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OPTION PRICING WITH DYNAMICALLY CORRELATED STOCHASTIC INTEREST RATE

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Abstract. In this work we review several option pricing models with stochastic interest rate and extend this model by incorporating local time dependent correlation between the underlying and the interest rate. We compare the difference between using a constant and a dynamic correlation by analyzing some numerical benchmarks. Furthermore, we conduct an experiment on fitting the pricing model to the market price. Our analysis shows that the option pricing within the Black-Scholes framework can not really be improved by incorporating stochastic interest rate even when using a nonlinear correlation.

1. Introduction

The Black-Scholes model [**2**] defining the fair price of European-style options is one of the most famous models. However, due to the assumption that the stock log-return follows a geometric Brownian motion (with constant volatility), the widening gap between model and market data could exist almost all the time. For this reason, the Black-Scholes model have been generalized to allow stochastic volatility, see e.g. [**5**] and [**6**], the pricing performance has been thus improved.

The other strong assumption of constant interest rate is also not realistic. The first work on incorporating a stochastic interest rate into the Black-Scholes model is provided by Merton [**8**]. Afterwards, a couple of work on option pricing under stochastic interest rate was published, e.g. [**1**], [**3**], [**4**] and [**9**]. However, some empirical findings showed that stochastic interest rates may be not important for the pricing and hedging of short term options, see e.g. [**4**] and [**7**]. Besides, the paper [**3**] concluded that allowing interest rates to be stochastic does not necessarily improve pricing performance any further, even for long-term options, once the model has accounted for stochastically varying volatility.

We have seen that the correlation between interest rates process and underlying process in the works mentioned above has been assumed to be constant. Unfortunately, this assumption is also dubious due to the fact that financial quantities are correlated always in a nonlinear way, even may be correlated stochastically, see [**10**], [**11**] and [**12**]. Besides, it has been inferred in [**12**] and [**13**] that the Heston model and the model of Quanto-option pricing can be better fitted to the market data using dynamic (only time-dependent) correlation than using constant correlation. Thus, it is interesting to ask whether stochastic interest rates could be important for the hedging and pricing of options if the correlation between interest rates and underlying asset is not considered as a constant.

Motivated by this question, in this work, we review and extend some option pricing models with stochastic interest rate by allowing nonconstant correlation. Firstly, we compare the option pricing between using constant and nonconstant correlation by analyzing some numerical results. Secondly, we conduct an experiment on fitting the pricing models to the market data, in

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order to check, whether stochastic interest rates are important for option pricing while allowing nonconstant correlation.

The paper is organized as follows. In the next section, we review and extend two different pricing models with stochastic interest rate and dynamic correlation. Section 3 is devoted to investigate the difference of model calibration between using constant and dynamic correlation. Finally, Section 4 concludes this work.

2. Option Pricing with dynamically correlated Stochastic Interest Rate

In this Section, we consider two pricing models with stochastic interest rate. First, we review and extend the Merton model [**8**] of pricing European option where bond price dynamics is allowed. Besides, we study the option pricing model with stochastic interest rate given by Vasicek stochastic differential equation in [**9**] and [**7**].

2.1. The Merton model

We use the following stochastic differential equation (SDE) to describe stock price S_t and bond price dynamics P_t, respectively as

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t^1 \tag{1}$$

$$\frac{dP_t}{P_t} = \mu_P dt + \sigma_P \rho_t dW_t^1 + \sigma_P \sqrt{1-\rho_t^2} dW_t^2 \tag{2}$$

with the instantaneous expected return μ_S, μ_P, the instantaneous variance σ_S^2, σ_P^2 and the two independent Brownian motions W_t^1, W_t^2. We denote the European option price function by $H(S, P, \tau; K)$ for using the constant correlation $\rho_t = \rho$ between the return on the stock and on the bond and by $V(S, P, \rho_\tau, \tau; K)$ for using the corresponding dynamic correlation ρ_t, where K is the strike price. Merton [**8**] has shown that $H(S, P, \tau; K)$ must satisfy

$$\frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 H}{\partial S^2} + \rho \sigma_S \sigma_P SP \frac{\partial^2 H}{\partial S \partial P} + \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 H}{\partial P^2} - \frac{\partial H}{\partial \tau} = 0 \tag{3}$$

subject to the boundary conditions

$$\begin{cases} H(0, P, \tau; K) = 0 \\ H(S, 1, 0; K) = \max(0, S-K), \end{cases} \tag{4}$$

which is a second-order, linear partial differential equation (PDE) of parabolic type. Since ρ_t is a function only dependent on time (without stochasticity), it is thus straightforward that

$$\frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \underline{\rho_t} \sigma_S \sigma_P SP \frac{\partial^2 V}{\partial S \partial P} + \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 V}{\partial P^2} - \frac{\partial V}{\partial \tau} = 0, \tag{5}$$

subject to the boundary conditions

$$\begin{cases} V(0, P, \rho_\tau, \tau; K) = 0 \\ V(S, 1, \rho_0, 0; K) = \max(0, S-K). \end{cases} \tag{6}$$

Following the methodologies [**8**], we define the $x = \frac{S}{KP_\tau}$ which can be described with the aid of Itô lemma as

$$\frac{dx}{x} = [\mu_S - \mu_P + \sigma_P^2 - \rho_t \sigma_P \sigma_S]dt + \sigma_S dW_t^1 - \sigma_P \rho_t dW_t^1 - \sigma_P \sqrt{1-\rho_t^2} dW_t^2, \tag{7}$$

from which we obtain the instantaneous variance of the return on x given by

$$\sigma_t^2 := \sigma_P^2 + \sigma_S^2 - 2\rho_t \sigma_P \sigma_S. \tag{8}$$

Furthermore, we define $v = \frac{V}{KP}$ and substitute x and v in (5) to get

$$\frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial \tau} = 0. \tag{9}$$

Finally, we consider a new time variable $T := \int_0^\tau \sigma_t^2 dt$ and define $y(x,T) := v(x,\tau)$ which can be substituted into (9) to obtain the famous heat equation:

$$\frac{1}{2}x^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial \tau} = 0, \tag{10}$$

subject to the boundary conditions, $y(0,T) = 0$ and $y(x,0) = \max(0, x-1)$. We know the fact that the heat equation (10) can be solved analytically, the solution of $V(S, P, \rho_\tau, \tau; K)$ can thus be found as:

$$V(S, P, \rho_\tau, \tau; K) = S\Phi(d_1) - KP\Phi(d_2) \tag{11}$$

with

$$d_1 := \frac{\ln \frac{S}{K} - \ln P + \frac{1}{2}\int_0^\tau \sigma_s^2 ds}{\sqrt{\int_0^\tau \sigma_s^2 ds}}, \quad d_2 := d_1 - \sqrt{\int_0^\tau \sigma_s^2 ds}$$

and where σ_t is defined in (8) and $\Phi(x)$ denote the standard normal cumulative distribution function. So far, in oder to compute the European call option price we need to know the formula of P_τ and a reasonable local correlation function ρ_t.

Following the methodologies [**8**] we assume that the short rate r_t follows a Gauss-Wiener process [1]

$$dr_t = \mu_r dt + \sigma_r \rho_t dW_t^1 + \sigma_r \sqrt{1-\rho_t^2} dW_t^2. \tag{12}$$

Applying Itô lemma with $P(\tau; r)$ we obtain

$$dP = \frac{\partial P}{\partial \tau} d\tau + \frac{\partial P}{\partial r} dr_t + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr_t)^2. \tag{13}$$

Substituting (12) into (13) leads to

$$dP = \left(-\frac{\partial P}{\partial \tau} + \mu_r \frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2}\right) dt + \sigma_r \rho_t \frac{\partial P}{\partial r} dW_t^1 + \sigma_r \sqrt{1-\rho_t^2}\frac{\partial P}{\partial r} dW_t^2. \tag{14}$$

By comparing the coefficients in (14) and (2) we get

$$-\frac{\partial P}{\partial \tau} + \mu_r \frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} = P\mu_P \text{ and } \sigma_r \frac{\partial P}{\partial r} = P\sigma_P \tag{15}$$

which gives

$$\sigma_P = -\tau\sigma_r \tag{16}$$

and

$$P(\tau; r) = \exp(-r\tau - \frac{\mu_r}{2}\tau^2 + \frac{\sigma_r^2}{6}\tau^3). \tag{17}$$

For ρ_t we employ the local correlation function proposed in [**12**], see also [**13**],

$$\rho_t := E\left[\tanh(X_t)\right] \tag{18}$$

for the dynamic correlation function, where X_t is any mean-reverting process with positive and negative values. For a fixed parameter of X_t, the correlation function ρ_t depends only on t.

[1] The limitation: the probability of negative interest rates is positive.

Furthermore, it is obvious that ρ_t takes values only in $(-1,1)$ for all t and converges for $t\to\infty$. By choosing X_t in (18) to be the *Ornstein-Uhlenbeck process* [**14**]

$$dX_t=\kappa(\mu-X_t)dt+\sigma dW_t,\quad t\geq 0, \tag{19}$$

the closed-form expression for ρ_t has been derived as

$$\rho_t=1-\frac{\exp(-A-\frac{B}{2})}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\frac{\pi u}{2})}\cdot\exp(iu(A+B)+u^2\frac{B}{2})du \tag{20}$$

with

$$A = \exp(-\kappa t)\tanh^{-1}(\rho_0)+\mu(1-\exp(-\kappa t)) \tag{21}$$

$$B = -\frac{\sigma^2}{2\kappa}(1-\exp(-2\kappa t)), \tag{22}$$

where $\kappa\geq 0,\ \sigma\geq 0,\ \mu\in\mathbb{R}$ and $\rho_0\in(-1,1)$.

Substituting (16), (17) and (20) into (11), we obtain the European Call-option price with dynamically correlated stochastic interest rate which is given by

$$V(S,P,\rho_\tau,\tau;K)=S\Phi(d_1)-KP\Phi(d_2) \tag{23}$$

$$d_1:=\frac{\ln\frac{S}{K}-\ln P+\frac{1}{2}\int_0^\tau\sigma_s^2ds}{\sqrt{\int_0^\tau\sigma_s^2ds}},\quad d_2:=d_1-\sqrt{\int_0^\tau\sigma_s^2ds}$$

and

$$\sigma_t^2=\tau^2\sigma_r^2+\sigma_S^2+2\rho_t\tau\sigma_r\sigma_S, \tag{24}$$

where P and ρ_t are defined in (17) and (20), respectively. The price of European Put-options are directly available using Put-Call parity.

2.2. Option pricing with Vasicek Interest rate-The Rabinovitch model

Rabinovitch [**9**] investigated the pricing of European option with Vasicek stochastic interest rates and derived the formula in a closed form. The comparison of pricing formulas of European Call-option with different stochastic interest rate processes can be found in [**7**]. In this section, we consider the pricing of European Call-options with Vasicek stochastic interest rate and incorporate dynamic correlation.

Again, we need the following SDEs for the stock price and bond price dynamics

$$\frac{dS_t}{S_t}=\mu^S dt+\sigma_S dW_t^1 \tag{25}$$

$$dr_t=\kappa_r(\mu^r-r_t)dt+\sigma_r\rho_t dW_t^1+\sigma_r\sqrt{1-\rho_t^2}dW_t^2 \tag{26}$$

where W_t^1 and W_t^2 are independent. The pricing formula of European Call-option according to (25) and (26) but with a constant correlation has been already given in [**9**], see also [**7**].

Furthermore, if we compare the pricing formula of the Merton model between using constant and dynamic correlation in Section 2.1, we see that incorporating dynamic correlation does not change the original pricing formula (with constant correlation) to a large extent, the new pricing formula with dynamic correlation has just the formel which can be obtained directly by fitting in the dynamic correlation function instead of constant correlation with the original formula.

We can observe that incorporating a dynamic correlation function into the pricing formula with the Vasicek stochastic interest rate provided in [**9**] and [**7**] also in this case. In order to adopt the approach in [**7**] to directly get the pricing formula with dynamic correlation, we need to rewrite (25) and (26) with respect to the Brownian Motions under a risk-neutral probability measure $\mathbb{Q}$ as

$$\frac{dS_t}{S_t} = \underbrace{(\mu^S - \sigma_S \lambda_S)}_{:=\mu_S} dt + \sigma_S d\tilde{W}_t^1 \tag{27}$$

$$dr_t = \kappa_r [\underbrace{(\mu^r - \frac{\sigma_r \sqrt{1-\rho_t^2}\lambda_t^r}{\kappa_r})}_{:=\mu_r} - r_t]dt + \sigma_r \rho_t d\tilde{W}_t^1 + \sigma_r \sqrt{1-\rho_t^2} d\tilde{W}_t^2, \tag{28}$$

where λ_S and λ_t^r are the market price of risk. Whilst we assume that the market price of risk λ_t^r to be a constant, this is to say we set $\mu^r - \frac{\sigma_r \sqrt{1-\rho_t^2}\lambda_t^r}{\kappa_r} = \mu_r$. The pricing formula with constant correlation in [**7**] can be thus straightforwardly adopted to find the pricing formula using dynamic correlation. Therefore, we omit the exact derivation and give the pricing formula using dynamic correlation as follows:

$$V(S, P, \rho_\tau, \tau; K) = S\Phi(d_1) - KP\Phi(d_2) \tag{29}$$

with

$$d_1 := \frac{\Sigma_{11}^\tau + \Sigma_{12}^\tau - C_\tau}{\sqrt{D_\tau}}, \quad d_2 := d_1 - \sqrt{D_\tau}$$

where

$$C_\tau := \frac{\Sigma_{11}^\tau}{2} - B_\tau + \ln\frac{K}{S}, \quad D_\tau := \Sigma_{11}^\tau + 2\Sigma_{12}^\tau + \Sigma_{22}^\tau, \quad \Sigma_{11}^\tau := \sigma_S^2 \tau,$$

$$\Sigma_{22}^\tau := \frac{\sigma_r^2}{\kappa_r^2}\left[\tau - \frac{3 + \mathrm{e}^{-\kappa_r \tau}(\mathrm{e}^{-\kappa_r \tau} - 4)}{2\kappa_r}\right], \quad \Sigma_{12}^\tau := \frac{\sigma_r \sigma_S}{\kappa_r}\int^\tau \rho_s (1 - \mathrm{e}^{(s-\tau)\kappa_r}) ds$$

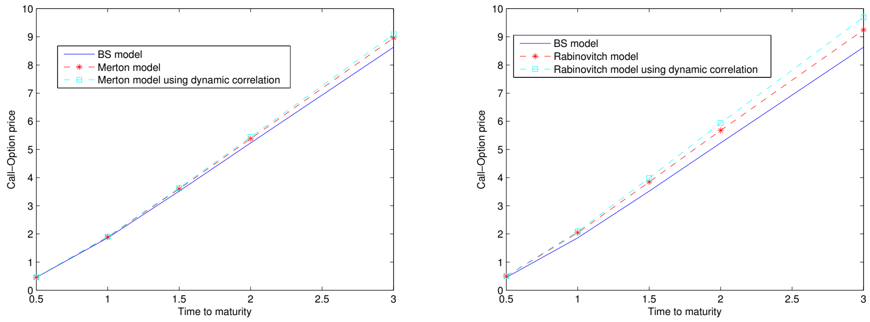
and

$$B_\tau := \frac{1}{\kappa_r}\left[\kappa_r \mu_r \tau - (r - \mu_r)(\mathrm{e}^{-\kappa_r \tau} - 1)\right], \quad P_\tau := \mathrm{e}^{\frac{1}{2}\Sigma_{22}^\tau - B_\tau}, \tag{30}$$

ρ_t has been defined in (20).

2.3. Numerical Results

In this section, we compare the option prices between using constant and dynamic correlation in the both models above by analyzing numerical results in this section. We assume that $S = 80$, $K = 100$, $\sigma_S = 0.2$, constant correlation: $\rho_c = 0.2$, parameters of dynamic correlation function: $\rho_0 = 0.2$, $\kappa_\rho = 2$, $\mu_\rho = 0.5$, $\sigma_\rho = 0.2$, constant interest rate for the Black-Scholes model: $r_c = 0.05$, stochastic rate for the Merton model: $r_0 = 0.05$, $\mu_r = 0.001$, $\sigma_r = 0.1$ and for the Rabinovitch model: $r_0 = 0.05$, $\kappa_r = 2$, $\mu_r = 0.001$, $\sigma_r = 0.1$. We compute the prices of the European Call-option using the Black-Scholes model, using the Merton model and the Rabinovitch model with constant and dynamic correlation for the different maturities $T = [0.5, 1, 1.5, 2, 2.5, 3]$, and display them in Figure 1. We can easily see the difference between the Black-Scholes model and the model using stochastic interest rate. However, as mentioned in the introduction, some empirical findings showed us that stochastic interest rates (with constant correlation) may not be important for the pricing. From Figure 1 we can also observe, the prices in the both models have been changed because of incorporating nonconstant correlation. Thus, one could ask whether stochastic interest rates with nonconstant correlation can contribute to the performance improvement of the Black-Scholes model. For this question, we run a calibration test in the next section.



(a) The Merton model (b) The Rabinovitch model

Figure 1. Comparison of pricing European Call-option using different models.

3. Calibration to the market data

Both works [**12**] and [**13**] showed that using dynamic correlation can improve the model calibration. In the following, we examine both the Merton model and the Rabinovitch model whether incorporating stochastic interest rate do contribute to the performance improvement of pring due to allowing dynamic correlation.

We have seen that the bond price formula is on hand in both models, see (17) for the Merton model and (30) for the Rabinovitch model. Thus, one can directly estimate the parameters of the short rate model using the market yield curve Y_τ with the aid of the relation

$$Y_\tau = -\frac{1}{\tau}\ln P_\tau. \tag{31}$$

We consider the overnight rate on July 30, 2013, $r_0 = 0.26\%$, and use the treasury yield curve[2] of this day to obtain the estimates: $\mu_r = 0.005$, $\sigma_r = 0.017$ (Merton short rate) and $\kappa_r = 0.111$, $\mu_r = 0.052$, $\sigma_r = 0.001$ (Vasicek short rate).

The parameters, which we do still need to estimate, are σ_S, ρ_c (for the case of using constant correlation) or correlation function parameters (for using dynamic correlation). For this purpose, we pick the market option prices on the $S\&P$ 500 on July 30, 2013 with the spot price $S = 169.1$, for the maturities $T = [30, 90, 180, 360]$ days and the strikes $K/S = [0.9, 1, 1.1]$. Then, we fit the model prices $V_{Mod}(T_i, K_j)$ to the market prices $V_{Mkt}(T_i, K_j)$ by minimizing the relative mean error sum of squares (RMSE) given by

$$\frac{1}{N}\sum_{i,j}\omega_{i,j}\frac{(V_{Mkt}(T_i,K_j) - V_{Mod}(T_i,K_j))^2}{V_{Mkt}(T_i,K_j)} \tag{32}$$

where $\omega_{i,j}$ is an optional weight and N is number of prices. While minimizing we need to add some constraints on the parameters: the implied volatility σ_S must be positive, the constant correlation ρ_c must belong to the intervall $(-1, 1)$. We know that the correlation function (20) stems from the expectation of the transformed Ornstein-Uhlenbeck process by tanh. As mentioned before the parameters of the correlation function must statisfy the following conditions

$$\kappa > 0, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad \rho_0 \in (-1, 1). \tag{33}$$

So we set the upper limit for κ to be 20 and the interval for μ to be $[-6, 6]$.

[2] available on http://www.treasury.gov

For this optimization problem we used the standard method of nonlinear optimization and report our results in Table 1 for using constant correlation and in Tabel 2 for using dynamic correlation. First, we look at Table 1 and find that the constant correlation ρ_c in both models

Model	σ_S	ρ_c	RMSE
The Merton model	0.12	0.99	0.115
The Rabinovitch model	0.12	0.99	0.141

Table 1. Parameter estimation for using constant correlation.

Model	σ_S	ρ_0	κ	μ	σ	RMSE
The Merton model	0.12	-0.99	18.14	6	5×10^{-4}	0.114
The Rabinovitch model	0.12	-0.99	20	6	2×10^{-3}	0.141

Table 2. Parameter estimation for using dynamic correlation.

tends to attain the boundary 1. However, we all know that the correlation between interest rate and stock process should not tend towards to the boundary 1. This is to say that both models with constant correlation can not be calibrated well. One could thus think the cause could be the assumption of constant correlation. However, from Table (2) we see although the dynamic correlation has changed from the initial value -0.99 to the boundary 1 with the time, there is no improvement of RMSE compared to the RMSE in Tabel 1, and the both RMSEs are quite large by the way.

Furthermore, we can observe that κ is attaining its upper limit 20 and the value of σ is quite small, this means that the dynamic correlation will rapidly attend to the boundary 1. To confirm this, we compare the model prices using constant and dynamic correlation to the market prices in Figure 2 for the Merton model and in Figure 3 for the Rabinovitch model. As expected, in both models there is almost no difference between prices using constant and dynamic correlation, especially, for the longer maturity.

Thus, we can also conclude that allowing dynamic correlation in this example does not improve the calibration as in [**12**] and [**13**]. Incorporating a stochastic volatility could probably solve this calibration problem. This means also that our experiment results do not only coincide with the statement that only incorporating stochastic interest rate does not improve pricing performance; furthermore, they show that the calibration is not getting better for allowing dynamic correlation between stochastic interest rate and stock process.

4. Conclusion

In this work, we reviewed two European option pricing models with stochastic interest rate: the Merton model (interest rate given by Gauss-Wiener process) and the Rabinovitch model (interest rate given by Vasicek process). We extend both models by incorporating local time dependent correlation between the underlying process and the stochastic interest rate. We presented the numerical results to show the difference between using a constant and a dynamic correlation. Furthermore, we conducted an experiment on fitting the model to the market price. As a result, the option pricing within the Black-Scholes framework can not really be improved by incorporating stochastic interest rate even when using nonlinear correlation.

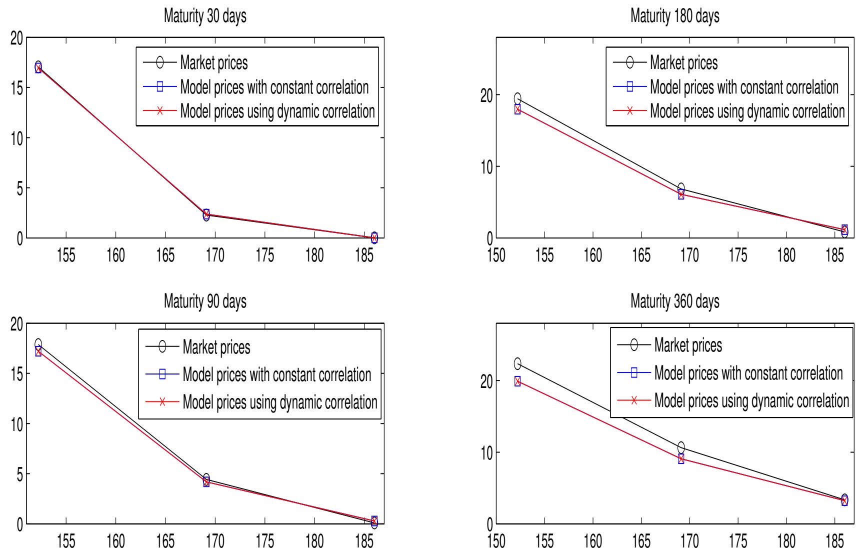


Figure 2. Comparison of the market and model prices in the Merton model.

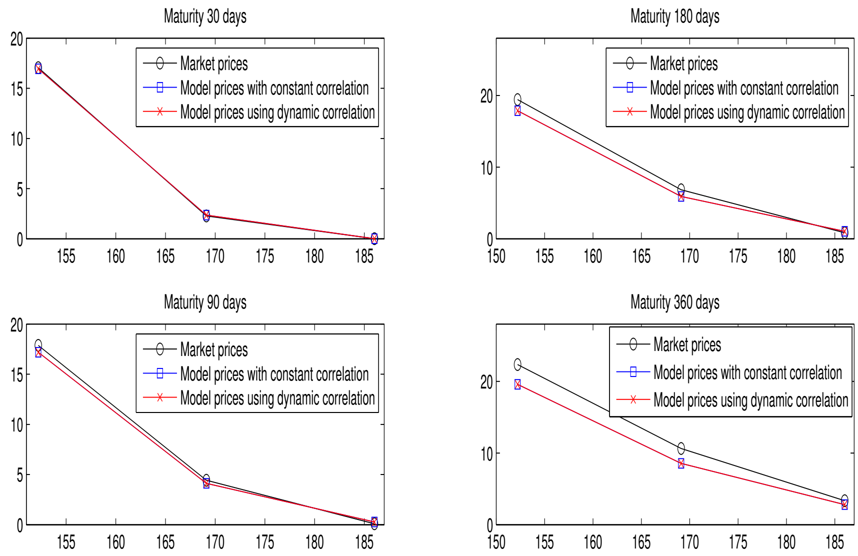


Figure 3. Comparison of the market and model prices in the Rabinovitch model.

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