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Abstract

High-order parabolic approximations for the Helmholtz equation can be obtained by the multiple-scale method. These approximations have the form of the hierarchies of parabolic equations, where the solution of the n-th equation is used as an input term for the $n + 1$-th equation. The transparent boundary conditions for such systems of coupled parabolic equations are derived. The well-posedness of the initial boundary value problem with the derived boundary conditions is established and a finite difference scheme for its solution is proposed.

Keywords: parabolic equation method, multiple-scale method, transparent boundary conditions
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1. Introduction

The wide-angle parabolic equations (WAPEs) presently are considered a main computational tool for many problems of wave propagation [1, 2]. The most important applications probably include acoustics [1], geophysics [3] and radiowave propagation problems [2] to mention only a few. The WAPEs are traditionally derived by means of the operator square root approximation with a Padé series (hereafter they are referred to as Padé WAPE). Recently

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another approach to the wide-angle parabolic approximations was proposed in [4]. This derivation relies upon the systematic use of the multiple-scale expansion method, and the resulting high-order parabolic approximations have the form of the system of parabolic equations (PEs), where the input term of the n-th PE is obtained from the solution of $n-1$-th PE [4]. It is important that for such parabolic approximations consistent interface and boundary conditions may be easily derived using the same multiple-scale asymptotic expansions [4]. In order to solve numerically the practical problems of wave propagation on the unbounded domains using these new parabolic approximations one has to truncate the domain with artificial boundaries.

For more than two decades many research efforts were made to develop the methods of the artificial domain truncation for the Schrödinger-type equations (including the acoustical parabolic equation and optical paraxial equation). This domain truncation may be accomplished either by imposing the transparent boundary conditions (TBC) or by extending the computational domain with so-called perfectly matching layers (PMLs). For a review of different approaches to TBCs and PMLs see [5] and numerous references therein. There are also some works concerning the TBCs for the conventional Padé WAPEs, e.g. the TBCs for the rational-linear WAPEs were derived in [6, 7] while the case of general Padé WAPE is dealt with in [8, 9, 10].

In this paper we derive the TBCs for the parabolic approximations proposed in [4]. These conditions are basically a natural but non-trivial generalization of the classical Baskakov-Popov TBCs [11]. We also show that the initial-boundary value problems (IBVPs) for the coupled PEs constituting the system from [4] with the derived TBCs are well posed and that its solution coincides with the solution of the same system on the unbounded domain. The derived TBCs may be also used for the solution of the wide angle mode parabolic equations.

We also propose a finite-difference scheme for the solution of the PEs system from [4] supplied with the derived TBCs. It is again a generalization of the numerical scheme of Baskakov and Popov [11]. In the interior of the computational domain the PEs are discretized using a second-order implicit Crank-Nicholson finite-difference method which is unconditionally stable for the unbounded domain or homogeneous Dirichlet conditions [5]. The incorporation of the TBCs into a numerical scheme may however render it only conditionally stable [12]. Sun and Wu [13] proved however that the Baskakov and Popov discretization of the TBCs leads to an unconditionally stable scheme. We adapted their proof for our new numerical scheme which also turns out to be unconditionally stable.

2. Wide angle parabolic approximations

Let us consider the problem of sound propagation in a 2D acoustical waveguide $\Omega = \{(x,z)|z \geq 0\}$ consisting of the water layer and one or more layers of bottom (which is assumed to be liquid), where z is the depth and x is the horizontal variable (here we use the acoustical notation following [4], although the same results may be reproduced for the open waveguides in optics and radiowave propagation theory). The acoustical pressure $p(x,z)$ due to a point source located at $x = 0$, $z = z_s$ then satisfies the Helmholtz equation

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{1}{\rho}\kappa^2 p = \frac{1}{\rho}\delta(x)\delta(z - z_s)\,, \tag{2.1}$$

where the medium parameters are the density $\rho = \rho(x,z)$ and the wavenumber $\kappa = \kappa(x,z) = \omega^2/c^2$. A pressure-release Dirichlet-type boundary condition

$$p(x,0) = 0 \tag{2.2}$$

is usually imposed at the ocean surface $z = 0$, while for sufficiently large values of depth (say for $z \geq L$) the medium is assumed to be homogeneous, i.e. $\rho(x,z) = \rho_b$ and $\kappa(x,z) = \kappa_b$ for all $z \geq L$. In order to correctly set a BVP for the equation (2.1) one also requires certain radiation conditions to be satisfied at infinity $R = \sqrt{x^2+z^2} \to \infty$ (see e.g. [1]). The propagation problems in the shallow-water acoustics usually feature additional complication associated with the presence of interfaces, i.e. surfaces $z = H(x)$ where media parameters have finite jump discontinuities (e.g. water-bottom interface). The following coupling conditions are imposed at the interface $z = H(x)$ (see e.g. [1]):

$$p|_{z=H(x)+0} = p|_{z=H(x)-0}\,, \qquad \left.\frac{1}{\rho}\frac{\partial p}{\partial \mathbf{n}}\right|_{z=H(x)+0} = \left.\frac{1}{\rho}\frac{\partial p}{\partial \mathbf{n}}\right|_{z=H(x)-0}\,. \tag{2.3}$$

The bottom relief described by the function $z = H(x)$ in practical problems is often very complicated.

Usually the BVP (2.1)-(2.2)-(2.2) is too complicated to be solved directly, and the high-order parabolic equations are used [1] to approximate the solution of the Helmholtz equation (2.1). While sacrificing some relatively unimportant propagation features, thus we obtain mathematical formulation of the problem which is much more efficient and easier for the numerical implementation.

Among the shortcomings of the traditional approach to the PEs derivation (the one based on the operator square root approximation) is its inability to systematically account for the sloping and variable bottoms. Strictly speaking the very Helmholtz operator factorization which leads to the equations for the forward- and backward-propagating waves (containing the operator square root) relies upon the assumption of waveguide range-independence [1]. It is also well-known that even the simplest boundary and interface conditions for the case of the sloping bottom may lead to the ill-posedness of the IBVP for the parabolic equation [14]. Although some efforts we made to derive the proper interface and boundary conditions for the PEs in the case of the sloping bottoms [15, 14], to our knowledge they were only partially successful. For example, in the paper [15] authors consider the case of the rational-linear wide-angle PE and the sloping pressure-release bottom, while for the general n-th order Padé WAPE and the arbitrarily sloping penetrable bottom no interface conditions were proposed so far. Usually the piecewise-linear approximation of the bottom is used for the computations involving these high-order WAPEs [1].

At the same time, the consistent interface and boundary conditions may be successfully obtained within another approach to the WAPE derivation [4]. This approach is based on the multiple-scale asymptotic expansion of the acoustic pressure $p(x, z)$. Here we outline the main results of this theory following closely [4].

2.1. System of PEs

We assume that the medium properties variation in x is much slower than in z and that $\kappa = \kappa_0(x) + \nu(x, z)$ ($|\nu(x, z)| \ll \kappa_0(x)$). Under these assumptions the complex acoustical pressure $p(x, z)$ may be approximated by the formula (see [4])

$$p_n(x, z) = \exp\left(\int_0^x \kappa_0(x)dx\right) \sum_{j=0}^{j=n} A_j(x, z)\,,$$

where

$$2\mathrm{i}\frac{1}{\rho}\kappa_0\frac{\partial A_j}{\partial x}+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_j}{\partial z}\right)_z+\left[\mathrm{i}\left(\frac{1}{\rho}\kappa_0\right)_x+\frac{1}{\rho}\nu\right]A_j+\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial A_{j-1}}{\partial x}\right)=0\,, \tag{2.4}$$

and $A_{-1}(x,z)=0$. System (2.4) consists of Schrödinger-type equations, and the input term of the j-th equation is determined from the solution of the $j-1$-th equation. Hence the whole system may be solved iteratively. The quantity $p_n(x,z)$ is called hereafter the *wide-angle parabolic approximation of the order* n to the solution $p(x,z)$ of the equation (2.1). It requires the solution of $n+1$ parabolic equations (2.4) to be computed. It is interesting to note that the system of PEs (2.4) for the case of the Gaussian beams propagation in the free space was derived earlier in the paper of Grikurov and Kiselev [16].

2.2. Initial conditions

To approximate the point-source solution of the Helmholtz equation, i.e. the solution of (2.1) with the right-hand side replaced by $\delta(x,z-z_s)$, one needs to define a proper Cauchy problem for the system (2.4). The Cauchy initial conditions $A_j(0,z)$ may be set up as follows. For the first equation of (2.4) which is solved for A_0 one may use any of the standard analytical or numerical starters $S(z)$ developed for the narrow-angle PEs or Padé WAPEs (e.g. Gaussian starter, Greene source, modal starter, etc, see [1] for a comprehensive study). All other equations of the PE system (2.4) are subsequently solved with zero Cauchy initial conditions:

$$\begin{aligned} &A_0(0,z)=S(z)\,,\\ &A_j(0,z)=0, \quad j=1,2,\ldots\,. \end{aligned} \tag{2.5}$$

It is shown [4] that for any fixed x_{max} approximations $p_n(x,z)$ obtained from such Cauchy problem for the system (2.4) converge to the normal mode solution of (2.1) as $n\to\infty$ uniformly in $x\in[0,x_{max}]$.

2.3. Interface conditions

Using the same asymptotic expansion as was used to derive (2.4), we obtain from (2.3) the following interface conditions for the parabolic equations

$$
\begin{aligned}
&A_j|_{z=H(x)+0} = A_j|_{z=H(x)-0} \,, \\
&\left[\frac{1}{\rho}\left(A_{j,z} + \mathrm{i}n_0 H_x A_j + H_x A_{j-1,x}\right)\right]\Big|_{z=H(x)+0} \\
&= \left[\frac{1}{\rho}\left(A_{j,z} + \mathrm{i}n_0 H_x A_j + H_x A_{j-1,x}\right)\right]\Big|_{z=H(x)-0} \,.
\end{aligned}
\tag{2.6}
$$

Similarly the boundary conditions at the sloping nonpenetrable bottom may be derived [4]. These conditions are the generalization of the Abrahamsson-Kreiss BCs [14].

It is also worthwhile to note that for parabolic approximations $p_n(x,z)$ and the consistent interface and boundary conditions (2.6) the proof of the asymptotic energy flux conservation may be established [4] in a very natural way. It is interesting to observe that the WAPA of the order n requires $n+1$ interface conditions (or boundary conditions at the nonpenetrable bottom). This is the reason why the authors of [15] found that an additional condition is necessary to establish the well-posedness of the IBVP for the rational-linear WAPE (and for higher-order Padé approximants there should be more depending on the degrees of the numerator and the denominator).

Also we note that the solutions to (2.4) are sought in $C([0, x_{max}], L^2([0,\infty)))$.

3. Transparent boundary conditions for the system of PE

The modeling of sound propagation in the typical problems of underwater acoustics requires the solution of the PEs system (2.4) in the domain $\Omega = \{(x,z) | z \geq 0, 0 \leq x \leq x_{max}\}$, where the field vanishes at the ocean surface $z = 0$ and the medium is assumed to have constant acoustical properties for sufficiently large values of depth z (i.e. there exists L such that $\rho(x,z) = \rho_b$, $\kappa(x,z) = \kappa_b$ and $\nu(x,z) = \nu_b$ for $z \geq L$). For computing the numerical solution of (2.4) it is therefore natural to truncate the computational domain at $z = L$ introducing a TBC at this fictitious boundary.

Without any loss of generality we consider in this paper the following problem. Let $\mathbf{A} = (A_0(x,z), A_1(x,z), \ldots, A_n(x,z))$ be a solution to the reference IBVP for the system

$$
\begin{aligned}
&2\mathrm{i}\kappa_0 A_{j,x} + A_{j,zz} + \nu A_j + A_{j-1,xx} = 0\,, \\
&A_0(0,z) = S(z)\,, \quad A_j(0,z) = 0\,, \quad j = 1, 2, \ldots\,, \\
&A_j(x,0) = 0\,, \\
&\lim_{z\to\infty} |A_j(x,z)| = 0\,.
\end{aligned}
\tag{3.1}
$$

(here the subscript indices x, z denote partial derivatives) in the domain Ω with the initial conditions $S(z)$ compactly supported on $[0, L]$.

We seek to construct the artificial boundary conditions of the form

$$
\mathcal{B}(A_j) = 0\,, \tag{3.2}
$$

for (3.1) at $z = L$ such that the solution $\mathbf{A}^t = (A_0^t(x,z), A_1^t(x,z), \ldots, A_n^t(x,z))$ to the IBVP for the system (3.1) on the truncated domain $\Omega^t = \{(x,z)|0 \le z \le L, 0 \le x \le x_{max}\}$ with initial conditions (2.5) and boundary conditions (3.2) at $z = L$ coincides with the solution of the reference IBVP. Recall that conditions (3.2) ensuring $\mathbf{A}^t = \mathbf{A}$ for all $(x,z) \in \Omega^t$ are called transparent boundary conditions (TBCs) for (3.1).

Note that we use the simplified form (3.1) of the equation (2.4) since in the halfspace $z \ge 0$ ρ and κ are assumed to be independent on z, x. From the TBC derivation in the next section it will be clear that it can be used without any changes for general case of the system (2.4). We assume also that the initial condition $S(z)$ vanishes for all $z \ge L$.

3.1. Construction of the TBCs

The reference IBVP in the halfspace $z \ge 0$ is obviously equivalent to the two coupled systems of IBVPs:

$$
\left\{
\begin{aligned}
&2\mathrm{i}\kappa_0 A^t_{j,x} + A^t_{j,zz} + \nu A^t_j + A^t_{j-1,xx} = 0\,, \quad (x,z) \in \Omega^t\,, \\
&A^t_0(0,z) = S(z)\,, \quad A^t_j(0,z) = 0, \quad j = 1, 2, \ldots\,, \\
&A^t_j(x,0) = 0, \quad j = 0, 1, \ldots\,, \\
&A^t_{j,z}(x,L) = A_{j,z}(x,L)\,,
\end{aligned}
\right.
\tag{3.3}
$$

and

$$
\left\{
\begin{aligned}
&2\mathrm{i}\kappa_0 A^r_{j,x} + A^r_{j,zz} + \nu_b A^r_j + A^r_{j-1,xx} = 0\,, \quad (x,z) \in \Omega^r\,, \\
&A^r_j(0,z) = 0, \quad j = 0, 1, \ldots\,, \\
&A^r_j(x,L) = A^t_j(x,L)\,, \\
&\lim\nolimits_{z\to\infty} |A^r_j(x,z)| = 0\,.
\end{aligned}
\right.
\tag{3.4}
$$

Here (3.3) is the problem on the truncated domain Ω^t and (3.4) is the right exterior problem on the halfspace $\Omega^r = [0, x_{max}] \times [L, \infty)$. For the given boundary input $A_j^t(x, L)$ the right exterior problem may be solved explicitly and $A_{j,z}^t(x, L)$ is then computed from the solution. Thus we consider the solution of (3.4) as an operator relating $A_j^t(x, L)$ to $A_{j,z}^t(x, L)$. This relation is called Dirichlet-to-Neumann map, it could be used as a boundary condition for (3.3). Note that according to (3.3),(3.4) there are no density discontinuities at $z = L$. The technique allowing to account for the density jumps is described in [6, 7].

For a given function $A_j^t(x, L)$ the problem (3.4) may be solved explicitly. We now apply the Laplace transform $\mathcal{L} : f(x) \mapsto \hat{f}(\xi)$ to all equations and BCs in (3.4) and arrive at the following BVP:

$$\begin{cases} \hat{A}_{j,zz}^r + (2\mathrm{i}\kappa_0\xi + \nu_b)\hat{A}_j^r = -\xi^2 \hat{A}_{j-1}^r\,, \quad z \in [L, \infty)\,, \\ \hat{A}_j^r(\xi, L) = \hat{A}_j^t(\xi, L)\,, \\ \lim_{z\to\infty} |\hat{A}_j^r(\xi, z)| = 0\,. \end{cases} \tag{3.5}$$

Introducing a new variable $t = z - L$ and setting $\hat{A}_j^r(\xi, z) = u_j(t)$, $2\mathrm{i}\kappa_0\xi + \nu_b = -w^2$, $\xi^2 = v$, $\hat{A}_j^t(\xi, L) = a_j$ in (3.5), we rewrite the BVP (3.5) as

$$\begin{cases} u_j'' - w^2 u_j = -v u_{j-1}\,, \quad t \in [0, \infty)\,, \\ u_j(0) = a_j\,, \\ \lim_{t\to\infty} |u_j| = 0\,. \end{cases} \tag{3.6}$$

Note that v, w are independent of t, and $w = \sqrt[+]{-2\mathrm{i}\kappa_0\xi - \nu_b}$ denotes the branch of the square root with the positive real part. This system of BVPs may be easily solved using the variation of parameters, and its solution has the general form

$$u_j(t) = \mathrm{e}^{-wt}\left(a_j + a_{j-1} v P_1(t, w) + a_{j-2} v^2 P_2(t, w) + \cdots + a_0 P_j(t, w)\right)\,, \tag{3.7}$$

where $P_k(t)$ for $k \geq 1$ are polynomials such that $u(t) = \mathrm{e}^{-wt} P_k(t)$ is a particular solution to the BVP with homogeneous BCs, i.e.:

$$\begin{cases} u'' - w^2 u = -\mathrm{e}^{-wt} P_{k-1}(t)\,, \\ u(0) = 0\,, \\ \lim_{t\to\infty} |u| = 0\,. \end{cases}$$

Substituting $u(t)$ into the latter BVP, we obtain a BVP for $P_k(t)$

$$\begin{cases} P_k'' - 2wP_k' = -P_{k-1}\,, \\ P_k(0) = 0\,, \\ \lim_{t\to\infty} |P_k(t)\mathrm{e}^{-wt}| = 0\,. \end{cases} \tag{3.8}$$

It is easy to verify explicitly that the BVPs (3.8) have solutions of the form

$$P_k(t) = \sum_{j=1}^{k} \alpha_{k,j} \frac{t^j}{w^{2k-j}}\,, \tag{3.9}$$

where the set of coefficients $\bar{\alpha}_{k+1} = \{\alpha_{k+1,j}\}$ may be computed from the set $\bar{\alpha}_k$ by solving the following system of linear algebraic equations:

$$\begin{cases} -2(k+1)\alpha_{k+1,k+1} = -\alpha_{k,k}\,, \\ -2(j+1)(j+2)\alpha_{k+1,j+2} - 2(j+1)\alpha_{k+1,j+1} = -\alpha_{k,j}, \quad j = 1, \ldots, k-1\,, \\ 2\alpha_{k+1,2} - 2\alpha_{k+1,1} = 0\,. \end{cases} \tag{3.10}$$

The solution of the system (3.10) by Gaussian elimination is an easy task; it requires the recursive evaluation of $\alpha_{k+1,j}$ from $\alpha_{k+1,j+1}$ by the formulae

$$\begin{cases} \alpha_{k+1,k+1} = \frac{1}{2(k+1)}\alpha_{k,k}\,, \\ \alpha_{k+1,j} = \frac{j+1}{2}\alpha_{k+1,j+1} + \frac{1}{2j}\alpha_{k-1,j}, \quad j = 2, \ldots, k\,, \\ \alpha_{k+1,1} = \alpha_{k+1,2}\,. \end{cases} \tag{3.11}$$

Now we have all necessary ingredients to solve the BVP (3.5). In the Laplace domain (ξ, z) we have the following expression for $\hat{A}_j^r$ in the halfspace $z \geq 0$:

$$\hat{A}_j^r(\xi, z) = \mathrm{e}^{-w(\xi)(z-L)} \left(\sum_{k=0}^{j} \xi^{2k} P_k(z-L, w(\xi)) \hat{A}_{j-k}^t(\xi, L) \right)\,,$$

where we write $P_k(z-L, w(\xi))$ instead of $P_k(z-L)$ to stress the dependence of the polynomial coefficients on ξ. To obtain a Dirichlet-to-Neumann (DtN) condition, we differentiate the last equation with respect to z

$$\frac{\partial \hat{A}_j^r(\xi, z)}{\partial z} = \mathrm{e}^{-w(z-L)} \left(\sum_{k=0}^{j} \xi^{2k} (P_k'(z-L) - wP_k(z-L)) \hat{A}_{j-k}^t(\xi, L) \right)\,. \tag{3.12}$$

Let us recall that the coupling condition in the IBVP (3.3) reads as $A^t_{j,z}(x,L) = A^r_{j,z}(x,L)$. Next we substitute the expression for $A^r_{j,z}(\xi,z)$ from (3.12) into this condition. Observing that

$$w(\xi)\mathrm{e}^{-w(\xi)(z-L)}\left(\sum_{k=0}^{j}\xi^{2k}P_k(z-L,w(\xi))\hat{A}^t_{j-k}(\xi,L)\right)\Bigg|_{z=L} = w(\xi)\hat{A}^t_j(\xi,L)\,,$$

and

$$\begin{aligned}
&\sum_{k=1}^{j}\xi^{2k}P_k'(z-L,w)\hat{A}^t_{j-k}(\xi,L)\Bigg|_{z=L}\\
&=\sum_{k=1}^{j}\xi^{2k}\left(\frac{\partial}{\partial z}\sum_{m=1}^{k}\alpha_{k,m}\frac{(z-L)^m}{w^{2k-m}}\right)\Bigg|_{z=L}\hat{A}^t_{j-k}(\xi,L)\\
&=\sum_{k=1}^{j}\xi^{2k}\left(\frac{\alpha_{k,1}}{w^{2k-1}}+\sum_{m=2}^{k}m\alpha_{k,m}\frac{(z-L)^m}{w^{2k-m}}\right)\Bigg|_{z=L}\hat{A}^t_{j-k}(\xi,L)\\
&=\sum_{k=1}^{j}\xi^{2k}\frac{\alpha_{k,1}}{w^{2k-1}}\hat{A}^t_{j-k}(\xi,L)\,,
\end{aligned}$$

we obtain the following DtN TBC in the Laplace domain:

$$\frac{\partial\hat{A}^t_j(\xi,z)}{\partial z}\Bigg|_{z=L} = -w(\xi)\hat{A}^t_j(\xi,L)+\sum_{k=1}^{j}\xi^{2k}\frac{\alpha_{k,1}}{w^{2k-1}}\hat{A}^t_{j-k}(\xi,L)\,. \tag{3.13}$$

For the sake of completeness we note that an analogous condition for the left artificial boundary at $z=0$ reads

$$\frac{\partial\hat{A}^t_j(\xi,z)}{\partial z}\Bigg|_{z=0} = w(\xi)\hat{A}^t_j(\xi,0)-\sum_{k=1}^{j}\xi^{2k}\frac{\alpha_{k,1}}{w(\xi)^{2k-1}}\hat{A}^t_{j-k}(\xi,0)\,. \tag{3.14}$$

In order to obtain the TBCs in the physical domain we now apply the inverse Laplace transform $\mathcal{L}^{-1}:\hat{f}(\xi)\mapsto f(x)$ to the conditions (3.13)-(3.14) using the well-known properties of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}(\xi^{2k}\hat{f}(\xi)) = \frac{d^{2k}f}{dx^{2k}}\,,$$

$$\mathcal{L}^{-1}(\hat{g}(\xi - a)\hat{f}(\xi)) = \mathrm{e}^{ax}\int_0^x g(x-y)\mathrm{e}^{-ay}f(y)dy = \mathrm{e}^{ax}\mathcal{L}^{-1}(\hat{g}(\xi)\mathcal{L}(\mathrm{e}^{-ax}f(x)))\,,$$

$$\mathcal{L}^{-1}\left(\frac{\sqrt[+]{\xi}}{\xi^k}\hat{f}(\xi)\right) = \frac{d}{dx}\int_0^x \frac{dy}{\sqrt{x-y}}\int_0^y\int_0^{y_k}\dots\int_0^{y_2} f(y_1)dy_1dy_2\dots dy_k\,,$$

and observing that

$$\frac{1}{w(\xi)^{2k-1}} = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}(2k-1)}}{(2\kappa_0)^{k-\frac{1}{2}}}\frac{\sqrt[+]{\xi - \frac{\mathrm{i}\nu_b}{2\kappa_0}}}{\left(\xi - \frac{\mathrm{i}\nu_b}{2\kappa_0}\right)^k}\,.$$

Thus we straightforwardly arrive at the following TBCs

$$\frac{\partial A_j^t(x,z)}{\partial \mathbf{n}} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\mathrm{e}^{\mathrm{i}\frac{\nu_b}{2\kappa_0}x}\frac{d}{dx}\int_0^x\frac{dy}{\sqrt{x-y}}\Bigg(A_j^t(y,z)\mathrm{e}^{-\mathrm{i}\frac{\nu_b}{2\kappa_0}y}$$
$$-\sum_{k=1}^{j}\alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k}\int_0^y\int_0^{y_k}\dots\int_0^{y_2}\mathrm{e}^{-\mathrm{i}\frac{\nu_b}{2\kappa_0}y_1}\,\frac{\partial^{2k}A_{j-k}^t(y_1,z)}{\partial y_1^{2k}}dy_1dy_2\dots dy_k\Bigg) \tag{3.15}$$

at $z = L$ and $z = 0$ ($\mathbf{n}$ denotes the outward unit normal vector at $z = L$, $z = 0$ respectively). It is reasonable to tabulate the first coefficients $\alpha_{k,1}$:

$$\alpha_{1,1} = \frac{1}{2}\,,\quad \alpha_{2,1} = \frac{1}{8}\,,\quad \alpha_{3,1} = \frac{1}{16}\,,\quad \alpha_{4,1} = \frac{5}{128}\,,\quad \alpha_{5,1} = \frac{7}{256}\,,\dots$$

since in practice it is enough to keep only the first few terms of the sum in (3.15).

The TBCs (3.15) simplify significantly if we choose the reference wavenumber κ_0 in such a way that $\nu_b = 0$ (this condition is fulfilled if we set $\kappa_0(x) = \kappa_b$ in (2.4)). Under this assumption the multiple integral on the right-hand side of (3.15) vanishes, and the TBC becomes

$$\frac{\partial A_j^t(x,z)}{\partial \mathbf{n}} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sum_{k=0}^{j}\alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k}\frac{d}{dx}\int_0^x\frac{\partial^k A_{j-k}^t(y,z)}{\partial y^k}\frac{dy}{\sqrt{x-y}}\,,$$
$$\text{at } z = 0, L\,, \tag{3.16}$$

where $\alpha_{0,1} = -1$.

Note that the TBCs (3.15) are a natural generalization of the TBC for the single parabolic equation (see e.g. [5]), while its simplified form (3.16) is a generalization of the TBC for the PE with the potential vanishing at the artificial boundary [11, 13].

4. Well-posedness of the IBVP and the uniqueness of its solution

In this section we first prove the uniqueness of the solution of the IBVP supplied with the TBC (3.15) and then show that the resulting problems for $A_j^t(x,z)$ are well-posed. For the sake of simplicity we restrict our attention to the problem with the right transparent boundary condition:

$$\begin{aligned}
&2\mathrm{i}\kappa_0 A^t_{j,x} + A^t_{j,zz} + \nu A^t_j + A^t_{j-1,xx} = 0\,,\ (x,z)\in\Omega^t\,,\\
&A^t_0(0,z) = S(z)\,,\quad A^t_j(0,z) = 0,\ j = 1,2,\dots\,,\\
&A^t_j(x,0) = 0,\ j = 0,1,\dots\,,\\
&\frac{\partial A^t_j(x,z)}{\partial z} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sum_{k=0}^{j}\frac{\alpha_{k,1}}{(-2i)^k\kappa_0^k}\frac{d}{dx}\int_0^x\frac{\partial^k A^t_{j-k}(y,z)}{\partial y^k}\frac{dy}{\sqrt{x-y}}\ \text{at } z = L.
\end{aligned}\tag{4.1}$$

4.1. Existence and uniqueness

The uniqueness proof relies on the following standard result for the homogeneous Schrödinger-type equation [7]

Lemma 4.1. *[7] An IBVP*

$$\begin{aligned}
&2\mathrm{i}\kappa_0 B_x + B_{zz} + \nu B = 0\,,\ (x,z)\in\Omega^t\,,\\
&B(0,z) = S(z)\,,\quad S(z) = 0 \textit{ for } z\ge L\,,\\
&B(x,0) = 0\,,\\
&\frac{\partial B(x,z)}{\partial z} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\alpha_{0,1}\frac{d}{dx}\int_0^x\frac{B(y,z)dy}{\sqrt{x-y}}\ \textit{at } z = L
\end{aligned}\tag{4.2}$$

for the standard PE with the TBC at $z = L$ and compactly supported initial data $S(z)\in H^2[0,L]$ has unique solution $B(x,z)\in L^2[0,L]$.

This lemma immediately implies the following result:

Proposition 4.1. *If there exist two solutions $A_j^1(x,z)$ and $A_j^2(x,z)$ of the IBVP (4.1) then $A_j^1(x,z) = A_j^2(x,z)$ for all $(x,y)\in\Omega^t$.*

Proof. If $A_j^1(x,z)$ and $A_j^2(x,z)$ satisfy (4.1) then $A_j^1(x,z) - A_j^2(x,z)$ satisfies IBVP (4.2) with $S(z) = 0$ for all z. From the uniqueness of the solution of IBVP (4.2) it follows that $A_j^1(x,z) - A_j^2(x,z) \equiv 0$. $\square$

To prove the existence of the solution of IBVP (4.1) and its well-posedness we first consider a general inhomogeneous IBVP of the form

$$\begin{aligned} &2\mathrm{i}\kappa_0 A_x + A_{zz} + \nu A + \zeta = 0\,,\\ &A(0,z) = 0\,,\\ &A(x,0) = 0\,,\\ &\lim_{z\to\infty} |A(x,z)| = 0\,, \end{aligned} \tag{4.3}$$

The following result may be then established [17, 18]:

Lemma 4.2. *If the input term $\zeta(x,z) \in C([0,x_{max}], H^2(\mathbb{R}))$ and the potential $\nu(x,z) \in C([0,x_{max}], H^2(\mathbb{R}))$ and $\nu(x,z) = 0$ for $z \geq L$ then the IBVP (4.3) has a unique classical solution*

$$A(x,z) \in C([0,x_{max}], H^2(\mathbb{R})) \cap C^1([0,x_{max}], L^2(\mathbb{R}))\,.$$

Note that S.-i. Doi [18] proved that the solution also belongs to $C([0,x_{max}], H^k(\mathbb{R}))$ for sufficiently smooth initial data and potential. Hereafter we assume that these conditions are fulfilled, i.e. that $\zeta(x,z) \in C([0,x_{max}], C^M(\mathbb{R}))$ and $\nu(x,z) \in C([0,x_{max}], C^M(\mathbb{R}))$ for sufficiently large M such that $A_{jxx}(x,z)$ in (3.1) belongs to $C([0,x_{max}], H^2(\mathbb{R}))$ for all $j = 1, 2, \ldots, n$. Although this requirement is somewhat too restrictive, the more tedious formulation of necessary conditions for $S(z)$ and $\nu(x,z)$ would lead us too far from the subject of this study. The function $S(z)$ in most PE starters is in $C^\infty(\mathbb{R})$, but the potential may actually be piecewise smooth in z (usually piecewise linear), and the well-posedness proof given here does not cover this case. Summarizing the arguments above we formulate

Lemma 4.3. *For $\nu(x,z) \in C([0,x_{max}], C^M(\mathbb{R}))$ and $S(z) \in C^M(\mathbb{R})$ where M is sufficiently large the IBVP (3.1) has a unique classical solution $\mathbf{A} = (A_0(x,z), A_1(x,z), \ldots, A_n(x,z))$ where $A_j(x,z) \in C([0,x_{max}], H^2(\mathbb{R}))$.*

This lemma immediately implies

Proposition 4.2. *If the conditions of Lemma 4.3 are fulfilled and additionally $\nu(x,z) = 0$ and $S(z) = 0$ for all $z \geq L$ then there exist a classical solution of the IBVP (4.1) (which is unique according to Proposition 4.1). This solution coincides with the solution of (3.1) for all $(x,z) \in \Omega^t$.*

Proof. By construction of the conditions (3.16), any solution $A_j(x,z)$ of the halfspace problem (3.1) satisfies the TBC (3.16). Thus the solution from the Lemma 4.3 also solves (4.1). $\square$

4.2. Well-posedness

Since the solution of the IBVP with the TBC (4.1) is merely a restriction of the halfspace problem (3.1) solution to the truncated domain Ω^t, the well-posedness of (4.1) also follows from the well-posedness of the IBVP (3.1). The latter fact is established by the following proposition

Proposition 4.3. *For the solution of the IBVP (3.1) under the assumptions of Lemma 4.3 the following inequality holds for* $j = 1, 2, \dots$

$$2\kappa_0 \left(N(x) - N(0)\right) \leq \int_0^x \eta(\xi) d\xi \,, \tag{4.4}$$

where

$$N(x) = \|A_j(x,z)\|_{L_z^2} = \left(\int_0^\infty |A_j(x,z)|^2 dz\right)^{1/2} ,$$

$$\eta(x) = \|\partial^2 A_{j-1}(x,z)/\partial x^2\|_{L_z^2} = \left(\int_0^\infty |\partial^2 A_{j-1}(x,z)/\partial x^2|^2 dz\right)^{1/2} .$$

Proof. For convenience we use the notation from (4.3): $A(x,z) = A_j(x,z)$ and $\zeta(x,z) = \partial^2 A_{j-1}(x,z)/\partial x^2$.

Let us recast the parabolic equation from (4.3) in the form

$$2\kappa_0 A_x = \mathrm{i} A_{zz} + \mathrm{i}\nu A + \mathrm{i}\zeta \,,$$

multiply it by A^* (the star denotes complex conjugation) and take the real part of the resulting expression. After obvious transformations we obtain

$$\kappa_0 \frac{\partial}{\partial x} |A|^2 = \Re(\mathrm{i} A^* A_{zz}) + \Re(\mathrm{i}\nu |A|^2) + \Re(\mathrm{i} A^* \zeta) \,.$$

Now let us integrate the last equation with respect to z:

$$\kappa_0 \frac{d}{dx} \|A\|_{L_z^2}^2 = \Re\left(\int_0^\infty \mathrm{i} A^* A_{zz} dz\right) + \int_0^\infty \Im(\nu) |A|^2 dz + \Re\left(\mathrm{i} \int_0^\infty A^* \zeta dz\right) .$$

Firstly using integration by parts we find that

$$\Re\left(\int_0^\infty \mathrm{i}A^* A_{zz} dz\right) = \Re\left(\mathrm{i}A^* A_z|_0^\infty - \mathrm{i}\int_0^\infty A_z^* A_z dz\right)$$
$$= \Re\left(\mathrm{i}\int_0^\infty |A_z|^2 dz\right) = 0\,.$$

Secondly we observe that $\Im(\nu) = 0$ for the medium with no attenuation, while for the lossy one $\Im(\nu) < 0$. We therefore conclude that

$$\kappa_0 \frac{d}{dx}\|A\|^2_{L^2_z} \leq \Re\left(\mathrm{i}\int_0^\infty A^* \zeta dz\right) \leq \|A\|_{L^2_z}\|\zeta\|_{L^2_z}\,.$$

This inequality may be rewritten using our notation as

$$2\kappa_0 N'(x)N(x) \leq N(x)\eta(x)\,,$$

and this concludes the proof of the proposition

$$2\kappa_0(N(x) - N(0)) = 2\kappa_0 \int_0^x N'(\xi)d\xi \leq \int_0^x \eta(\xi)d\xi\,.$$

□

5. Numerical scheme for the solution of the IBVP and its stability

In this section we construct a finite difference numerical scheme for the solution of the IBVP (4.1). It is a simple generalization of the scheme devised by Sun and Wu [13] which in turn reduces to the numerical method of Baskakov and Popov [11]. This scheme enjoys a remarkable property of the unconditional stability, and this fact was first proven in [13]. Our scheme also inherits this property as we will show here.

We introduce a uniform grid $x^n = n\Delta x$, $z^m = m\Delta z$, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$, where $\Delta z M = L$, $\Delta x N = x_{max}$. Hereafter upper indices are used to denote values of any given function $U(x, z)$ in the grid points:

$$U^{n,m} \equiv U(x^n, z_m)\,, \quad U^{n+1/2,m} \equiv \frac{1}{2}\left(U(x^{n+1}, z^m) + U(x^n, z^m)\right)\,.$$

For the parabolic equations (2.4) in the problem (4.1) we use the Crank-Nicholson finite difference scheme modified to include the input term:

$$
\begin{aligned}
2\mathrm{i}\kappa_0 \frac{A_j^{n+1,m} - A_j^{n,m}}{\Delta x} + & \frac{A_j^{n+1/2,m+1} - 2A_j^{n+1/2,m} + A_j^{n+1/2,m-1}}{\Delta z^2} \\
& + \nu^{n+1/2,m} A_j^{n+1/2,m} + (A_{j-1,xx})^{n+1/2,m} = 0\,,
\end{aligned}
\tag{5.1}
$$

where the input term $(A_{j-1,xx})^{n+1/2,m} = \left((A_{j-1,xx})^{n+1,m} + (A_{j-1,xx})^{n,m}\right)/2$ is computed from the solution of the equation for A_{j-1} using the standard 3-point finite difference approximation

$$
(A_{j-1,xx})^{n,m} \approx \frac{A_{j-1}^{n+1,m} - 2A_{j-1}^{n,m} + A_{j-1}^{n-1,m}}{\Delta x^2}\,.
$$

The scheme (5.1) is marching in x, i.e. it computes the vector $\mathbf{A}_j^n = (A_j^{n,0}, A_j^{n,1}, \ldots, A_j^{n,M})$ from the vector $\mathbf{A}_j^{n-1}$. The initial condition (4.1) at $x = 0$ yields the starter $\mathbf{A}_j^0$:

$$
A_0^{0,m} = S(z^m)\,, \quad A_j^{0,m} = 0, \quad j = 1, 2, \ldots\,. \tag{5.2}
$$

The Dirichlet initial condition $A_j(x, 0) = 0$ at $z = 0$ turns into

$$
A_j^{n,0} = 0\,, \quad j = 0, 1, 2, \ldots\,. \tag{5.3}
$$

We now describe the approximation of the TBC (3.16) in (4.1). Let us first recast the convolution terms of the TBC in the following form

$$
\frac{d}{dx}\int_0^x \frac{\partial^k A_{j-k}^t(y,z)}{\partial y^k}\frac{dy}{\sqrt{x-y}} = \frac{d^k}{dx^k}\int_0^x \frac{\partial A_{j-k}^t(y,z)}{\partial y}\frac{dy}{\sqrt{x-y}}\,.
$$

For the the latter integral we use the approximation developed by Baskakov and Popov [11]:

$$
\int_0^{(n+1)\Delta x} \frac{\partial A_j(y,L)}{\partial y}\frac{dy}{\sqrt{x-y}} \approx \sum_{k=0}^{n} \gamma_k A_j^{n+1-k,M}\,, \tag{5.4}
$$

where $\gamma_0 = \sqrt{2}/\sqrt{\Delta x}$, while for $k = 1, 2, \ldots$

$$
\gamma_k = \frac{-2}{\sqrt{\Delta x}\left(\sqrt{k+1}+\sqrt{k}\right)\left(\sqrt{k-1}+\sqrt{k}\right)\left(\sqrt{k+1}+\sqrt{k-1}\right)}\,.
$$

To further simplify the notation, we define $\beta_k = \sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\gamma_k$ and introduce the following approximation for the input terms in the BC at $z = L$:

$$\begin{aligned}\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \sum_{k=1}^{j} \alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k} \frac{d}{dx} \int_0^{\Delta x(n+1)} \frac{\partial^k A^t_{j-k}(y,z)}{\partial y^k} \frac{dy}{\sqrt{x-y}} \\ \approx \sum_{k=1}^{j} \alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k}(D_x)^k \sum_{s=0}^{n} \beta_k A_j^{n+1-s,M}\,,\end{aligned} \tag{5.5}$$

where $(D_x)^k$ is the k-th degree of the central difference operator D_x defined by

$$D_x U^{n,m} = \frac{U^{n+1,m} - U^{n-1,m}}{2\Delta x}$$

for any function $U^{n,m} = U(x^n, z^m)$ on the grid x^n, z^m. The right-hand side of (5.5) is denoted hereafter as W_j^n. The discretized form of the TBC (3.16) from the IBVP (4.1) can be written as

$$\frac{1}{2\Delta z}\left(A_j^{n+1/2,M+1} - A_j^{n+1/2,M-1}\right) = -\sum_{k=0}^{n} \beta_s A_j^{n+1/2-s,M} - W_j^{n+1/2}\,. \tag{5.6}$$

Expressing the quantity $A_j^{n+1/2,M+1}$ from (5.6) and substituting it into the formula (5.1) (for $m = M$), we finally arrive at

$$\begin{aligned}2\mathrm{i}\kappa_0 \frac{A_j^{n+1,M} - A_j^{n,M}}{\Delta x} + 2\frac{A_j^{n+1/2,M-1} - A_j^{n+1/2,M}}{\Delta z^2} + \nu^{n+1/2,M} A_j^{n+1/2,M} \\ - \frac{2}{\Delta z}\left(\sum_{k=0}^{n} \beta_s A_j^{n+1/2-s,M} + W_j^{n+1/2}\right) + (A_{j-1,xx})^{n+1/2,M} = 0\,,\end{aligned} \tag{5.7}$$

Equations (5.1), (5.7) and (5.3) constitute together a linear system which allows to obtain $\mathbf{A}_j^{n+1}$ from $\mathbf{A}_j^n$.

Sun and Wu [13] have proved the following result for the numerical scheme (5.1)-(5.7)-(5.3) with no input (i.e. for the case of $A_{j-1,xx} = 0$ and $W_j = 0$):

Proposition 5.1. *In the case of the IBVP (4.2) the numerical scheme (5.1)-(5.7)-(5.3) is unconditionally stable, i.e. the following inequality holds for all n:*

$$\|\mathbf{B}^n\|_{\ell^2} \le \|\mathbf{B}^0\|_{\ell^2}\,,$$

where $\mathbf{B}^n = (B^{n,1}, \ldots, B^{n,M})$*, and* $\|\cdot\|_{\ell^2}$ *denotes the discrete* ℓ^2*-norm:*

$$\|\mathbf{B}^n\|_{\ell^2}^2 = \sum_{m=0}^{M} |B^{n,m}|^2 \,.$$

It is easy to see that in the case of the equation for $A_j(x^n, z^m)$ the input terms may be considered as additional initial conditions (introduced at $x = x^1$, $x = x^2, \ldots$). The resulting solutions for all of these additional initial conditions are combined in the inhomogeneous equation for A_j and by Proposition 5.1 we conclude the following:

Proposition 5.2. *The numerical solution of the IBVP (4.1) obtained using the numerical scheme (5.1)-(5.7)-(5.3) is unconditionally stable, and the following inequality is satisfied for any* $N = 1, 2, \ldots$*:*

$$\|\mathbf{B}^N\|_{\ell^2} \leq \|\mathbf{B}^0\|_{\ell^2} + C_1 \sum_{n=0}^{N} |W_j^n|^2 + C_2 \sum_{n=0}^{N} \sum_{m=0}^{M} |(A_{j-1,xx})^{n,M}|^2 \,,$$

where the constants C_1 *and* C_2 *are independent on* W_j*,* $A_{j-1,xx}$ *and* N*.*

The proof of this result obviously follows from Proposition 5.1.

6. Test examples

In this section we conduct some numerical experiments in order to test the derived TBCs for the equations (2.4). Since the performance of the TBCs does not depend on the inhomogeneities in the interior of the computational domain, we study here the simplest case of sound propagation in the homogeneous medium. Let us consider a halfspace $z \geq 0$ with the parameters $c(x, z) = c$ and $\rho(x, z) = 1$. The source of the frequency f is located at $x = 0$, $z = z_s$ (hence $\kappa_0 = 2\pi f/c$). We set up an artificial boundary at $z = L = 2z_s$ and solve the problem of sound propagation using the system of the IBVPs (4.1) and the finite difference scheme (5.1)-(5.7)-(5.3).

We use the Gaussian initial condition [1] of the form

$$A_0(0, z) = S(z) = \bar{A} \mathrm{e}^{-\frac{(z-z_s)^2}{\sigma^2}} \tag{6.1}$$

in the IBVP (4.1) for A_0 in order to simulate an acoustical field produced by the point source. It easy to see that for such initial conditions all the

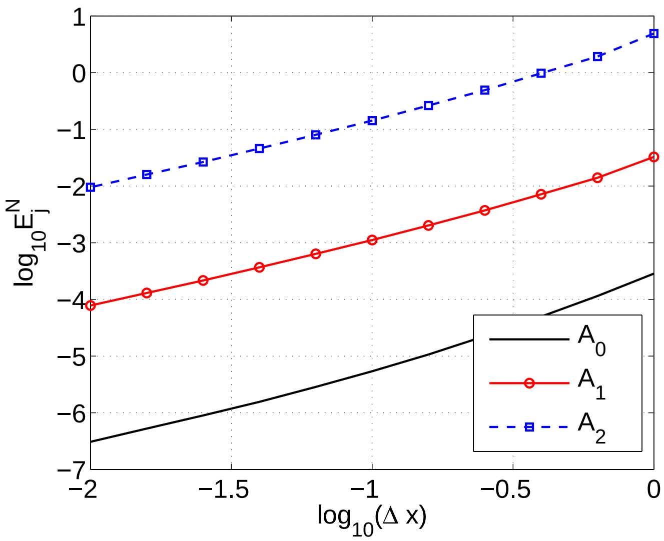

Figure 1: ℓ^2-errors E_j^N computed for different mesh sizes Δx for the solutions $A_j(x,y)$, $j = 0, 1, 2$.

equations of the system (3.1) in the homogeneous halfspace $z \geq 0$ may be solved analytically using the Fourier transform in z (cf. [1]). The solution has a very simple form, e.g. A_0^A and A_1^A (subscript A for "analytical") writes as

$$
\begin{aligned}
A_0^A(x,z) &= U_0(x, z - z_s) - U_0(x, z + z_s)\,, \\
A_1^A(x,z) &= U_1(x, z - z_s) - U_1(x, z + z_s)\,,
\end{aligned}
\tag{6.2}
$$

where $U_0(x,z), U_1(x,z)$ are the solutions for the unbounded media (i.e. for $-\infty \leq z \leq \infty$) with the source located at $z = 0$:

$$
\begin{aligned}
U_0(x,z) &= \bar{A}\sqrt{\frac{\sigma^2\kappa_0}{\sigma^2\kappa_0 + 2\mathrm{i}x}}\exp\left(-\frac{\kappa_0 z^2}{\sigma^2\kappa_0 + 2\mathrm{i}x}\right)\,, \\
U_1(x,z) &= \mathrm{i}x\frac{6x^2 - 6\mathrm{i}\sigma^2\kappa_0 x + 12\mathrm{i}\kappa_0 z^2 x - 2\kappa_0^2 z^4 + 6\sigma^2\kappa_0^2 z^2 - \frac{3}{2}\sigma^4\kappa_0^2}{\kappa_0(\sigma^2\kappa_0 + 2\mathrm{i}x)^4}U_0(x,z)\,.
\end{aligned}
\tag{6.3}
$$

The solutions of higher-order equations have similar form and may be obtained in the same straightforward way. In the book [1], it is shown that the

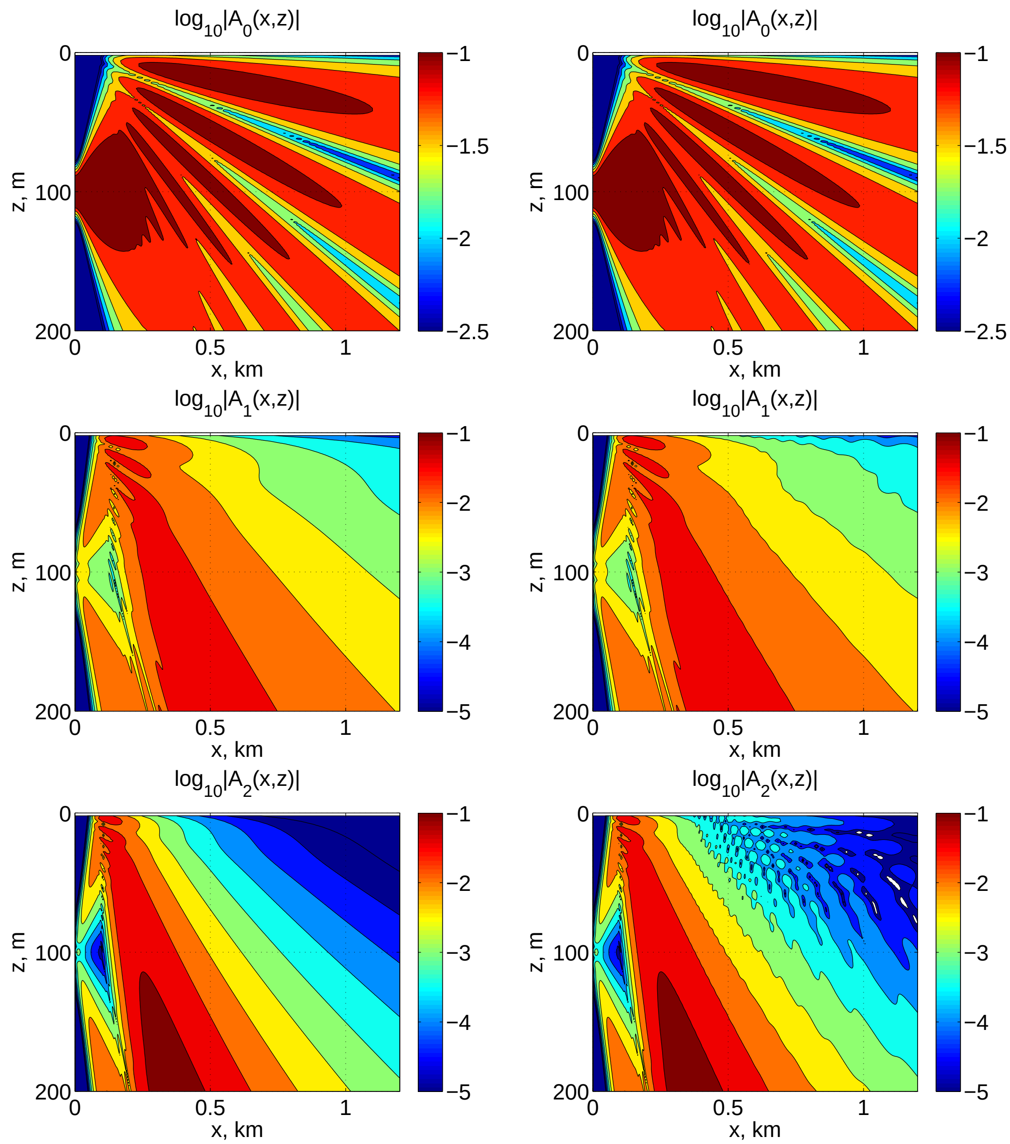

Figure 2: Contour plots of $\log_{10}|A_j(x,z)|$ for $j=0$ (top), $j=1$ (middle), $j=2$ (bottom). The numerical solutions of the IBVP (4.1) are shown on the right, while their analytical counterparts are presented on the left.

parameters in the initial condition (6.1) must be chosen as

$$\bar{A} = \sqrt{\kappa_0}\,, \quad \sigma = \frac{\sqrt{2}}{\kappa_0}\,.$$

We may now compare the numerical solutions $A_0, A_1, \ldots$ of the IBVP (4.1) against the analytical solutions $A_0^A, A_1^A, \ldots$ (6.2) of the halfspace problem (3.1). The numerical solutions were computed for different pairs of $\Delta x_q, \Delta z_q$, where

$$\Delta x_q = \frac{1}{10^{0.2q}} \Delta x_0 \,, \quad \Delta z_q = \frac{1}{10^{0.1q}} \Delta z_0 \,,$$

and $\Delta z_0 = 1$ m, $\Delta x_0 = 1$ m. The ratio $(\Delta z_q)^2/\Delta x_q = (\Delta z_0)^2/\Delta x_0$ is therefore kept constant (this is standard practice for finite difference schemes for parabolic equations). In order to estimate the quality of approximation, we compute E_j^N, the ℓ^2-error at $x_{max} = N\Delta x$ divided by the ℓ^2-norm of the exact solution in the domain $[0, L]$:

$$E_j^N = \frac{\|\mathbf{A}_j^N - \mathbf{A}_j^{A,N}\|_{\ell^2}}{\|\mathbf{A}_j^{A,N}\|_{\ell^2}} \,, \tag{6.4}$$

where $\mathbf{A}_j^{A,n} = \left(A_j^A\left(x^n, z^0\right), A_j^A\left(x^n, z^0\right), \ldots, A_j^A\left(x^n, z^0\right)\right)$

All the numerical solutions A_j converge steadily to their analytical counterparts, and the rate of convergence is the same as predicted by the estimates of Sun and Wu [13] (i.e. the error is approximately linearly dependent on Δz). This convergence is illustrated in the Fig. 1, where errors E_j^N are plotted against the mesh sizes Δz_q for $j = 0, 1, 2$. The computed solutions A_0, A_1, A_2 and their analytical counterparts are shown in Fig. 2 in the form of the log-contour plots (i.e. we plotted the contours $0, -1, -2, \ldots$ for the function $\log_{10} |A_j(x, y)|$). These log-contours allow us to detect very small levels of the reflection.

Note that although the level of accuracy decreases as j increases, this fact is related mostly to the numerical differentiation on the right-hand side of (2.4). In the paper [4] we suppressed these errors using the Lanczos low-noise differentiation formulae. Here we did not use this technique, since the proper TBCs (3.15) allowed us to use standard 3-point central difference formula.

7. Conclusion

In this work the TBCs for the system of the parabolic equations from [4] were derived. The solutions of the equations (2.4) form the high-order parabolic approximation to the solution of the Helmholtz equation describing the propagation of acoustic waves. Therefore, in many practical situations

TBCs are required in order to solve them numerically on the artificially truncated domains. The TBCs were derived by the same method based on the Laplace transform as was used in several classical papers (see [5]). We considered the IBVP (4.1) for the system of PEs (2.4) with the presented TBCs and proved the existence and uniqueness of its solution. Also the well-posedness of the latter IBVP was established. We developed a finite difference scheme for the numerical solution of this IBVP with the derived TBC. This scheme, being a natural generalization of the one proposed by Baskakov and Popov, is shown to be unconditionally stable. The efficiency of the presented scheme and the convergence of the computed solutions to the analytical ones (as $\Delta z \to \infty$) is confirmed by a numerical example.

It is important to note that since we must differentiate the numerical solution $A_j(x, z)$ with respect to x in order to obtain an input term in the equation for $A_{j+1}(x, z)$, even small reflections at the fictitious boundary may destroy the method's order of accuracy (small errors are inflated by differentiation). This is the reason why it is necessary to make these undesirable reflections as small as possible. One possible way to almost completely suppress them is to use the fully-discrete TBCs consistent with the Crank-Nicolson scheme. Such conditions for the usual parabolic equation (or Schrödinger equation) were derived in [7], and in our future work we will attempt to generalize the TBCs form [7] to the case of the system (2.4).

It would be also interesting to generalize the equations (2.4) to the 3D case, and to investigate the TBCs for this case using e.g. the method of Schädle [19].

8. Acknowledgements

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