator and $\lim_{n \rightarrow \infty} \|G_n P_n - G\|_1 = 0$, then*

$$\hat{\ell}_{1,n}(t) \rightarrow \hat{\ell}_1(t), \quad \hat{r}_n \rightarrow \hat{r}, \quad n \rightarrow \infty.$$

Proof. From Theorem 2.7 and Theorem 3.5, we have $\lim_{r \rightarrow r_0} \|\Pi_r(t) - \Pi_{r_0}(t)\|_1 = 0$ and $\lim_{n \rightarrow \infty} \|\Pi_{r,n}(t) \mathbf{P}_n - \Pi_r(t)\|_1 = 0$. The same with Theorem 3.3, we have $\hat{\ell}_{1,n}(t) \leq \|\Pi_{\hat{r}}(t)\|_1 = \hat{\ell}_1(t)$, $n \rightarrow \infty$. Besides, there exists a subsequence $\{\hat{\ell}_{1,n_k}(t)\}$ such that

$$\liminf_{n \rightarrow \infty} \hat{\ell}_{1,n}(t) = \lim_{k \rightarrow \infty} \hat{\ell}_{1,n_k}(t) = \lim_{k \rightarrow \infty} \|\Pi_{\hat{r}_{n_k}, n_k}(t)\|_1 = \|\Pi_{\hat{r}}(t)\|_1 \geq \|\Pi_{\hat{r}}(t)\|_1 = \hat{\ell}_1(t).$$

Therefore, $\lim_{n \rightarrow \infty} \hat{\ell}_{1,n}(t) = \hat{\ell}_1(t)$ and $\lim_{k \rightarrow \infty} \|\Pi_{\hat{r}_{n_k}, n_k}(t)\|_1 = \|\Pi_{\hat{r}}(t)\|_1$. By the continuity in Theorem 2.7, $\lim_{k \rightarrow \infty} \hat{r}_{n_k} = \hat{r}$. \square

4 Kalman filter in Hilbert spaces and the duality of LQ optimal control problem

There are several literatures [5], [6], [13], [18] discussing the Kalman filter in different approaches and the duality to the linear-quadratic optimal control. However, to author's knowledge, these derivations involve the generator of semigroups or evolution operators. In this section, without the differentiability of evolution operators, we first derive the Kalman filter in real separable Hilbert spaces. Further, we will discuss the duality between Kalman filter and linear-quadratic optimal control.

Let $(\Omega, \mathcal{B}, \mu)$ be a complete probability space and $\mathcal{X}, \mathcal{E}, \mathcal{Y}$ be a real separable Hilbert spaces. First, we define some basic concepts of probability theory in Hilbert spaces [6], [25].

Definition 4.1. *The map $x : \Omega \rightarrow \mathcal{X}$ is a \mathcal{X} -valued random variable if it is strong measurable with respect to a measure μ .*

Definition 4.2. μ is a totally finite measure on \mathcal{X} if for any \mathcal{X} -valued random variable x , $\int_{\Omega} \|x\| d\mu < \infty$. Further, if there exists $\bar{x} \in \mathcal{X}$ such that

$$\langle \bar{x}, h \rangle = \mathbf{E}\langle \bar{x}, h \rangle = \int_{\Omega} \langle \bar{x}, h \rangle \mu(dx), \quad \forall h \in \mathcal{X},$$

\bar{x} is called the mean or expectation of x and denoted by $\mathbf{E}x$.

Definition 4.3. *For any \mathcal{X} -valued random variable x with mean $\mathbf{E}x$, the covariance operator P of x , also denoted by $\text{Cov}(x)$, if it exists, is given by*

$$\langle Ph_1, h_2 \rangle = \langle h_1, Ph_2 \rangle = \int_{\Omega} \langle x - \mathbf{E}x, h_1 \rangle \langle x - \mathbf{E}x, h_2 \rangle \mu(dx), \quad \forall h_1, h_2 \in \mathcal{X}.$$

Definition 4.4. *The random variables x, y whose expectations exist are independent if $\mathbf{E}(\langle x, y \rangle) = \langle \mathbf{E}(x), \mathbf{E}(y) \rangle$.*

Definition 4.5. Let μ be a probability measure on \mathcal{X} . If for any $x \in \mathcal{X}$, the random variable $\langle x, \cdot \rangle$ has a Gaussian distribution, then μ is called a Gaussian measure. Further, we denote x of the Gaussian measure with mean \bar{x} and covariance P by $x \sim N(\bar{x}, P)$.

Definition 4.6. $\{\omega(t), t \in \mathbb{R}\}$ is a set of white noises if for any $t \in [0, +\infty]$, there exists a covariance operator $W(t)$ such that $\omega(t) \sim N(0, W(t))$ and for any $t \neq s$, $\omega(t)$ and $\omega(s)$ are independent.

We consider time-varying systems on Hilbert spaces given by

$$x(t) = M(t, t_0)x(t_0) + \int_{t_0}^t M(t, s)[B(s)u(s) + D(s)\omega(s)]ds, \quad (t, t_0) \in \Gamma_{t_0}^b, \quad (9)$$

where $M(\cdot, \cdot)$ is a mild evolution operator on \mathcal{X} . $x(t)$ and $\omega(t)$ are random variables with values in \mathcal{X} and \mathcal{E} , respectively and $\omega(t) \sim N(0, W(t))$ is the white noise. Further, we assume $u \in L^2(t_0, b; U)$, $B \in L_s^\infty(t_0, b; U, \mathcal{X})$, $B^* \in L_s^\infty(t_0, b; \mathcal{X}, U)$, $D \in L_s^\infty(t_0, b; \mathcal{E}, \mathcal{X})$.

We consider the following observation system

$$y(t) = H(t)x(t) + E(t)\nu(t), \quad t \in [t_0, b], \quad (10)$$

where $H \in L_s^\infty(t_0, b; \mathcal{X}, \mathcal{Y})$, $E \in L_s^\infty(t_0, b; \mathcal{E}, \mathcal{Y})$, $y(t)$ and $\nu(t)$ are random variables with values in \mathcal{Y} and \mathcal{E} , respectively and $\nu(t) \sim N(0, V(t))$ is the white noise and $V(t)$ is a coercive operator.

In our paper, we only consider the integral form of time-varying systems. Let $Y_t = \{y(s), t_0 \leq s \leq t\}$, the linear unbiased estimation of the filter problem $\hat{x}(t|t)$ of $x(t)$ [19] has the form

$$\begin{aligned} \hat{x}(t|t) &= M(t, t_0)\hat{x}(t_0|t_{-1}) \\ &+ \int_{t_0}^t M(t, s)B(s)u(s)ds + \int_{t_0}^t K_f(t, s)[y(s) - H(s)\hat{x}(s|s)]ds, \end{aligned} \quad (11)$$

where $\hat{x}(t_0|t_{-1}) = \mathbf{E}(x(t_0))$, $Cov(x(t_0)) = P(t_0|t_{-1})$ and $K_f(\cdot, \cdot)$ is an unknown linear gain operator.

Denoting $\tilde{x}(t|t) := x(t) - \hat{x}(t|t)$, $P(t|t) := Cov(\tilde{x}(t))$ and $R(t) := E(t)V(t)E^*(t)$, we obtain the following theorem

Theorem 4.7. For the time-varying system (9) with the observation system (10), the linear unbiased estimation of the filter problem $\hat{x}(t|t)$ of $x(t)$ is optimal if the linear gain operator in (11) is given by $K_f(t, \tau) = M(t, \tau)P(\tau|\tau)H^*(\tau)R^{-1}(\tau)$, $\tau \leq t$.

Proof. By Wiener-Hopf's equation [13], [19], $\hat{x}(t|t)$ minimizes the minimal covariance if and only if $\mathbf{E}\langle \tilde{x}(t), h_1 \rangle \langle y(\tau) - H(\tau)\hat{x}(\tau|\tau), h_2 \rangle = 0$, $\tau < t$, $h_1, h_2 \in \mathcal{X}$. Further, according to [6, Corollary 6.3], $\mathbf{E}\langle \hat{x}(t|t), h_1 \rangle \langle \tilde{x}(t|t), h_2 \rangle = 0$. Hence, on one hand,

$$\begin{aligned} &\mathbf{E}\langle x(t), h_1 \rangle \langle y(\tau) - H(\tau)\hat{x}(\tau|\tau), h_2 \rangle \\ &= \mathbf{E}\langle x(t), h_1 \rangle \langle H(\tau)\tilde{x}(\tau|\tau), h_2 \rangle + \mathbf{E}\langle x(t), h_1 \rangle \langle E(\tau)\nu(\tau), h_2 \rangle \\ &= \mathbf{E}\langle M(t, \tau)x(\tau), h_1 \rangle \langle H(\tau)\tilde{x}(\tau|\tau), h_2 \rangle - \mathbf{E}\langle M(t, \tau)\hat{x}(\tau|\tau), h_1 \rangle \langle H(\tau)\tilde{x}(\tau|\tau), h_2 \rangle \\ &= \mathbf{E}\langle M(t, \tau)\tilde{x}(t|t), h_1 \rangle \langle H(\tau)\tilde{x}(\tau|\tau), h_2 \rangle = \langle h_1, M(t, \tau)P(\tau|\tau)H^*(\tau)h_2 \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \mathbf{E}\langle \hat{x}(t|t), h_1 \rangle \langle y(\tau) - H(\tau)\hat{x}(\tau|\tau), h_2 \rangle \\
&= \mathbf{E}\langle \int_{\tau}^t K_f(t, s)[H(s)\tilde{x}(s|s) + E(s)\nu(s)]ds, h_1 \rangle \langle y(\tau) - H(\tau)\hat{x}(\tau|\tau), h_2 \rangle \\
&= \mathbf{E}\langle \int_{\tau}^t K_f(t, s)E(s)\nu(s)ds, h_1 \rangle \langle H(\tau)\tilde{x}(\tau|\tau) + E(\tau)\nu(\tau), h_2 \rangle \\
&= \mathbf{E}\langle \int_{\tau}^t K_f(t, s)E(s)\nu(s)ds, h_1 \rangle \langle E(\tau)\nu(\tau), h_2 \rangle = \langle h_1, K_f(t, \tau)R(\tau)h_2 \rangle
\end{aligned}$$

Therefore, $K_f(t, \tau)R(\tau) = M(t, \tau)P(\tau|\tau)H^*(\tau)$. Since $R(t)$ is coercive, we obtain

$$K_f(t, \tau) = M(t, \tau)P(\tau|\tau)H^*(\tau)R^{-1}(\tau), \quad \tau < t.$$

If $t = \tau$, by the strong continuity of $K_f(t, \cdot)$, $K_f(t, t) = P(t|t)H^*(t)R^{-1}(t)$. \square

Defining $K(t) := K_f(t, t) = P(t|t)H^*(t)R^{-1}(t)$, Theorem 4.7 implies that

$$\begin{aligned}
\tilde{x}(t|t) &= M(t, t_0)\tilde{x}(t_0|t_{-1}) - \int_{t_0}^t M(t, s)K(s)H(s)\tilde{x}(s|s)ds \\
&\quad + \int_{t_0}^t M(t, s)[D(s)\omega(s) - K(s)E(s)\nu(s)]ds. \tag{12}
\end{aligned}$$

Theorem 4.8. Equation (12) is equivalent to

$$\tilde{x}(t|t) = M_K(t, t_0)\tilde{x}(t_0|t_{-1}) + \int_{t_0}^t M_K(t, s)(D(s)\omega(s) - K(s)E(s)\nu(s))ds, \tag{13}$$

where $M_K(t, \tau)x = M(t, \tau)x - \int_{\tau}^t M_K(t, s)K(s)H(s)M(s, \tau)xds$, $(t, \tau) \in \Gamma_{t_0}^b$

Proof. From (12),

$$\begin{aligned}
& \tilde{x}(t|t) \\
&= M_K(t, t_0)\tilde{x}(t_0|t_{-1}) + \int_{t_0}^t M_K(t, s)K(s)H(s)M(s, t_0)\tilde{x}(t_0|t_{-1})ds \\
&\quad - \int_{t_0}^t M_K(t, s)K(s)H(s)\tilde{x}(s|s)ds \\
&\quad - \int_{t_0}^t \int_s^t M_K(t, \eta)K(\eta)H(\eta)M(\eta, s)K(s)H(s)\tilde{x}(s|s)d\eta ds \\
&\quad + \int_{t_0}^t M_K(t, s)[D(s)\omega(s) - K(s)E(s)\nu(s)]ds \\
&\quad + \int_{t_0}^t \int_s^t M_K(t, \eta)K(\eta)H(\eta)M(\eta, s)(D(s)\omega(s) - K(s)E(s)\nu(s))d\eta ds
\end{aligned}$$

$$\begin{aligned}
&= M_K(t, t_0)\tilde{x}(t_0|t_{-1}) + \int_{t_0}^t M_K(t, s) (D(s)\omega(s) - K(s)E(s)\nu(s)) ds \\
&\quad - \int_{t_0}^t M_K(t, s)K(s)H(s)\tilde{x}(s|s)ds + \int_{t_0}^t M_K(t, s)K(s)H(s)M(s, t_0)\tilde{x}(t_0|t_{-1})ds \\
&\quad - \int_{t_0}^t M_K(t, s)K(s)H(s) \int_{t_0}^s M(s, \eta)K(\eta)H(\eta)\tilde{x}(\eta|\eta)d\eta ds \\
&\quad + \int_{t_0}^t M_K(t, s)K(s)H(s) \int_{t_0}^s M(s, \eta) (D(\eta)\omega(\eta) - K(\eta)E(\eta)\nu(\eta)) d\eta ds \\
&= M_K(t, t_0)\tilde{x}(t_0|t_{-1}) + \int_{t_0}^t M_K(t, s) (D(s)\omega(s) - K(s)E(s)\nu(s)) ds.
\end{aligned}$$

□

For finite-dimensional systems, the trace of the covariance of $\tilde{x}(t|t)$ is considered as an evaluation of the estimation errors. For systems on Hilbert spaces, similarly we consider the nuclear norm of the covariance of $\tilde{x}(t|t)$. Defining $Q(t) := D(t)W(t)D^*(t)$, we obtain the following theorem.

Theorem 4.9. *The covariance (if exists) of $\tilde{x}(t|t)$ satisfies the IRE*

$$\begin{aligned}
P(t|t) &= M_K(t, t_0)P(t_0|t_{-1})M_K^*(t, t_0) \\
&\quad + \int_{t_0}^t M_K(t, s) [Q(s) + P(s|s)H^*(s)R^{-1}(s)H(s)P(s|s)] M_K^*(t, s)ds.
\end{aligned} \tag{14}$$

Proof. For $\tilde{x}(t|t)$ in (13), assume its covariance $P(t|t)$ exists and define $\mathcal{Q}_t: L^2(t_0, t; \mathcal{E} \times \mathcal{E}) \rightarrow \mathcal{X}$ by

$$\mathcal{Q}_t \begin{pmatrix} \omega \\ \nu \end{pmatrix} = \int_{t_0}^t [M_K(t, s)D(s), -M_K(t, s)K(s)E(s)] \begin{pmatrix} \omega(s) \\ \nu(s) \end{pmatrix} ds.$$

Its adjoint operator $\mathcal{Q}_t^*: \mathcal{X} \rightarrow L^2(t_0, t; \mathcal{E} \times \mathcal{E})$ is given by

$$\mathcal{Q}_t^* x = \begin{pmatrix} D^*(\cdot)M_K^*(t, \cdot) \\ -E^*(\cdot)K^*(\cdot)M_K^*(t, \cdot) \end{pmatrix} x, \quad x \in \mathcal{X}.$$

Then, we obtain

$$\begin{aligned}
&\mathbf{E}\langle \tilde{x}(t|t), h_1 \rangle \langle \tilde{x}(t|t), h_2 \rangle \\
&= \mathbf{E}\langle M_K(t, t_0)\tilde{x}(t_0|t_{-1}), h_1 \rangle \langle M_K(t, t_0)\tilde{x}(t_0|t_{-1}), h_2 \rangle \\
&\quad + \mathbf{E}\langle \mathcal{Q}_t \begin{pmatrix} \omega \\ \nu \end{pmatrix}, h_1 \rangle \langle \mathcal{Q}_t \begin{pmatrix} \omega \\ \nu \end{pmatrix}, h_2 \rangle \\
&= \langle M_K(t, t_0)P(t_0|t_{-1})M_K^*(t, t_0)h_1, h_2 \rangle + \langle \mathcal{Q}_t Cov \left(\begin{pmatrix} \omega \\ \nu \end{pmatrix} \right) \mathcal{Q}_t^* h_1, h_2 \rangle \\
&= \langle M_K(t, t_0)P(t_0|t_{-1})M_K^*(t, t_0)h_1, h_2 \rangle
\end{aligned}$$

$$+ \langle \int_{t_0}^t M_K(t, s) [Q(s) + K(s)R(s)K^*(s)] M_K^*(t, s) h_1 ds, h_2 \rangle,$$

Hence, for any $x \in \mathcal{X}$

$$\begin{aligned} P(t|t)x &= M_K(t, t_0)P(t_0|t_{-1})M_K^*(t, t_0)x \\ &\quad + \int_{t_0}^t M_K(t, s) [Q(s) + K(s)R(s)K^*(s)] M_K^*(t, s)x ds \\ &= M_K(t, t_0)P(t_0|t_{-1})M_K^*(t, t_0)x \\ &\quad + \int_{t_0}^t M_K(t, s) [Q(s) + P(s|s)H^*(s)R^{-1}(s)H(s)P(s|s)] M_K^*(t, s)x ds. \end{aligned}$$

□

A comparison with the main results of the linear-quadratic optimal control problem in Section 2 yields: By observing the similarity between (14) and the second integral Riccati equation related to the linear quadratic optimal control problem, it is clear that to consider the covariance of $\tilde{x}(t|t)$ of the time-varying system (9) with the observations (10) is equivalent to consider the Riccati operator $\Pi(b-t)$ in (4) corresponding to the time-varying system

$$x(t) = T(t, t_0)x(t_0) + \int_{t_0}^t T(t, s)B(s)u(s)ds.$$

with the cost functional

$$J(t, x, u) = \langle x(b), Gx(b) \rangle + \int_t^b \langle C(s)x(s), C(s)x(s) \rangle + \langle u(s), F(s)u(s) \rangle ds,$$

where $T(t, s) = M^*(b-s, b-t)$, $B(s) = H^*(b-s)$, $G = P(t_0|t_{-1})$, $C(s) = Q^{\frac{1}{2}}(b-s)$, $F(s) = R(b-s)$, $(t, s) \in \Gamma_{t_0}^b$.

Then, by the duality between the linear quadratic control problem and Kalman filters, Corollary 2.4 implies the following condition to guarantee the existence and nuclearity of $P(t|t)$.

Theorem 4.10. *For the time-varying system (9) with the observation system (10), if \mathcal{E} and \mathcal{Y} are finite dimensional and $P(t_0|t_{-1})$ is a nuclear operator, then the covariance of $\tilde{x}(t|t)$ based on Y_t satisfying (14) exists and is a nuclear operator.*

5 Kalman smoother in Hilbert spaces

In this section, we study the optimal linear unbiased estimation of $x(\tau)$ based on Y_t by $\hat{x}(\tau|t)$, $\tau \leq t$. We still constrain the linear estimation of $x(\tau|t)$ has the form

$$\hat{x}(\tau|t) = \int_{t_0}^t K_s(\tau, s)[y(s) - H(s)\hat{x}(s|s)]ds, \quad \tau \leq t, \quad (15)$$

where $K_s(\cdot, \cdot)$ is an unknown linear operator.

Since in the case $\tau = t$, (15) with the minimal covariance is equivalent to the optimal linear unbiased estimation based on Kalman filter, in order to determine the optimal estimation of $\hat{x}(\tau|t)$, $\tau \leq t$, we can rewrite (15) as

$$\hat{x}(\tau|t) = \hat{x}(\tau|\tau) + \int_{\tau}^t K_s(\tau, s)[y(s) - H(s)\hat{x}(s|s)]ds. \quad (16)$$

Theorem 5.1. *For the time-varying system (9) with the observation system (10), the linear unbiased estimation of the filter problem $\hat{x}(\tau|t)$ of $x(\tau)$ is optimal if $K_s(\cdot, \cdot)$ in (11) is given by*

$$K_s(\tau, \eta) = P(\tau|\tau)M_K^*(\eta, \tau)H^*(\eta)R^{-1}(\eta), \quad \tau \leq \eta \leq t.$$

Proof. By Wiener-Hopf's equation [19], [13], $E\langle \tilde{x}(\tau|t), h_1 \rangle \langle y(\eta) - H(\eta)\hat{x}(\eta|\eta), h_2 \rangle = 0$, $h_1 \in \mathcal{X}, h_2 \in \mathcal{Y}$, for any $\eta < t$. It is clear for any $\eta < \tau$, $E\langle \tilde{x}(\tau|t), h_1 \rangle \langle y(\eta) - H(\eta)\hat{x}(\eta|\eta), h_2 \rangle = 0$ holds. Now we assume $\tau \leq \eta < t$. On one hand,

$$\begin{aligned} & \mathbf{E}\langle x(\tau), h_1 \rangle \langle y(\eta) - H(\eta)\hat{x}(\eta|\eta), h_2 \rangle \\ &= \mathbf{E}\langle x(\tau) - \hat{x}(\tau|\eta), h_1 \rangle \langle H(\eta)\tilde{x}(\eta|\eta), h_2 \rangle \\ &= \mathbf{E}\langle \tilde{x}(\tau|\tau) - \int_{\tau}^{\eta} K_s(\tau, s)[y(s) - H(s)\hat{x}(s|s)]ds, h_1 \rangle \langle H(\eta)\tilde{x}(\eta|\eta), h_2 \rangle \\ &= \mathbf{E}\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle H(\eta)M_K(\eta, \tau)H^*(\eta)\tilde{x}(\tau|\tau), h_2 \rangle = \langle h_1, P(\tau|\tau)M_K^*(\eta, \tau)H^*(\eta)h_2 \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{E}\langle \hat{x}(\tau|t), h_1 \rangle \langle y(\eta) - H(\eta)\hat{x}(\eta|\eta), h_2 \rangle \\ &= \mathbf{E}\langle \hat{x}(\tau|\eta), h_1 \rangle \langle H(\eta)\tilde{x}(\eta|\eta) + E(\eta)\nu(\eta), h_2 \rangle \\ & \quad + \mathbf{E}\langle \int_{\eta}^t K_s(\tau, s)[y(s) - H(s)\hat{x}(s|s)]ds, h_1 \rangle \langle y(\eta) - H(\eta)\hat{x}(\eta|\eta), h_2 \rangle \\ &= \mathbf{E}\langle \int_{\tau}^t K_s(s, \tau)E(s)\nu(s)ds, h_1 \rangle \langle E(\eta)\nu(\eta), h_2 \rangle = \langle h_1, K_s(\tau, \eta)R(\eta)h_2 \rangle. \end{aligned}$$

By the coercivity of $R(t)$, we obtain $K_s(\tau, \eta) = P(\tau|\tau)M_K^*(\eta, \tau)H^*(\eta)R^{-1}(\eta)$. \square

Defining $\tilde{x}(\tau|t) = x(\tau) - \hat{x}(\tau|t)$, Theorem 5.1 implies

$$\tilde{x}(\tau|t) = \tilde{x}(\tau|\tau) - P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau)H^*(s)R^{-1}(s)[y(s) - H(s)\hat{x}(s|s)]ds.$$

Thus, its covariance can be derived by

Theorem 5.2. *The covariance (if exists) of $\tilde{x}(\tau|t)$, $(t, \tau) \in \Gamma_{t_0}^b$ is*

$$\begin{aligned} P(\tau|t)x &= P(\tau|\tau)x \\ & - P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau)H^*(s)R^{-1}(s)H(s)M_K(s, \tau)P(\tau|\tau)x ds, \quad x \in \mathcal{X}. \end{aligned} \quad (17)$$

Proof. Denoting the covariance of $\tilde{x}(\tau|t)$ by $P(\tau|t)$, we obtain

$$\begin{aligned}
\langle h_1, P(\tau|t)h_2 \rangle &= \mathbf{E}\langle \tilde{x}(\tau|t), h_1 \rangle \langle \tilde{x}(\tau|t), h_2 \rangle \\
&= \mathbf{E}\langle \tilde{x}(\tau|\tau) - P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) [y(s) - H(s)\hat{x}(s)] ds, h_1 \rangle \langle \tilde{x}(\tau|t), h_2 \rangle \\
&= \mathbf{E}\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle \tilde{x}(\tau|\tau) \\
&\quad - P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) [y(s) - H(s)\hat{x}(s|s)] ds, h_2 \rangle \\
&= \mathbf{E}\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle \tilde{x}(\tau|\tau), h_2 \rangle \\
&\quad - \mathbf{E}\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) H(s) \tilde{x}(s|s) ds, h_2 \rangle \\
&= \langle h_1, P(\tau|\tau)h_2 \rangle - E\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) H(s) \\
&\quad \cdot \left(M_K(s, \tau) \tilde{x}(\tau|\tau) + \int_{\tau}^s M_K(s, \eta) (D(\eta)\omega(\eta) - K(\eta)E(\eta)\nu(\eta)) d\eta \right) ds, h_2 \rangle \\
&= \langle h_1, P(\tau|\tau)h_2 \rangle \\
&\quad - \mathbf{E}\langle \tilde{x}(\tau|\tau), h_1 \rangle \langle P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) H(s) M_K(s, \tau) \tilde{x}(\tau|\tau) ds, h_2 \rangle \\
&= \langle h_1, P(\tau|\tau)h_2 \rangle - \langle h_1, P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) H(s) M_K(s, \tau) P(\tau|\tau) h_2 ds \rangle.
\end{aligned}$$

Hence, for any $x \in \mathcal{X}$, $(t, \tau) \in \Gamma_{t_0}^b$, we get

$$P(\tau|t)x = P(\tau|\tau)x - P(\tau|\tau) \int_{\tau}^t M_K^*(s, \tau) H^*(s) R^{-1}(s) H(s) M_K(s, \tau) P(\tau|\tau) x ds.$$

□

Theorem 5.3. *For the time-varying system (9) with the observation system (10), if \mathcal{E} and \mathcal{Y} are finite dimensional and $P(t_0|t_{-1})$ is a nuclear operator, then $P(\tau|t)$, $(t, \tau) \in \Gamma_{t_0}^b$ satisfying (17) exists and is a nuclear operator.*

Proof. By Theorem 4.10 and the uniform boundedness of M_K , H and R^{-1} in $[t_0, b]$,

$$\|P(\tau|t)\|_1 \leq \|P(\tau|\tau)\|_1 + \|P(\tau|\tau)\|_1^2 \int_{\tau}^t \|M_K(s, \tau)\|^2 \|R^{-1}(s)\| \|H(s)\|^2 ds < \infty,$$

so $P(\tau|t)$ is a nuclear operator for any $(t, \tau) \in \Gamma_{t_0}^b$. □

6 Optimal locations of observations based on Kalman filter and smoother

In this section, we also take the observation location problem into account. The location parameter r is defined as in Section 2. The following theorems show the continuity of

$P_r(t|t)$ and $P_r(\tau|t)$, $(t, \tau) \in \Gamma_{t_0}^b$ in nuclear norm. For the filter problem, due to the duality and Theorem 2.7, we obtain the following theorem.

Theorem 6.1. *Consider the filter problem of the time-varying system (9) with location-dependent output operators and the observation system (10). If H_r is of the property that $\lim_{r \rightarrow r_0} \|H_r - H_{r_0}\|_\infty = 0$, \mathcal{E} and \mathcal{Y} are finite-dimensional, and $P(t_0|t_{-1})$ is nuclear, then*

$$\lim_{r \rightarrow r_0} \|P_r(t|t) - P_{r_0}(t|t)\|_1 = 0, \quad t \in [t_0, b],$$

and there exists an optimal location \hat{r}^f such that,

$$\hat{\ell}_1^f(t) = \|P_{\hat{r}^f}(t|t)\|_1 = \inf_{r \in \Omega^m} \|P_r(t|t)\|_1.$$

Theorem 6.2. *Consider the smoother problem of the time-varying system (9) with the location-dependent output operators and the observation system (10). H_r has the property that $\lim_{r \rightarrow r_0} \|H_r - H_{r_0}\|_\infty = 0$. If \mathcal{E} and \mathcal{Y} are finite-dimensional, and $P(t_0|t_{-1})$ is nuclear, then,*

$$\lim_{r \rightarrow r_0} \|P_r(\tau|t) - P_{r_0}(\tau|t)\|_1 = 0, \quad (t, \tau) \in \Gamma_{t_0}^b,$$

and there exists an optimal location \hat{r}^s such that for any initial time $\tau \in [t_0, b]$, $\tau \leq t$,

$$\hat{\ell}_1^s(\tau|t) = \|P_{\hat{r}^s}(\tau|t)\|_1 = \inf_{r \in \Omega^m} \|P_r(\tau|t)\|_1.$$

Proof. From Lemma 5.3, $P_r(\tau|t)$, $r \in \Omega^m$ are nuclear operators. Hence,

$$\begin{aligned} & \|P_r(\tau|t) - P_{r_0}(\tau|t)\|_1 \leq \|P_r(\tau|\tau) - P_{r_0}(\tau|\tau)\|_1 \\ & + \int_{t_0}^t \|P_{r_0}(\tau|\tau)M_{K,r_0}^*(s, \tau)H_{r_0}^*(s) - P_r(\tau|\tau)M_{K,r}^*(s, \tau)H_r^*(s)\| \\ & \cdot \|R^{-1}(s)H_{r_0}(s)M_{K,r_0}(s, \tau)P_{r_0}(\tau|\tau)\|_1 ds + \int_{t_0}^t \|P_r(\tau|\tau)M_{K,r}^*(s, \tau)H_r^*(s)R^{-1}(s)\|_1 \\ & \cdot \|H_{r_0}(s)M_{K,r_0}(s, \tau)P_{r_0}(\tau|\tau) - H_r(s)M_{K,r}(s, \tau)P_r(\tau|\tau)\| ds, \end{aligned}$$

Since $P_r(t)$, $r \in \Omega^m$ are nuclear operators and $R^{-1}(t)$, $H_r(t)$, $M_{K,r_0}(t, \tau)$ are uniformly bounded for $(t, \tau) \in \Gamma_{t_0}^b$, then $\|R^{-1}(s)H_{r_0}(s)M_{K,r_0}(s, \tau)P_{r_0}(\tau|\tau)\|_1 < \infty$ and so is its adjoint.

By Theorem 6.1 and dominated convergence theorem, we obtain

$$\|P_r(\tau|t) - P_{r_0}(\tau|t)\|_1 \rightarrow 0, \quad r \rightarrow r_0.$$

Because of the compactness of Ω^m , there exists the optimal location of observations such that $\hat{\ell}_1^s(\tau|t) = \|P_{\hat{r}^s}(\tau|t)\|_1 = \inf_{r \in \Omega^m} \|P_r(\tau|t)\|_1$. \square

Next we consider a sequence of approximations of time-varying systems in order to study the convergence of optimal observation locations based on Kalman filter and smoother. Let \mathcal{X}_n be a family of finite-dimensional subspaces of \mathcal{X} and \mathbf{P}_n be the corresponding

orthogonal projection of \mathcal{X} onto \mathcal{X}_n . The finite spaces $\{\mathcal{X}_n\}$ inherit the norm from \mathcal{X} . For $n \in \mathbb{N}$, let $M_n(\cdot, \cdot)$ be a mild evolution operator on \mathcal{X}_n , $D_n(t) = \mathbf{P}_n D(t)$ and $H_n(t) = H(t)\mathbf{P}_n$, $t \in [t_0, b]$. In order to guarantee that $P_n(t|t)$ converges to $P(t|t)$, the following assumptions are needed in the approximation of observation problems for partial differential equations. For each $x \in X$, $\omega \in \mathcal{E}$, $y \in \mathcal{Y}$

(A1) (i) $M_n(t, s)\mathbf{P}_n x \rightarrow M(t, s)x$; (ii) $M_n^*(t, s)\mathbf{P}_n x \rightarrow M^*(t, s)x$

and $\sup_n \|M_n(t, s)\| < \infty$, for any $(t, s) \in \Gamma_{t_0}^b$.

(A2) (i) $D_n(t)\omega \rightarrow D(t)\omega$; (ii) $D_n^*(t)\mathbf{P}_n x \rightarrow D^*(t)x$, a.e. $t \in [t_0, b]$.

(A3) (i) $H_n(t)\mathbf{P}_n x \rightarrow H(t)x$; (ii) $H_n^*(t)y \rightarrow H^*(t)y$, a.e. $t \in [t_0, b]$.

(A4) $P_n(t_0|t_{-1})\mathbf{P}_n x \rightarrow P(t_0|t_{-1})x$ and $\sup_n \|P_n(t_0|t_{-1})\| < \infty$.

The next theorem shows the uniform convergence of the approximations of covariances of the Kalman filter and smoother in nuclear norm.

Theorem 6.3. *Assume that the assumptions (A1) – (A4) are satisfied. If \mathcal{E} and \mathcal{Y} are finite-dimensional, $\lim_{n \rightarrow \infty} \|P_n(t_0|t_{-1})\mathbf{P}_n - P(t_0|t_{-1})\|_1 = 0$ and $P(t_0|t_{-1})$ is nuclear, then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n(t|t)\mathbf{P}_n - P(t|t)\|_1 &= 0, \\ \lim_{n \rightarrow \infty} \|P_n(\tau|t)\mathbf{P}_n - P(\tau|t)\|_1 &= 0, \quad (t, \tau) \in \Gamma_{t_0}^b. \end{aligned}$$

Proof. Due to the duality between Kalman filter and LQ optimal control problem, according to Theorem 3.5, we have

$$\lim_{n \rightarrow \infty} \|P_n(t|t)\mathbf{P}_n - P(t|t)\|_1 = 0, \quad (t, \tau) \in \Gamma_{t_0}^b. \quad (18)$$

Then,

$$\begin{aligned} &\|P_n(\tau|t)\mathbf{P}_n - P(\tau|t)\|_1 \leq \|P_n(\tau|\tau)\mathbf{P}_n - P(\tau|\tau)\|_1 \\ &+ \int_{\tau}^t \|P(\tau|\tau)M_K^*(s, \tau)H^*(s) - \mathbf{P}_n P_n(\tau|\tau)M_{K,n}^*(s, \tau)H_n^*(s)\| \\ &\cdot \|R^{-1}(s)H(s)M_K(s, \tau)P(\tau|\tau)\|_1 ds + \int_{\tau}^t \|\mathbf{P}_n P_n(\tau|\tau)M_{K,n}^*(s, \tau)H_n^*(s)R^{-1}(s)\|_1 \\ &\cdot \|H(s)M_K(s, \tau)P(\tau|\tau) - H_n(s)M_{K,n}(s, \tau)P_n(\tau|\tau)\mathbf{P}_n\| ds. \end{aligned}$$

where, according to Lemma 2.5 and (18),

$$\begin{aligned} &\|H(s)M_K(s, \tau)P(\tau|\tau) - H_n(s)M_{K,n}(s, \tau)P_n(\tau|\tau)\mathbf{P}_n\| \\ &\leq \|H(s)M_K(s, \tau)\| \|P(\tau|\tau) - P_n(\tau|\tau)\mathbf{P}_n\| + \|H(s) - H_n(s)\| \|M_K(s, \tau)P_n(\tau|\tau)\mathbf{P}_n\| \\ &+ \|H_n(s)\| \|(M_K(s, \tau) - M_{K,n}(s, \tau))P_n(\tau|\tau)\mathbf{P}_n\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

and so is its adjoint operator.

By the uniform boundedness of $P(t|t)$, $M_K(t, s)$, $H_n(t)$ for $t \in [t_0, b]$, we have $\|P_n(\tau|t)\mathbf{P}_n - P(\tau|t)\|_1 \rightarrow 0$, $n \rightarrow \infty$. \square

Now let us take the location of observations into account and show the convergence of optimal observation locations of approximated covariance of Kalman filter and smoother.

Theorem 6.4. *Assume that the assumptions (A1) – (A4) are held and that $H_{r,n} = H_r \mathbf{P}_n$ and $\lim_{r \rightarrow r_0} \|H_r - H_{r_0}\|_\infty = 0$. If \mathcal{E} and \mathcal{Y} are finite-dimensional, $P(t_0|t_{-1})$ is nuclear and $\lim_{n \rightarrow \infty} \|P_n(t_0|t_{-1})\mathbf{P}_n - P(t_0|t_{-1})\|_1 = 0$, then*

$$\hat{\ell}_{1,n}^f(t) \rightarrow \hat{\ell}_1^f(t), \quad \hat{r}_n^f \rightarrow \hat{r}^f, \quad \hat{\ell}_{1,n}^s(\tau|t) \rightarrow \hat{\ell}_1^s(\tau|t), \quad \hat{r}_n^s \rightarrow \hat{r}^s, \quad (t, \tau) \in \Gamma_{t_0}^b, \quad n \rightarrow \infty.$$

Proof. Follows by duality and Theorem 3.6. \square

7 Application

As a popular data assimilation method, the ensemble Kalman filter and smoother are widely applied in meteorology. Hence, we consider a linear advection-diffusion model with $\Omega := (0, 5) \times (0, 5) \times (0, 1)$ on a fixed time interval $[0, 3]$ based on the Kalman filter and smoother, the theoretical foundation of the ensemble Kalman filter and smoother, as an example:

$$\begin{aligned} \frac{\partial \delta c}{\partial t} &= -v_x \frac{\partial \delta c}{\partial x} - v_y \frac{\partial \delta c}{\partial y} + \frac{\partial}{\partial z} \left(K(z) \frac{\partial \delta c}{\partial z} \right) + \delta e - \delta d, \\ \delta c(t_0) &= \delta c_0, \quad \delta e(t_0) = \delta e_0, \quad \delta d(t_0) = \delta d_0, \end{aligned}$$

where δc , δe and δd are the perturbations of the concentration, the emission rate and deposition rate of a species, respectively. v_x and v_y are constants and $K(z)$ is a continuous differentiable function of z .

Defining $A_x := -v_x \frac{\partial}{\partial x}$, $A_y := -v_y \frac{\partial}{\partial y}$ and $D_z := \frac{\partial}{\partial z} \left(K(z) \frac{\partial}{\partial z} \right)$ with domains

$$\begin{aligned} \mathcal{D}(A_x) &= \{f \in L^2(\Omega) \mid A_x f \in L^2(\Omega), f(0, y, z) = f(5, y, z)\}, \\ \mathcal{D}(A_y) &= \{f \in L^2(\Omega) \mid A_y f \in L^2(\Omega), f(x, 0, z) = f(x, 5, z)\}, \\ \mathcal{D}(D_z) &= \{f \in L^2(\Omega) \mid D_z f \in L^2(\Omega), f_z(x, y, 0) = f_z(x, y, 1) = 0\} \end{aligned}$$

and denote by S_x , S_y and S_z the semigroups generated by A_x , A_y and D_z . S is the semigroup generated by $A_x + A_y + D_z$ with the domain $\mathcal{D} = \mathcal{D}(A_x) \cap \mathcal{D}(A_y) \cap \mathcal{D}(D_z)$.

In particular, in order to include the emission rate into the state vector as optimized parameter, the dynamic model for the emission rate with constant emission factors [12] is established as

$$\delta e(t) = M_e(t, s) \delta e(s),$$

where $M_e(t, s) = \frac{e_b(t)}{e_b(s)} \in L(L^2(\Omega))$, $e_b(\cdot) \in L^2(\Omega)$ is termed as the background knowledge of the emission rate, which is continuous in time and $\sup_{(t,s) \in \Gamma_0^3} \left\| \frac{e_b(t)}{e_b(s)} \right\| < \infty$. According to Definition 2.1, it is easy to show that $M_e(\cdot, \cdot)$ is a self-adjoint mild evolution operator.

Ignoring the model error, the model extended with emission rate is given by

$$\begin{pmatrix} \delta c(t + \Delta t) \\ \delta e(t + \Delta t) \end{pmatrix}$$

$$= M(t + \Delta t, t) \begin{pmatrix} \delta c(t) \\ \delta e(t) \end{pmatrix} - \begin{pmatrix} \int_t^{t+\Delta t} S(t + \Delta t - s) \delta d(s) ds \\ 0 \end{pmatrix} \quad (19)$$

where

$$M(t + \Delta t, t) = \begin{pmatrix} S(\Delta t) & \int_t^{t+\Delta t} S(t + \Delta t - s) M_e(s, t) ds \\ 0 & M_e(t + \Delta t, t) \end{pmatrix}$$

also satisfies Definition 2.1.

The numerical solution is based on the symmetric operator splitting technique [2], [27] with space discretization via finite difference method with discretized intervals Δx , Δy and Δz in three dimensions. We assume that the grid points $\{r_i\}_{i=1}^n$ have the coordinates $\{(x_{r_i}, y_{r_i}, z_{r_i})\}$ and define the projection $\mathbf{P}_n : L^2(\Omega) \rightarrow \mathbb{R}^n$

$$(\mathbf{P}_n f)_i := \frac{1}{V_i} \int_{\Omega_i} f(\omega) d\omega, \quad i = 1, \dots, n. \quad (20)$$

where $\Omega_i = [x_{r_i} - \frac{\Delta x}{2}, x_{r_i} + \frac{\Delta x}{2}] \times [y_{r_i} - \frac{\Delta y}{2}, y_{r_i} + \frac{\Delta y}{2}] \times [z_{r_i} - \frac{\Delta z}{2}, z_{r_i} + \frac{\Delta z}{2}]$, V_i is the volume of Ω_i . Defining $S_n(\Delta t) := S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(\Delta t) S_{y,n}(\frac{\Delta t}{2}) S_{x,n}(\frac{\Delta t}{2})$, according to [1, Theorem 3.17], we obtain

$$\lim_{n \rightarrow \infty, \Delta t \rightarrow 0} \|(S_n(\Delta t))^{\frac{t}{\Delta t}} \mathbf{P}_n f - \mathbf{P}_n S(t) f\| = 0, \quad f \in L^2(\Omega). \quad (21)$$

With the same space discretization for δc , the approximation of the emission rate is given by $\mathbf{P}_n \delta e(t) = M_{e,n}(t, s) \mathbf{P}_n \delta e(s)$, where $M_{e,n}(t, s)$ is a diagonal matrix with the diagonal given by

$$\text{diag}(M_{e,n}(t, s)) = \left(\frac{\int_{\Omega_1} e_b(t, \omega) d\omega}{\int_{\Omega_1} e_b(s, \omega) d\omega}, \dots, \frac{\int_{\Omega_n} e_b(t, \omega) d\omega}{\int_{\Omega_n} e_b(s, \omega) d\omega} \right).$$

Then, we can easily get

$$\left\| \frac{\int_{\Omega_i} e_b(t, \omega) d\omega}{\int_{\Omega_i} e_b(s, \omega) d\omega} (\mathbf{P}_n f)_i - (\mathbf{P}_n M_e(t, s) f)_i \right\| \rightarrow 0, \quad n \rightarrow \infty, \quad f \in L^2(\Omega),$$

so is the adjoint of $M_e(t, s)$. The extended model with operator splitting discretized in space can be written as

$$\begin{pmatrix} \delta c_n(t + \Delta t) \\ \delta e_n(t + \Delta t) \end{pmatrix} = M_n(t + \Delta t, t) \begin{pmatrix} \delta c_n(t) \\ \delta e_n(t) \end{pmatrix} - \begin{pmatrix} S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) \int_t^{t+\Delta t} S_{z,n}(t + \Delta t - s) \delta d_n(s) ds \\ 0 \end{pmatrix},$$

where $\delta c_n(t) = \mathbf{P}_n \delta c(t)$, $\delta e_n(t) = \mathbf{P}_n \delta e(t)$ and $\delta d_n(t) = \mathbf{P}_n \delta d(t)$

$$M_n(t + \Delta t, t) = \begin{pmatrix} S_n(\Delta t) & S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) \int_t^{t+\Delta t} S_{z,n}(t + \Delta t - s) M_{e,n}(s, t) ds \\ 0 & M_{e,n}(t + \Delta t, t) \end{pmatrix}.$$

For any pair of time $(t, s) \in \Gamma_0^3$, assume $m = \frac{t-s}{\Delta t} \in \mathbb{N}$, we have

$$\begin{aligned} & \prod_{i=1}^m M_n(s + i\Delta t, s + (i-1)\Delta t) \\ &= \begin{pmatrix} (S_n(\Delta t))^m & \sum_{i=1}^m \int_{s+(i-1)\Delta t}^{s+i\Delta t} S_{ce,n}^i(t-h) M_{e,n}(h, s) dh \\ 0 & M_{e,n}(t, s) \end{pmatrix}, \end{aligned}$$

where $S_{ce,n}^i(t-h) = (S_n(\Delta t))^{m-i} S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h)$, $h \in [s+(i-1)\Delta t, s+i\Delta t]$. In order to show that $\prod_{i=1}^m M_n(s+i\Delta t, s+(i-1)\Delta t) \mathbf{P}_n$ is strongly convergent to $\mathbf{P}_n M(t, s)$, we only need to show

$$\|S_{ce,n}^i(t-h) M_{e,n}(h, s) \mathbf{P}_n f - \mathbf{P}_n S(t-h) M_e(h, s) f\| \rightarrow 0, \quad m, n \rightarrow \infty.$$

In fact,

$$\begin{aligned} & \|S_{ce,n}^i(t-h) M_{e,n}(h, s) \mathbf{P}_n f - \mathbf{P}_n S(t-h) M_e(h, s) f\| \\ & \leq \|S_{ce,n}^i(t-h) M_{e,n}(h, s) \mathbf{P}_n f - S_{ce,n}^i(t-h) \mathbf{P}_n M_e(h, s) f\| \\ & \quad + \|S_{ce,n}^i(t-h) \mathbf{P}_n M_e(h, s) f - \mathbf{P}_n S(t-h) M_e(h, s) f\|, \end{aligned}$$

where, clearly, $\|S_{ce,n}^i(t-h) M_{e,n}(h, s) \mathbf{P}_n f - S_{ce,n}^i(t-h) \mathbf{P}_n M_e(h, s) f\| \rightarrow 0$, $m, n \rightarrow \infty$. Moreover, we have

$$\begin{aligned} & \|S_{ce,n}^i(t-h) \mathbf{P}_n M_e(h, s) f - \mathbf{P}_n S(t-h) M_e(h, s) f\| \\ & \leq \|((S_n(\Delta t))^{m-i} - S(t-s-i\Delta t)) S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h) \mathbf{P}_n M_e(h, s) f\| \\ & \quad + \|S(t-s-i\Delta t) (S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h) \mathbf{P}_n \\ & \quad - \mathbf{P}_n S(s+i\Delta t-h)) M_e(h, s) f\|, \end{aligned}$$

where, according to (21), $\|((S_n(\Delta t))^{m-i} - \mathbf{P}_n S(t-s-i\Delta t)) S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h) \mathbf{P}_n M_e(h, s) f\| \rightarrow 0$ and

$$\begin{aligned} & \|(S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h) \mathbf{P}_n - \mathbf{P}_n S(s+i\Delta t-h)) M_e(h, s) f\| \\ & \leq \|S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(s+i\Delta t-h)\| \\ & \quad \cdot \|(I - S_{z,n}(h-s-(i-1)\Delta t)) \mathbf{P}_n M_e(h, s) f\| \\ & \quad + \|(S_{x,n}(\frac{\Delta t}{2}) S_{y,n}(\frac{\Delta t}{2}) S_{z,n}(\Delta t) - S_n(\Delta t)) \mathbf{P}_n M_e(h, s) f\| \\ & \quad + \|(S_n(\Delta t) \mathbf{P}_n - \mathbf{P}_n S(\Delta t)) M_e(h, s) f\| \\ & \quad + \|(S(h-s-(i-1)\Delta t) - I) S(s+i\Delta t-h) M_e(h, s) f\| \rightarrow 0, \quad \Delta t \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Further, we discretize the model in time by the Lax-Wendroff scheme for advection equations in horizontal directions and Crank-Nicolson scheme for the diffusion equation in

the vertical direction such that $S_{x/y/z,n}$ are approximated by

$$\begin{aligned}\tilde{S}_{x/y,n}\left(\frac{\Delta t}{2}\right) &= I + \frac{\Delta t}{2}A_{x/y,n} + \frac{\Delta t^2}{8}A_{x/y,n}^2, \\ \tilde{S}_{z,n}(\Delta t) &= \left(I - \frac{\Delta t}{2}D_{z,n}\right)^{-1}\left(I + \frac{\Delta t}{2}D_{z,n}\right), \\ \tilde{B}_{z,n}^e(t,s)f &= \left(I - \frac{\Delta t}{2}D_{z,n}\right)^{-1}\left(\frac{\Delta t}{2}M_{e,n}(t,s)f\right),\end{aligned}$$

where $A_{x/y,n}$ and $D_{z,n}$ is the approximate generators to n -dimensional state space based on finite difference methods.

It is well known [11] that the Lax-Wendroff scheme is consistent and conditional stable for A_x and A_y and the Crank-Nicolson scheme is consistent and stable for D_z , $\left(I - \frac{\Delta t}{2}D_{z,n}\right)^{-1}$ is the consistent and condition stable implicit Euler scheme, by Lax equivalence theorem, that is

$$\begin{aligned}\lim_{\Delta t \rightarrow 0} \|\left(\tilde{S}_{x/y/z,n}(\Delta t)\right)^{\frac{t}{\Delta t}}f - S_{x/y/z,n}(t)f\| &= 0, \quad f \in L^2(\Omega), \\ \lim_{\Delta t \rightarrow 0} \|\left(\left(I - \frac{\Delta t}{2}D_{z,n}\right)^{-1}\right)^{\frac{2t}{\Delta t}}f - S_{z,n}(t)f\| &= 0, \quad f \in L^2(\Omega).\end{aligned}$$

Similarly defining $\tilde{S}_n := \tilde{S}_{x,n}\tilde{S}_{y,n}\tilde{S}_{z,n}\tilde{S}_{y,n}\tilde{S}_{x,n}$,

$$\lim_{n \rightarrow \infty, \Delta t \rightarrow 0} \|\left(\tilde{S}_n(\Delta t)\right)^{\frac{t}{\Delta t}}\mathbf{P}_n f - \mathbf{P}_n S(t)f\| = 0, \quad f \in L^2(\Omega). \quad (22)$$

Since A_x , A_y and D_z are self-adjoint, $\tilde{S}_n^*(\Delta t)^{\frac{t}{\Delta t}}$ is also strongly convergent to $S^*(t)$.

Thus, (19) is approximated by

$$\begin{aligned}& \begin{pmatrix} \delta\tilde{c}_n(t + \Delta t) \\ \delta\tilde{e}_n(t + \Delta t) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{S}_n(\Delta t) & \tilde{S}_{x,n}\left(\frac{\Delta t}{2}\right)\tilde{S}_{y,n}\left(\frac{\Delta t}{2}\right)\tilde{B}_{z,n}^e(t + \Delta t, t) \\ 0 & M_{e,n}(t + \Delta t, t) \end{pmatrix} \begin{pmatrix} \delta\tilde{c}_n(t) \\ \delta\tilde{e}_n(t) \end{pmatrix} \\ &- \begin{pmatrix} \tilde{S}_{x,n}\left(\frac{\Delta t}{2}\right)\tilde{S}_{y,n}\left(\frac{\Delta t}{2}\right)\left(I - \frac{\Delta t}{2}D_{z,n}\right)^{-1}\left[\frac{\Delta t}{2}(\delta d_n(t + \Delta t) + \delta d_n(t))\right] \\ 0 \end{pmatrix}.\end{aligned}$$

Defining the above block evolution operator as $\tilde{M}_n(t,s)$, $(t,s) \in \Gamma_0^3$, we have

$$\begin{aligned}& \prod_{i=1}^m \tilde{M}_n(s + i\Delta t, s + (i-1)\Delta t) \\ &= \begin{pmatrix} (\tilde{S}_n(\Delta t))^m & \sum_{i=1}^m (\tilde{S}_n(\Delta t))^{m-i}\tilde{S}_{x,n}\left(\frac{\Delta t}{2}\right)\tilde{S}_{y,n}\left(\frac{\Delta t}{2}\right)\tilde{B}_{z,n}^e(s + i\Delta t, s) \\ 0 & M_{e,n}(t,s) \end{pmatrix}.\end{aligned}$$

We define by

$$B_{z,n}^e(s + i\Delta t, s + (i-1)\Delta t, s)f$$

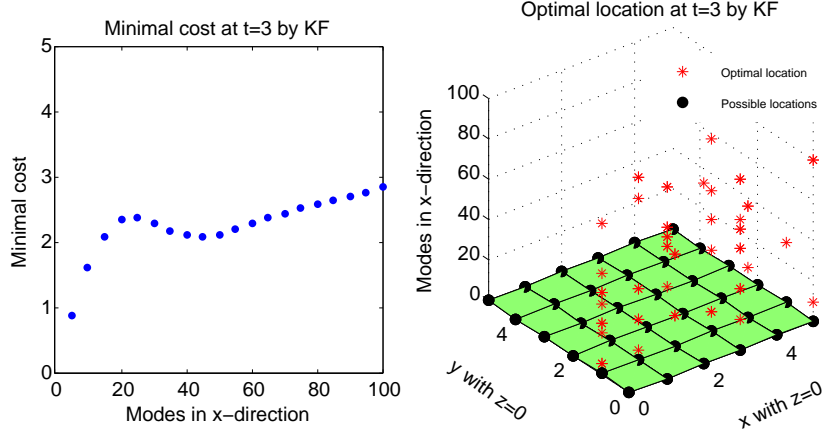


Figure 1: Minimal cost and optimal location based on Kalman filter without the nuclearity of $P(t_0|t_{-1})$.

$$:= \int_{s+(i-1)\Delta t}^{s+i\Delta t} S_{z,n}(s+i\Delta t-h)M_{e,n}(h,s)fdh, \quad f \in L^2(\Omega).$$

By the trapezoidal rule and convergence of the implicit Euler scheme, we have

$$\begin{aligned} & \|\tilde{B}_{z,n}^e(s+i\Delta t,s)f - B_{z,n}^e(s+i\Delta t,s)f\| \\ & \leq \|((I - \frac{\Delta t}{2}D_{z,n})^{-1} - S_{z,n}(\Delta t))(\frac{\Delta t}{2}M_{e,n}(s+i\Delta t,s)f)\| \\ & + \|\frac{\Delta t}{2}M_{e,n}(s+i\Delta t,s)f\| + \|\frac{\Delta t}{2}S_{z,n}(\Delta t)M_{e,n}(s+(i-1)\Delta t,s)f \\ & + \frac{\Delta t}{2}M_{e,n}(s+i\Delta t,s)f - B_{z,n}^e(s+i\Delta t,s)f\| \\ & + \|\frac{\Delta t}{2}S_{z,n}(\Delta t)(M_{e,n}(s+i\Delta t,s)f - M_{e,n}(s+(i-1)\Delta t,s)f)\| \rightarrow 0, \Delta t \rightarrow 0. \end{aligned}$$

For the observation system, we assume there is only a single observation during the entire time interval and define the observation mapping $H_r : L^2(\Omega) \rightarrow \mathbb{R}$ by

$$H_r f := \frac{1}{V_r} \int_{\Omega_r} f(\omega)d\omega, \quad r = (x_r, y_r, z_r), \quad f \in L^2(\Omega),$$

where Ω_r and V_r are similarly defined as (20). Then, the observation system extended with the emission rate is given by

$$\delta y(t) = (H_r, 0) \begin{pmatrix} \delta c(t) \\ \delta e(t) \end{pmatrix} + \nu(t),$$

where $\delta y(t) \in \mathbb{R}$ and $\nu(t)$ is the white noise with distribution $N(0, 1)$.

According to the spatial discretization of the model, in the vertical direction, $[0, 1]$ is discretized into three layers $\{0, 0.5, 1\}$. Since the diffusion coefficient $K(z)$ is small, we

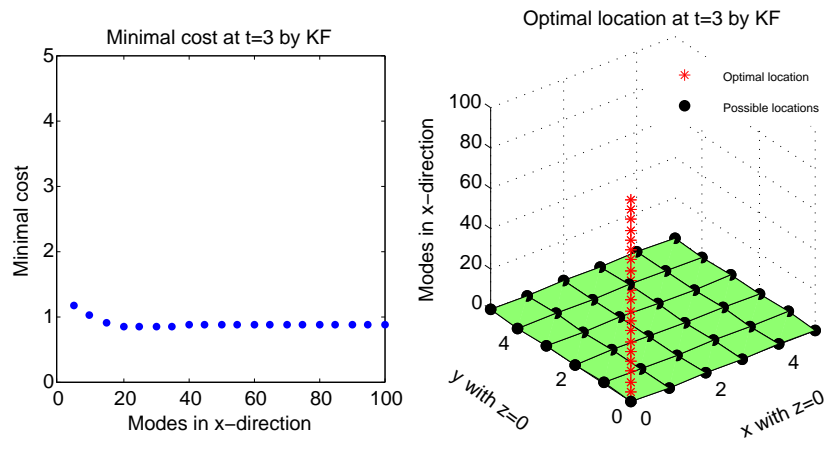


Figure 2: Minimal cost and optimal location of the estimation of the state at final time by Kalman filter.

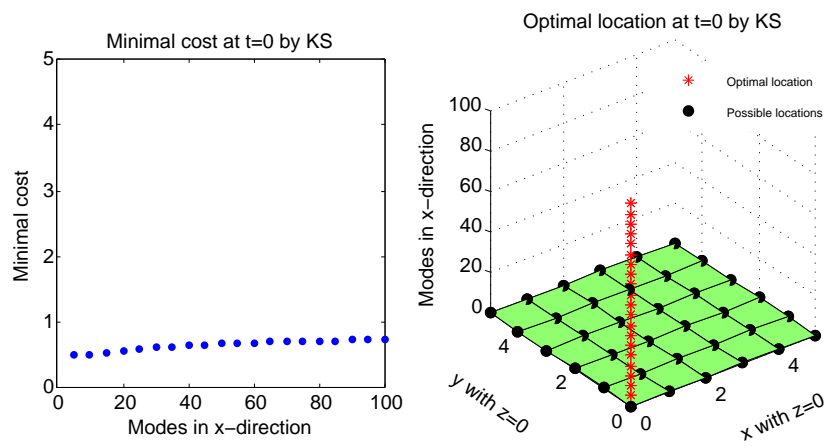


Figure 3: Minimal cost and optimal location of the estimation of the initial state by Kalman smoother.

assume possible locations of the single observation are around the grid points in the first layer $z = 0$.

We have already shown that the assumptions (A1) – (A3) in Section 6 and the compactness of the possible area of observation locations are satisfied.

In addition, according to the spatial discretization, we assume that the initial covariance is given by $P_n(t_0|t_{-1}) = e^{-8}I_n$, where I_n is the $n \times n$ identity matrix. It implies that $P_n(t_0|t_{-1})$ does not converge to a nuclear operator. It is shown in Figure 1 that the optimal location and minimal cost based on Kalman filter do not converge in this situation.

Then, according to (22) and dominated convergence theorem, we obtain that

$$\sum_{i=1}^m (\tilde{S}_n(\Delta t))^{m-i} \tilde{S}_{x,n}\left(\frac{\Delta t}{2}\right) \tilde{S}_{y,n}\left(\frac{\Delta t}{2}\right) \tilde{B}_{z,n}^e(s + i\Delta t, s)$$

is strongly convergent to

$$\sum_{i=1}^m (S_n(\Delta t))^{m-i} S_{x,n}\left(\frac{\Delta t}{2}\right) S_{y,n}\left(\frac{\Delta t}{2}\right) B_{z,n}^e(s + i\Delta t, s).$$

Further, $\prod_{i=1}^m \tilde{M}_n(s + i\Delta t, s + (i-1)\Delta t)$ is strongly convergent to $\prod_{i=1}^m M_n(s + i\Delta t, s + (i-1)\Delta t)$.

Next we define the initial covariance as

$$P(t_0|t_{-1})f = \sum_{i=1}^{\infty} e^{-i^2} \langle f, e_i \rangle e_i, \quad f \in L^2(\Omega),$$

where $\{e_i\}$ is an orthogonal basis of $L^2(\Omega)$. The n -dimensional approximation of $P(t_0|t_{-1})$ is given by

$$P_n(t_0|t_{-1})\mathbf{P}_n f = \sum_{i=1}^n e^{-i^2} \langle \mathbf{P}_n f, e_i \rangle e_i, \quad f \in L^2(\Omega).$$

With this choice, $P(t_0|t_{-1})$ is nuclear and the assumption (A4) in Section 6 is satisfied. By Theorem 6.4, the optimal location and minimal cost based on Kalman filter and smoother are convergent, which are shown in Figure 2 and Figure 3, respectively.

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