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Christian Hendricks, Matthias Ehrhardt and Michael Günther

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Christian Hendricks, Matthias Ehrhardt, Michael Günther

Bergische Universität Wuppertal, Chair of Applied Mathematics and Numerical Analysis (AMNA), Gaußstrasse 20, 42119 Wuppertal, Germany

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Abstract

In this paper we introduce high dimensional tensor product interpolation for the combination technique. In order to compute the sparse grid solution, the discrete numerical sub solutions have to be extended by interpolation. If unsuitable interpolation techniques are used, the rate of convergence is deteriorated. We derive the necessary framework to preserve the error structure of high order finite difference solutions of elliptic partial differential equations within the combination technique. This enables us to obtain high order sparse grid solutions on the full domain. Exemplary for the case of order four we illustrate our theoretical results by two test examples with up to four dimensions.

Keywords: high order, sparse grids, interpolation, finite difference schemes

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Email addresses: `hendricks@math.uni-wuppertal.de` (Christian Hendricks), `ehrhardt@math.uni-wuppertal.de` (Matthias Ehrhardt), `guenther@math.uni-wuppertal.de` (Michael Günther)

1 Introduction

In many of todays applications high dimensional problems arise. Especially in the field of computational finance partial differential equations (PDEs) with several dimensions have to be solved to evaluate the price of financial products. Since the number of grid points in a tensor based grid growth exponentially with the dimension, the so called *curse of dimensionality* shows its effects very quickly. *Sparse grids* (Zenger (1991)) and the *Combination Technique* (Griebel et al. (1992)) have proven their great ability to reduce the computational effort. Using a sparse grid representation of a function in d dimensions, $\mathcal{O}(h_n^{-1} \log(h_n^{-1})^{d-1})$ grid points are employed. Bungartz, Griebel & Rüde (1994) and Garcke (2008) have shown shown that the approximation accuracy is of order $\mathcal{O}(h_n^2 \log(h_n^{-1})^{d-1})$ under certain smoothness requirements. Compared to a tensor based full grid with $\mathcal{O}(h_n^{-d})$ grid points and an accuracy of $\mathcal{O}(h_n^2)$ the total number of nodes is significantly decreased. Thus the sparse grid approach only suffers from the curse of dimensionality in a much lower extent. Sparse grids have successfully been used by Griebel & Hamaekers (2007), Griebel & Thurner (1995), Gaikwad & Toke (2009) to solve PDEs with several dimensions. In order to construct a solution on the sparse grid, the combination technique can be used. It is based on linearly combining a sequence of solutions via interpolation. Since each solution can be computed independently of the others, the method is embarrassingly parallel. Hence it can be efficiently implemented on a cluster to accelerate the computation time (Gaikwad & Toke (2009)).

In the literature second order finite difference schemes are employed to solve each of the resulting subproblems and the solutions are combined to the sparse grid solution via multilinear interpolation. As far as we know, there exists only one article by Leentvaar & Oosterlee (2006), where fourth order stencils are used. But the question, which interpolation technique is suitable, remains open. From an intuitive point of view it is clear that linear interpolation cannot preserve the order of the highly accurate sub-solutions. In this paper we want to present interpolation techniques which do not interfere the error splitting within the combination technique. Since high dimensional problems shall be solved, we use a tensor product approach to extend the univariate interpolation to the multivariate case.

The outline is as follows: in Section 2 and 3 we give a short overview of the combination technique and motivate the need for high-order interpolation techniques. In Section 4 we take a closer look at the two dimensional test case. Here we can omit a complex notation and give the reader an idea of how the approach works. Later in Section 5 the framework is extended to the n dimensional case. Finally, numerical results are presented in Section 6.

2 Combination Technique in a Nutshell

Here we want to give a short introduction to the combination technique. It is based on linearly combining a sequence of discrete solutions to a more accurate solution. In order to achieve a higher accuracy, the error structure of the discrete solutions is exploited in such a way that low order errors cancel out. This can most easily be demonstrated in a two dimensional example. Let us consider the Poisson equation on the unit square $\Omega = (0, 1)^2$

$$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y) \text{ on } \Omega \tag{2.1}$$
$$u(x,y) = g(x,y) \text{ on } \partial\Omega,$$

with its discrete solution u_h on the grid Ω_h with mesh widths $h = (h_x, h_y)$ respectively. The discrete solution is computed via a standard second order finite difference scheme. Bungartz Bungartz, Griebel, Röschke & Zenger (1994) was the first one, who proved with help of Fourier series of discrete and semi-discrete solutions that the error of the discrete solution consists of second order errors from each of the directions and one mixed error

$$u_h(x,y) = u(x,y) + w_1(x,y;h_x)h_x^2 + w_2(x,y;h_y)h_y^2 + w_{1,2}(x,y;h_x,h_y)h_x^2h_y^2. \tag{2.2}$$

We see that the errors w_1 and w_2 either depend on h_x or h_y. If we compute the difference between two discrete solutions, which use the same step size in one of the two spatial dimensions, one error term cancels out. To simplify our notation, we write $u_h := u(i,j)$ where $h_x = 2^{-i}$ and $h_y = 2^{-j}$ for $i, j \in \mathbb{N}$. We obtain

$$\begin{aligned} u(2^{-i}, 2^{-j}) - u(2^{-i}, 2^{-(j-1)}) \\ &= w_2(x,y;2^{-j})2^{-j} - w_2(x,y;2^{-(j-1)})2^{-(j-1)} + w_{1,2}(x,y;2^{-i},2^{-j})2^{-i}2^{-j} \\ &\quad - w_{1,2}(x,y;2^{-i},2^{-(j-1)})2^{-i}2^{-(j-1)}. \end{aligned}$$

We see that the error w_1 has vanished. We can further exploit this idea by combining the sub-solutions according to the following formula, so that all lower order terms cancel out

$$u_n^s := \sum_{i+j=n+1} u(2^{-i}, 2^{-j}) - \sum_{i+j=n} u(2^{-i}, 2^{-j}). \tag{2.3}$$

This combined solution u_n^s is called *sparse grid* solution. Figure 1 (A) shows the sparse grid and (B) the solution of the Poisson problem (2.1) with $u(x,y) = \exp(x\,y)$. According to Bungartz, Griebel, Röschke & Zenger (1994) the pointwise error is given by

$$|u_n^s - u| \leq K h_n^2 (1 + \tfrac{5}{4}\log_2(h_n^{-1})) = \mathcal{O}(h_n^2 \log_2(h_n^{-1})),$$

where $h_n = 2^{-n}$ and w_1, w_2, $w_{1,2}$ are bounded by K. This can easily be verified by straightforward inserting (2.2) into the combination formula (2.3). Please note that an interpolation is needed to combine the sub solutions.

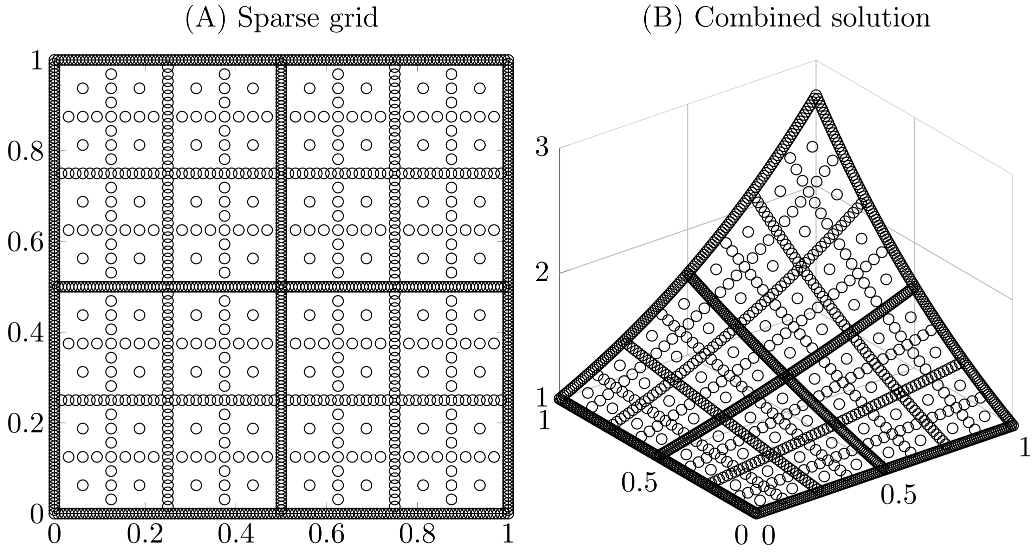

Figure 1: Sparse grid and combined solution with $u(x, y) = \exp(x\,y)$

3 Motivation

We now want to use a fourth order finite difference scheme to compute the discrete solution. If we assume that the error can be split as

$$u(x, y) - u_h(x, y) = c_1(x, y; h_x)h_x^4 + c_2(x, y; h_y)h_y^4 + c_{1,2}(x, y; h_x, h_y)h_x^4 h_y^4, \tag{3.1}$$

then we can combine the sub-solutions according to (2.3) and estimate the error of the sparse grid solution

$$e_n := |u_n^s - u| \le K h_n^4 (\tfrac{5}{4} + \tfrac{17}{16} \log_2(h_n^{-1})) = \mathcal{O}(h_n^4 \log_2(h_n^{-1})). \tag{3.2}$$

This error bound holds for all points, which are not subject to the interpolation scheme. These are the points, which belong to all underlying sub-grids. Since there is only one interior grid point (0.5,0.5), which fulfills this condition, the use of the convergence result (3.2) seems to be very limited. In the following we want to check numerically if the convergence result is also valid for other grid points. As a test example we solve the two dimensional Poisson equation (2.1)

$$\begin{aligned} &u_{xx}(x, y) + u_{yy}(x, y) = (x^2 + y^2) \exp(x\,y) \text{ on } \Omega = (0, 1)^2 \\ &u(x, y) = \exp(x\,y) \text{ on } \partial\Omega, \end{aligned}$$

with the combination technique and a standard fourth order difference scheme. The sub-solutions are combined via multi-linear interpolation. This example was also considered by Leentvaar & Oosterlee (2006). Thinking of the discrete solution u_n^s as a vector, we compute the error in the maximum-norm, which is appropriate for elliptic problems. The error is given by

$$e_n^\infty := \|u_n^s - R_s^n u\|_\infty,$$

where R_s^n is a *restriction operator* restricting the analytical solution to the sparse grid.

In Figure 2 we compare the convergence of the errors e_n at $(x, y) = (0.5, 0.5)$ and e_n^∞. The decline of the pointwise error is close to order four and thus in line with the theoretical result (3.2). The error in the maximum-norm exhibits a much lower numerical rate of convergence. This observation underlines that multi-linear interpolation cannot preserve the error structure (3.1) of the finite difference solution. Since in most practical applications a high order on the complete domain is requested, the question arises, which interpolation technique to use. In the remainder of this paper we want to determine suitable interpolation techniques.

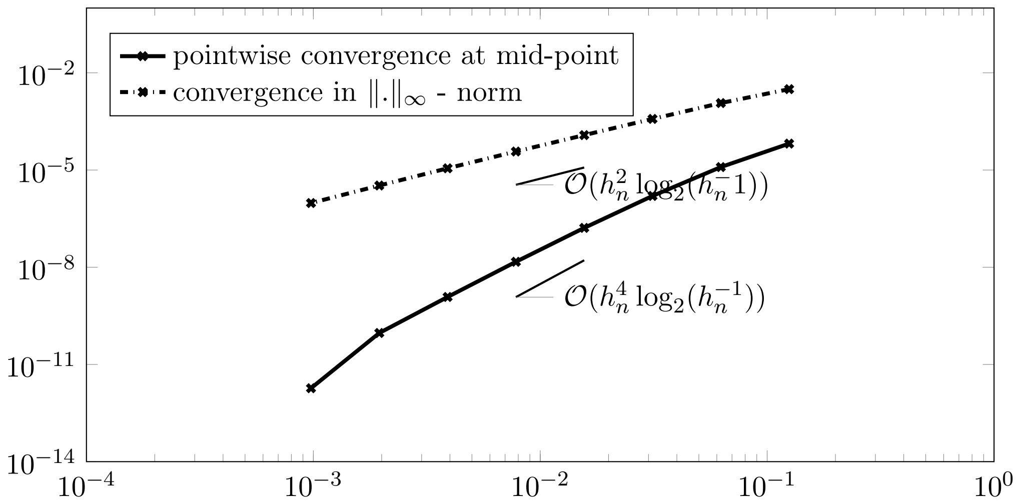

Figure 2: Convergence at mid-point and in $\|.\|_\infty$-norm

4 The 2-d case

In the last section it was demonstrated that the error splitting is interfered if unsuitable interpolation techniques are used. But what is a suitable interpolation technique? A suitable technique P preserves the error structure of the discrete solution u_h on the whole domain $(x, y) \in \Omega$ in such a way that it holds

$$u(x,y) - (Pu_h)(x,y) = \gamma_1(x,y;h_x)h_x^p + \gamma_2(x,y;h_y)h_y^p + \gamma_{1,2}(x,y;h_x,h_y)h_x^p h_y^p, \qquad (4.1)$$

where $p \in \mathbb{N}$ and γ_1, γ_2, $\gamma_{1,2}$ are bounded by some constant K. If (4.1) holds, the pointwise error in the combined solution holds for all points in Ω. Thus we have convergence of order $\mathcal{O}(h_n^4 \log_2(h_n^{-1}))$ on the whole domain and not only at the mid-point if $p = 4$. In the remainder of this section we want to derive such an interpolation technique. Reisinger (2012) proved in his seminal paper that a splitting of the form (4.1) with $p = 2$ holds, if multi-linear interpolations is applied. His proof is twofold: Under the assumption of a linear interpolation operator P the error can be split into an interpolation error of the analytical solution and the interpolation of the pointwise error at the grid Ω_h. The evaluation of the analytical solution at the discrete grid is denoted by u_{Ω_h}, i.e. we have

$$\begin{aligned} u(x,y) - &(Pu_h)(x,y) \qquad (4.2)\\ &= u(x,y) - (Pu_{\Omega_h})(x,y) + (Pu_{\Omega_h})(x,y) - (Pu_h)(x,y) \\ &= u(x,y) - (Pu_{\Omega_h})(x,y) + (P(u_{\Omega_h} - u_h))(x,y). \end{aligned}$$

Hence two errors can be analyzed separately and the final result for $u(x, y) - (Pu_h)(x, y)$ follows immediately. We want to follow this approach and start with the derivation of a tensor product based interpolant and investigate the structure of the interpolation error.

We construct the multidimensional interpolation P as the tensor product of univariate interpolation P_x, P_y respectively. P_x interpolates along the x direction, while P_y interpolates in coordinate direction of y. The univariate interpolant of a function g can be constructed in the form

$$P_x g = \sum_{j=1}^{n} \alpha_j(g) f_j$$

with a given function basis $\{f_j\}_{j=1,\dots,n}$ with respect to linear functionals $\{\lambda_j\}_{j=1,\dots,n}$, which fulfill

$$\lambda_j P_x g = \lambda_j g, \text{ for } j = 1, ..., n.$$

This general framework allows for a wide class of interpolation techniques such as linear or spline interpolation. Due to Lemma XVII.1 in de Boor (1978) the interpolant is unique with $\alpha(g) = A^{-1}(\lambda_j g)$, where A is the Gramian matrix $A := (\lambda_i f_j)_{i,j=1,\dots,n}$. The interpolation in y direction can be defined in the same way with the given basis $\{h_j\}_{j=1,\dots,m}$ and linear functionals $\{\mu_j\}_{j=1,\dots,m}$, so that the interpolant is given by

$$P_y g = \sum_{j=1}^{m} \beta_j(g) h_j,$$

where $\beta(g) = B^{-1}(\mu_j g)$ and $B := (\mu_i h_j)_{i,j=1,\dots,m}$. The tensor interpolation operator $P = P_x \otimes P_y$ has the form

$$Pg = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i(g) \beta_j(g) f_i h_j.$$

Thus the error of P_x and P_y can be expressed in terms of their remainders R_x, R_y,

$$(P_x g_{\Omega_h})(x,y) = g(x,y) + h_x^p R_x g(x,y),$$
$$(P_y g_{\Omega_h})(x,y) = g(x,y) + h_y^p R_y g(x,y),$$

where p is the order of the interpolant. The error of P can be given by separate univariate remainder terms

$$\begin{aligned} P g_{\Omega_h} = P_x \otimes P_y g_{\Omega_h} &= (I + h_x^p R_x) \otimes (I + h_y^p R_y) g \\ &= g + h_x^p R_x g + h_y^p R_y g + h_x^p h_y^p R_x \otimes R_y g. \end{aligned} \tag{4.3}$$

If the order $p = 2$ is desired, then linear interpolation is sufficient, while in the case of $p = 4$, a cubic spline interpolation is appropriate. If cubic spline interpolation is used and the function g has four continuous derivatives, then

$$\begin{aligned} \|R_x g\|_\infty &\leq \tfrac{5}{384} \|\tfrac{\partial^4 g}{\partial x^4}\|, \\ \|R_y g\|_\infty &\leq \tfrac{5}{384} \|\tfrac{\partial^4 g}{\partial y^4}\| \end{aligned} \tag{4.4}$$

holds, as proved by Hall (1968).

In the following we consider a function $u \in C_K^{(k,k)}(\Omega)$. Here $C_K^{(k,k)}(\Omega)$ denotes the function space, where all partial derivatives $\frac{\partial^{i+j}}{\partial x^i \partial y^j}$ with $i, j = 1, ..., k$ are continuous and bounded by K for an integer k. In the sequel we assume k to be large enough so that u is sufficiently smooth. If an interpolation scheme of order p is used, we get due to (4.3)

$$(P u_{\Omega_h})(x,y) = u(x,y) + h_x^p c_1(x,y;h_x) + h_y^p c_2(x,y;h_y) + h_x^p h_y^p c_{1,2}(x,y;h_x,h_y), \tag{4.5}$$

for $(x,y) \in \Omega$.

To analyze the second source of error in (4.2), it has to be shown that the finite difference solution has an error of the following form

$$u(x,y) - u_h(x,y) = h_x^p w_1(x,y;h_x) + h_y^p w_2(x,y;h_y) + h_x^p h_y^p w_{1,2}(x,y;h_x,h_y),$$

for $(x,y) \in \Omega_h$. While Bungartz, Griebel, Röschke & Zenger (1994) showed that such a splitting exists for the Laplace euqation in 2-d with second order finite difference schemes, Reisinger (2012) developed a framework for elliptic equations and linear finite difference schemes with arbitrary order. His work starts with a proof of consistency of the finite difference scheme. The order of consistency p of a PDE can be investigated with a straightforward Taylor expansion

$$L_h u(x,y) - f_h(x,y) = h_x^p \tau_1(x,y;h_x) + h_y^p \tau_2(x,y;h_y) + h_x^p h_y^p \tau_{1,2}(x,y;h_x,h_y),$$

for sufficiently smooth u and $(x,y) \in \Omega_h$. Here L_h denotes the discretization operator. Under suitable regularity assumptions, we can conclude from Reisinger (2012) that it holds

$$u_{\Omega_h}(x,y) - u_h(x,y) = h_x^p w_1(x,y;h_x) + h_y^p w_2(x,y;h_y) + h_x^p h_y^p w_{1,2}(x,y;h_x,h_y). \tag{4.6}$$

Due to (4.6) the interpolation of the pointwise error yields

$$\begin{aligned} &\big(P(u_{\Omega_h} - u_h)\big)(x,y) \\ &\quad = h_x^p (P w_1(.;h_x))(x,y) + h_y^p (P w_2(.;h_y))(x,y) + h_x^p h_y^p (P w_{1,2}(.;h_x,h_y))(x,y). \end{aligned}$$

Applying equation (4.5), we obtain

$$
\begin{aligned}
&\big(P(u_{\Omega_h} - u_h)\big)(x,y) \\
&\qquad = h_x^p \left(w_1(x,y;h_x) + h_x^p \hat{c}_1(x,y;h_x) + h_y^p \hat{c}_2(x,y;h_y) + h_x^p h_y^p \hat{c}_{1,2}(x,y;h_x,h_y)\right) \\
&\qquad\quad + h_y^p \left(w_2(x,y;h_x) + h_x^p \tilde{c}_1(x,y;h_x) + h_y^p \tilde{c}_2(x,y;h_y) + h_x^p h_y^p \tilde{c}_{1,2}(x,y;h_x,h_y)\right) \\
&\qquad\quad + h_x^p h_y^p \left(w_{1,2}(x,y;h_x) + h_x^p \bar{c}_1(x,y;h_x) + h_y^p \bar{c}_2(x,y;h_y) + h_x^p h_y^p \bar{c}_{1,2}(x,y;h_x,h_y)\right).
\end{aligned}
$$

If we sort the terms according to their order and condense, the error can be rewritten as

$$
\begin{aligned}
&\big(P(u_{\Omega_h} - u_h)\big)(x,y) \\
&\qquad = h_x^p \beta_1(x,y;h_x) + h_y^p \beta_2(x,y;h_y) + h_x^p h_y^p \beta_{1,2}(x,y;h_x,h_y),
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
\beta_1(x,y;h_x) &= w_1(x,y;h_x) + h_x^p \hat{c}_1(x,y;h_x) \\
\beta_2(x,y;h_y) &= w_2(x,y;h_y) + h_y^p \tilde{c}_2(x,y;h_y) \\
\beta_{1,2}(x,y;h_x,h_y) &= w_{1,2}(x,y;h_x,h_y) + \hat{c}_2(x,y;h_2) + \tilde{c}_1(x,y;h_x) + h_x^p \hat{c}_{1,2}(x,y;h_x,h_y) \\
&\quad + h_y^p \tilde{c}_{1,2}(x,y;h_x,h_y) + h_x^p \bar{c}_1(x,y;h_x) + h_y^p \bar{c}_2(x,y;h_y) \\
&\quad + h_x^p h_y^p \bar{c}_{1,2}(x,y;h_x,h_y),
\end{aligned}
\tag{4.8}
$$

From equations (4.5) and (4.7) we can finally conclude for all $(x,y) \in \Omega$ that

$$
\begin{aligned}
&u(x,y) - (Pu_h)(x,y) \\
&\qquad = h_x^p \left(\beta_1(x,y;h_x) - c_1(x,y;h_x)\right) + h_y^p \left(\beta_2(x,y;h_y) - c_2(x,y;h_y)\right) \\
&\qquad\quad + h_x^p h_y^p \left(\beta_{1,2}(x,y;h_x,h_y) - c_{1,2}(x,y;h_x,h_y)\right) \\
&\qquad = h_x^p \gamma_1(x,y;h_x) + h_y^p \gamma_2(x,y;h_y) + h_x^p h_y^p \gamma_{1,2}(x,y;h_x,h_y),
\end{aligned}
$$

with $\gamma_1 := \beta_1 - c_1$, $\gamma_2 := \beta_2 - c_2$, $\gamma_{1,2} := \beta_{1,2} - c_{1,2}$.

We now want to apply this framework to the Poisson equation (2.1) to derive a splitting of the form (4.1) with $p = 4$. As already mentioned, the cubic spline interpolation fulfills the desired order. The solution u has to be sufficiently smooth with bounded mixed derivatives. Since u needs to fulfill the consistency requirements of the finite difference solution and as the functions w_1, w_2, $w_{1,2}$ in (4.6) are interpolated via spline interpolation, we assume $u \in C_K^{(10,10)}(\Omega)$ and that u and its derivatives vanish at the boundaries of Ω. In order to show that a (semi-) discrete maximum principle holds for the finite difference solution, we cite Theorem 2 by Ciarlet (1970):

Theorem 2 by Ciarlet (1970) *The discrete finite difference operator satisfies the discrete maximum principle if the following two matrix conditions are satisfied:*

- *(I) The finite difference matrix A is monotone,*
- *(II) the row sums of the matrix A are all nonnegative.*

Since the row sum is zero in (4.11) and also in the case of a second order discretization at the boundary it is clear that condition (II) is fulfilled. The monotonicity of $-A$ directly follows from Lemma 3.1. in the paper by Bramble & Hubbard (1964), where the fourth order discretization to the Poisson equation $-\Delta u = f$ is considered. Rewriting our discretization to $-Au = -f$, we can conclude that a (semi-) discrete maximum principle holds if $-f \leq 0$ or a (semi-) discrete minimum principle if $-f \geq 0$ is fulfilled.

This enables us to apply Lemma 3.1. (Reisinger (2012)) for our finite difference stencil (4.11) and to get bounds for the Poisson problem with Dirichlet data and also for its (semi-) discrete solution

$$\begin{aligned} \|u\|_\infty &\le \tfrac{1}{8}\|f\|_\infty \\ \|u_h^{(k)}\|_\infty &\le \tfrac{1}{8}\|f\|_\infty, \end{aligned} \tag{4.9}$$

where $u_h^{(k)}$ is the semi-discrete solution in direction k for $k = 1, 2$. We obtain for $(x, y) \in \Omega$

$$(Pu_{\Omega_h})(x,y) = u(x,y) + h_x^4 c_1(x,y;h_x) + h_y^4 c_2(x,y;h_y) + h_x^4 h_y^4 c_{1,2}(x,y;h_x,h_y), \tag{4.10}$$

where $\|c_1\|_\infty \le \frac{5}{384}K$, $\|c_2\|_\infty \le \frac{5}{384}K$, $\|c_{1,2}\|_\infty \le \frac{5^2}{384^2}K$.

The standard fourth order finite difference scheme reads

$$\begin{aligned} &L_h u(x_i, y_j) \\ &\quad = \frac{-u(x_i+2h_x,y_j)+16u(x_i+h_x,y_j)-30u(x_i,y_j)+16u(x_i-h_x,y_j)-u(x_i-2h_x,y_j)}{12h_x^2} \\ &\quad + \frac{-u(x_i,y_j+2h_y)+16u(x_i,y_j+h_y)-30u(x_i,y_j)+16u(x_i,y_j-h_y)-u(x_i,y_j-2h_y)}{12h_y^2}. \end{aligned} \tag{4.11}$$

Via Taylor expansion we obtain for $(x, y) \in \Omega_h$

$$L_h u(x,y) - f_h(x,y) = h_x^4 \tau_1(x,y;h_x) + h_y^4 \tau_2(x,y;h_y).$$

The errors are bounded by $\|\tau_1\|_\infty \le \frac{1}{90}K$ and $\|\tau_2\|_\infty \le \frac{1}{90}K$. We can conclude from Reisinger (2012) that it holds

$$u_{\Omega_h}(x,y) - u_h(x,y) = h_x^4 w_1(x,y;h_x) + h_y^4 w_2(x,y;h_y) + h_x^4 h_y^4 w_{1,2}(x,y;h_x,h_y)$$

for $(x, y) \in \Omega_h$. The functions w_1, w_2 are defined as the solution of the auxiliary problems

$$\begin{aligned} L_h^{(x)} w_1(.;h_x) &= \tau_1(.;h_x) \\ L_h^{(y)} w_2(.;h_y) &= \tau_2(.;h_y), \end{aligned}$$

where $L_h^{(k)}$ is the semi-discretization operator in direction k for $k = x, y$. The function $w_{1,2}$ can be derived from

$$\begin{aligned} (L_h^{(x)} - L_h) w_1(.;h_x) &=: h_y^4 \sigma_{1;2}(.;h_x,h_y) \\ (L_h^{(y)} - L_h) w_2(.;h_y) &=: h_x^4 \sigma_{2:1}(.;h_x,h_y) \end{aligned}$$

and solving

$$L_h w_{1,2} = \sigma_{1;2} + \sigma_{2;1}.$$

For more details we refer to Reisinger (2012). The bounds are given by

$$\begin{aligned} \|w_1\|_\infty &\le \tfrac{1}{8}\tfrac{1}{90}K \\ \|w_2\|_\infty &\le \tfrac{1}{8}\tfrac{1}{90}K \\ \|w_{1,2}\|_\infty &\le 2\tfrac{1}{8^2}\tfrac{1}{90^2}K. \end{aligned}$$

Interpolation of the pointwise error of the finite difference solution and summation of both error terms in (4.2) yields for $(x, y) \in \Omega$

$$
\begin{aligned}
u(x, y) &- (Pu_h)(x, y) \\
&= h_x^4 \left(\beta_1(x, y; h_x) - c_1(x, y; h_x)\right) + h_y^4 \left(\beta_2(x, y; h_y) - c_2(x, y; h_y)\right) \\
&\quad + h_x^4 h_y^4 \left(\beta_{1,2}(x, y; h_x, h_y) - c_{1,2}(x, y; h_x, h_y)\right) \\
&= h_x^4 \gamma_1(x, y; h_x) + h_y^4 \gamma_2(x, y; h_y) + h_x^4 h_y^4 \gamma_{1,2}(x, y; h_x, h_y),
\end{aligned}
$$

with $\gamma_1 := \beta_1 - c_1$, $\gamma_2 := \beta_2 - c_2$, $\gamma_{1,2} := \beta_{1,2} - c_{1,2}$. The functions β_1, β_2, $\beta_{1,2}$ are defined according to (4.8) and bounded by

$$
\begin{aligned}
\|\beta_1\|_\infty &\leq \tfrac{389}{276480} K \\
\|\beta_2\|_\infty &\leq \tfrac{389}{276480} K \\
\|\beta_{1,2}\|_\infty &\leq \tfrac{1551721}{38220595200} K
\end{aligned}
$$

These bounds can be derived by straightforward computation. We illustrate it for $\beta_1 = w_1 + h_x^4 \hat{c}_1 = w_1 + h_x^4 R_x w_1$. Exploiting (4.4), (4.9) we obtain

$$
\begin{aligned}
\|\beta_1\|_\infty &\leq \|w_1\|_\infty + \|R_x w_1\|_\infty \\
&\leq \frac{1}{8}\frac{1}{90} K + \frac{5}{384} \|\frac{\partial^4}{\partial x^4} w_1\|_\infty \\
&\leq \frac{1}{8}\frac{1}{90} K + \frac{5}{384}\frac{1}{8} \|\frac{\partial^4}{\partial x^4} \tau_1\|_\infty \\
&= \frac{1}{8}\frac{1}{90} K + \frac{5}{384}\frac{1}{8}\frac{1}{90} \|\frac{\partial^{10}}{\partial x^{10}} u\|_\infty \\
&\leq \left(\frac{1}{8}\frac{1}{90} + \frac{5}{384}\frac{1}{8}\frac{1}{90}\right) K = \frac{389}{276480} K
\end{aligned}
$$

Thus we can derive bounds

$$
\begin{aligned}
\|\gamma_1\|_\infty &\leq \left(\tfrac{389}{276480} + \tfrac{5}{384}\right) K = \tfrac{3989}{276480} K \\
\|\gamma_2\|_\infty &\leq \tfrac{3989}{276480} K \\
\|\gamma_{1,2}\|_\infty &\leq \left(\tfrac{1551721}{38220595200} + \tfrac{5^2}{384^2}\right) K = \tfrac{8031721}{38220595200} K
\end{aligned}
$$

Hence we obtained a suitable interpolation technique of order four.

5 The d - dimensional case

In this section we extend the framework to the general d dimension case. To do so we split the error

$$
u(x) - (Pu_h)(x) = \underbrace{u(x) - \left(Pu_{\Omega_h}\right)(x)}_{I} + \underbrace{\left(P(u_{\Omega_h} - u_h)\right)(x)}_{II}. \tag{5.1}
$$

In a first step, we derive an expression for the interpolation error in I, if cubic spline interpolation is used in the tensor product approach. This corresponds to the case $p = 4$. Please note that the same structure of proofs can be followed to derive a similar error splitting for higher p. But since schemes with an order higher than four are usually not used in practice, we are satisfied with $p = 4$. Next, we take a closer look at the error structure of our fourth order finite difference solution. Then the knowledge of the interpolation error can be applied to the interpolation of the finite difference solution to obtain the structure in II (Lemma 3). In a final step, we deduce Theorem 1, which gives us an expression of the error (5.1).

Lemma 1. *Let $u \in C_K^{(4,...,4)}(\Omega)$ and univariate cubic spline interpolation P_i along the $i-th$ coordinate direction for $i = 1, ..., d$ with $P_i u = u + h_i^4 R_i u$ and remainder operator R_i be given. Then the error of the tensor product interpolation $P = P_1 \otimes P_2 \otimes ... \otimes P_d$ for all $x \in \Omega$ is*

$$(Pu_{\Omega_h})(x) - u(x) = \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 c_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}),$$

where $\|c_{j_1,...,j_m}\|_\infty \leq \frac{5^m}{384^m} K$ for $m = 1, ..., d$.

Proof.

$$\begin{aligned}(Pu_{\Omega_h})(x) &= (P_1 \otimes P_2 \otimes ... \otimes P_d u_{\Omega_h})(x) = (I + h_1^4 R_1) \otimes (I + h_2^4 R_2) \otimes ... \otimes (I + h_d^4 R_d) u(x) \\ &= u(x) + \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 R_{j_1} \otimes ... \otimes R_{j_m} u(x),\end{aligned}$$

where I is the identity. Defining $c_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}) := R_{j_1} \otimes ... \otimes R_{j_m} u(x)$ we obtain the desired form. Due to the findings by Hall (1968) it holds

$$\|R_{j_1} \otimes ... \otimes R_{j_m} u\|_\infty \leq \frac{5^m}{384^m} \|\frac{\partial^{(4,...,4)}}{\partial x_{j_1}^4 ... \partial x_{j_m}^4} u\|_\infty \leq \frac{5^m}{384^m} K.$$

□

Lemma 2. *Let $u \in C_K^{(6,...,6)}(\Omega)$ be the solution to the Poisson equation and let u_h denote its finite difference solution of order four at the grid point $x_h \in \Omega_h$ with step sizes $h = (h_1, ..., h_d)$. Then the pointwise error is*

$$u(x_h) - u_h = \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 w_{j_1,...,j_m}(x_h; h_{j_1}, ..., h_{j_m}),$$

where

$$|w_{j_1,...,j_m}(x_h; h_{j_1}, ..., h_{j_m})| \leq m!\, 8^{-m}\, 90^{-m} K \text{ for } m = 1, ..., d.$$

Proof. Consistency of finite difference approximation via Taylor expansion and application of the framework developed by Reisinger (2012). □

Before we investigate the interpolation of the pointwise error, we derive bounds for the remainder terms $R_{j_1} \otimes R_{j_2} \otimes ... \otimes R_{j_m} w_{i_1,i_2,...,i_n}$. Similar to the two dimensional one can compute bounds for the analytical and (semi-) discrete solution of the Poisson equation with homogenous Dirichlet data

$$\begin{aligned}\|u\|_\infty &\leq \tfrac{1}{8}\|f\|_\infty \\ \|u_h^{(i_1,...,i_m)}\|_\infty &\leq \tfrac{1}{8}\|f\|_\infty.\end{aligned} \tag{5.2}$$

Restricting ourself to function spaces with vanishing derivatives of sufficiently high order at the boundary, we can also derive bounds for the derivatives of f, u respectively. We cite from Reisinger (2012) the auxiliary problem with solution $w_{i_1,\ldots,i_n}$

$$L_h^{i_1,\ldots,i_n} w_{i_1,\ldots,i_n} = \tau_{i_1,\ldots,i_n}$$

and the definition of the terms $\tau_{i_1,\ldots,i_n}$

$$\tau_{i_1,\ldots,i_n} := \sum_{\substack{z_1,z_2,\ldots,z_{n-1},z \\ s.t.\ \{z_1,z_2,\ldots,z_{n-1}\}\cup\{z\} \\ =\{i_1,i_2,\ldots,i_n\}}} \sigma_{z_1,\ldots,z_{n-1};z}.$$

Please note that τ_{i_1} for $i_1 = 1,..,d$ is the truncation error of the finite difference stencil in coordinate direction i_1. The functions $\sigma_{z_1,\ldots,z_n}$ are obtained via the expansion

$$\left(L_h^{(i_1,\ldots,i_n)} - L_h\right) w_{i_1,\ldots,i_n} = \sum_{\substack{k\in\{1,..,d\} \\ k\notin\{i_1,\ldots,i_n\}}} \sigma_{i_1,\ldots,i_n;k}.$$

The terms $\sigma_{i_1,\ldots,i_n;k}$ can be expressed as the truncation error of the semi-discrete and fully discrete problem from above and thus $\|\sigma_{i_1,\ldots,i_n;k}\|_\infty = \frac{1}{90}\|\frac{\partial^6}{\partial x_k} w_{i_1,\ldots,i_n}\|_\infty$ holds.

$$\begin{aligned}
\|R_{j_1}\otimes R_{j_2}\otimes\ldots\otimes R_{j_m} w_{i_1,i_2,\ldots,i_n}\|_\infty &\le \frac{5^m}{384^m}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4} w_{i_1,i_2,\ldots,i_n}\|_\infty \\
&\le \frac{5^m}{384^m}\frac{1}{8}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4}\tau_{i_1,i_2,\ldots,i_n}\|_\infty \\
&= \frac{5^m}{384^m}\frac{1}{8}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4} \sum_{\substack{z_1,z_2,\ldots,z_{n-1},z \\ s.t.\ \{z_1,z_2,\ldots,z_{n-1}\}\cup\{z\} \\ =\{i_1,i_2,\ldots,i_n\}}} \sigma_{z_1,z_2,\ldots,z_{n-1};z}\|_\infty \\
&= \frac{5^m}{384^m}\frac{1}{8}\frac{1}{90}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4} \sum_{\substack{z_1,z_2,\ldots,z_{n-1},z \\ s.t.\ \{z_1,z_2,\ldots,z_{n-1}\}\cup\{z\} \\ =\{i_1,i_2,\ldots,i_n\}}} \frac{\partial^6}{\partial x_z^6} w_{z_1,z_2,\ldots,z_{n-1}}\|_\infty
\end{aligned}$$

The sum has n terms and we recursively repeat this procedure $n-1$ times until we can conclude the final result in the last step

$$\begin{aligned}
&\le \frac{5^m}{384^m}\frac{1}{8^{n-1}}\frac{1}{90^{n-1}}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4}(n-1)!\sum_{k=1}^{n}\frac{\partial^{6(n-1)}}{\partial x_{i_1}^6\cdot\ldots\cdot\partial x_{i_l\neq k}^6\cdot\ldots\cdot\partial x_{i_n}^6} w_k\|_\infty \\
&\le \frac{5^m}{384^m}\frac{1}{8^{n}}\frac{1}{90^{n-1}}\|\frac{\partial^{4m}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4}(n-1)!\sum_{k=1}^{n}\frac{\partial^{6n}}{\partial x_{i_1}^6\cdot\ldots\cdot\partial x_{i_n}^6}\tau_k\|_\infty \\
&\le \frac{5^m}{384^m}\frac{1}{8^{n}}\frac{1}{90^{n}} n!\|\frac{\partial^{4m+6n}}{\partial x_{j_1}^4\cdot\ldots\cdot\partial x_{j_m}^4\partial x_{i_1}^6\cdot\ldots\cdot\partial x_{i_n}^6} u\|_\infty \\
&\le \frac{5^m}{384^m}\frac{1}{8^{n}}\frac{1}{90^{n}} n! K. \qquad (5.3)
\end{aligned}$$

Here we see that $u \in C_K^{(10,\ldots,10)}(\Omega)$ has to be satisfied to ensure a bounded error.

Lemma 3. *Let $u \in C_K^{(10,...,10)}(\Omega)$ be the solution to the Poisson equation and let u_h denote its finite difference solution of order four on the grid Ω_h with step sizes $h = (h_1, ..., h_d)$. Using tensor product interpolation P with univariate cubic spline interpolation in each coordinate direction, then the interpolation of the pointwise error for all $x \in \Omega$ and the discrete grid Ω_h is*

$$(P(u_{\Omega_h} - u_h))(x) = \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 \beta_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}),$$

where

$$|\beta_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m})| \leq K\, C_m$$

for $m = 1, ..., d$ and constant $C_m \in \mathbb{R}$.

Proof. Interpolation of the pointwise error gives

$$\big(P(u_{\Omega_h} - u_h)\big)(x) = \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 (P w_{j_1,...,j_m}(\Omega_h; h_{j_1}, ..., h_{j_m}))(x), \tag{5.4}$$

where

$$\begin{aligned}\big(P w_{j_1,...,j_m}(\Omega_h; h_{j_1}, ..., h_{j_m})\big)(x) =& w_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}) \\ &+ \sum_{n=1}^{d} \sum_{\substack{\{l_1,...,l_n\} \\ \subset\{1,...,d\}}} h_{l_1}^4 \cdot ... \cdot h_{l_n}^4 R_{l_1} \otimes ... \otimes R_{l_n} w_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}).\end{aligned}$$

Condensing all terms in (5.4), which have the same leading step sizes, we define

$$\begin{aligned}&h_{l_1}^4 \cdot ... \cdot h_{l_k}^4 \beta_{l_1,...,l_k}(x; h_{l_1}, ..., h_{l_k}) := \\ &\quad \sum_{\substack{m,n\in\mathbb{N} \\ s.t.\, m,n\leq k \\ k\leq m+n}} \sum_{\substack{\{i_1,...,i_m\}\cup\{j_1,...,j_n\} \\ =\{l_1,...,l_k\}}} h_{i_1}^4 \cdot ... \cdot h_{i_m}^4 R_{i_1} \otimes ... \otimes R_{i_m}\, h_{j_1}^4 \cdot ... \cdot h_{j_n}^4 w_{j_1,...,j_n}(x; h_{j_1}, ..., h_{j_n}) \\ &\quad + h_{l_1}^4 \cdot ... \cdot h_{l_k}^4\, w_{l_1,...,l_k}(x; h_{l_1}, ..., h_{l_k}).\end{aligned}$$

We already know that it holds (cf. (5.3))

$$\|R_{i_1} \otimes ... \otimes R_{i_m} w_{j_1,...,j_n}\|_\infty \leq \tfrac{5^m}{384^m}\, n!\, 8^{-n} 90^{-n} K.$$

The inner sum has $\binom{k}{m}\binom{m}{n-(k-m)}$ elements and we obtain the estimate

$$\|\beta_{l_1,...,l_k}\|_\infty \leq K \sum_{\substack{m,n\in\mathbb{N} \\ s.t.\, m,n\leq k \\ k\leq m+n}} \tbinom{k}{m}\tbinom{m}{n-(k-m)} \tfrac{5^m}{384^m}\, n!\, 8^{-n} 90^{-n} + K k!\, 8^{-k} 90^{-k} =: K\, C_k$$

□

Table 1 states the constants C_m for the β functions in Lemma 3 for $m = 1, ..., 4$.

m	1	2	3	4
C_m	0.0014	4.0599e-5	8.8699e-7	1.7416e-8

Table 1: Bounds in Lemma 3 for different choices of m.

Theorem 1. *Let $u \in C_K^{(10,...,10)}(\Omega)$ be the solution to the Poisson equation and let u_h denote its finite difference solution of order four on the grid Ω_h with step sizes $h = (h_1, ..., h_d)$. Using tensor product interpolation P with univariate cubic spline interpolation in each coordinate direction, then the error between the analytical solution and the interpolation of the finite difference solution is*

$$u(x) - (Pu_h)(x) = \sum_{m=1}^{d} \sum_{\substack{\{j_1,...,j_m\} \\ \subset\{1,...,d\}}} h_{j_1}^4 \cdot ... \cdot h_{j_m}^4 \gamma_{j_1,...,j_m}(x; h_{j_1}, ..., h_{j_m}),$$

where $\|\gamma_{j_1,...,j_m}\|_\infty \leq (\frac{5^m}{384^m} + C_m)K$ for $m = 1, ..., d$.

Proof. The error can be rewritten as

$$u(x) - (Pu_h)(x) = \underbrace{u(x) - \big(Pu_{\Omega_h}\big)(x)}_{I} + \underbrace{\big(P(u_{\Omega_h} - u_h)\big)(x)}_{II}.$$

The proof immediately follows from Lemma 1 (I) and Lemma 3 (II) and by definition of $\gamma_{j_1,...,j_m} := \beta_{j_1,...,j_m} - c_{j_1,...,j_m}$. □

6 Numerical Experiments

We apply the high order combination technique to the Poisson equation to illustrate the theoretical considerations of the previous sections. The Poisson equation is given by

$$\begin{aligned}\Delta u &= f \text{ on } \Omega = (0,1)^d \\ u &= g \text{ on } \partial\Omega,\end{aligned}$$

and will be solved for several choices of g. In order to discretize the derivatives we use a standard fourth order scheme in each dimension $i = 1, ..., d$

$$\frac{\partial^2 u_j}{\partial x_i^2} = \frac{1}{12h_i^2}\left(-u_{j+2} + 16u_{j+1} - 30u_j + 16u_{j-1} - u_{j-2}\right) + \mathcal{O}(h_i^4),$$

for each node u_j. At points close to the boundary this five point stencil causes problems since grid points from outside the domain are involved. A standard approach is to use second order stencil of the form

$$\frac{\partial^2 u_j}{\partial x_i^2} = \frac{1}{h_i^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + \mathcal{O}(h_i^2)$$

near the boundary. Thus lowering the order of consistency from four to two. We also want to test a different approach and apply polynomial extrapolation. Therefore we consider the grid in one dimension as a sequence of $n+1$ points $x_0, x_1, ..., x_n$ and the unknowns at the corresponding grid points as $u_0, u_1, ..., u_n$. Then we can construct a polynomial P^+, which interpolates the first $m+1$ data points $(x_0, u_0), (x_1, u_1), ..., (x_m, u_m)$ via

$$P^+(x) = \sum_{i=0}^{m} u_i l_i(x)$$

with Lagrange basis functions

$$l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{m} \tfrac{x - x_j}{x_i - x_j}.$$

The polynomial P^-, which interpolates the last $m+1$ points $(x_n, u_n), (x_{n-1}, u_{n-1}), ...,$ (x_{n-m}, u_{n-m}) can be constructed in an analogue manner. The value of a ghost point $(-h, u_{-1})$, $(1+h, u_{n+1})$ can be computed by extrapolation of P^+, P^- respectively

$$\begin{aligned}u_{-1} &= P^+(x_{-1}) \;\; = P^+(-h) \\ u_{n+1} &= P^-(x_{n+1}) = P^-(1+h).\end{aligned}$$

Table 2 shows the coefficients of the involved unknowns u_i for different choices of m. To quote an example: we want to compute u_{-1} with the help of a cubic polynomial. The first line in the table ($m = 3$) states $u_{-1} = 4u_0 - 6u_1 + 4u_2 - 4u_3$. Hence we can replace the ghost point by a linear combination of unknowns. Please note that in the case of $m = 3$, the resulting finite difference stencil coincides with the second order stencil.

In the combination technique the PDE is solved on a sequence of anisotropic grids. If the number of grid points in one of the dimensions is lower than the number of needed grid nodes of the extrapolation technique, it cannot be applied.

m	$u_{0/n}$	$u_{1/n-1}$	$u_{2/n-2}$	$u_{3/n-3}$	$u_{4/n-4}$	$u_{5/n-5}$	$u_{6/n-6}$
3	4	-6	4	-4			
4	5	-10	10	-5	1		
5	6	-15	20	-15	6	-1	
6	7	-21	35	-35	21	-7	1

Table 2: Coefficients in the polynomial extrapolation

m	1	2	3	4	5	6
# grid points	2	3	4	5	6	7
minimal level	0	1	2	2	3	3

Table 3: Extrapolation and needed level

Table 3 states the possible extrapolation technique for different levels. In the case of $m = 2$, three grid points are needed to uniquely define and to extrapolate a quadratic polynomial. Hence a level of one is sufficient, since the grid then consists of three points. Since we are interested in the error on the whole grid we define the error of our numerical approximation with help of the maximum-norm

$$e_n^\infty := \|u_s^n - R_s^n u\|_\infty,$$

where R_s^n restricts the analytical solution to the sparse grid at level n.

6.1 Experiment 1

As a first test example we consider a smooth function $u \in C^\infty(\Omega)$. Furthermore we want to neglect any perturbations, which are introduced by the discretization near the boundary of the domain. Thus we choose a solution that vanishes at the boundary

$$u(x) = \prod_{i=1}^{d} \sin(\pi x_i).$$

Please note that here also all mixed even derivatives vanish for $x \in \partial\Omega$. The function f is then given by $f(x) = -d\pi^2 u(x)$.

In Figure 3 we compare the errors of the combined solution for different choices of m and $d = 2$. If the number of grid points needed for extrapolation exceeds the available number of grid points in one of the coordinate directions, we choose a maximal m according to Table 3 in this direction. Figure 4 shows the convergence in the case of $m = 6$ in two, three and four spatial dimensions. Both plots reveal that the extrapolation technique has no strong influence on the order of convergence for this test problem. Since the solution and its mixed even derivatives vanish at the boundary this observation is not surprising.

6.2 Experiment 2

As the second test case, we want to evaluate the influence of the extrapolation techniques. We have already seen in the first example that the order of convergence of the sparse grid solution is not deteriorated if the solution and its mixed even derivatives vanish at the boundary. In order to show what happens for non vanishing solutions and derivatives we

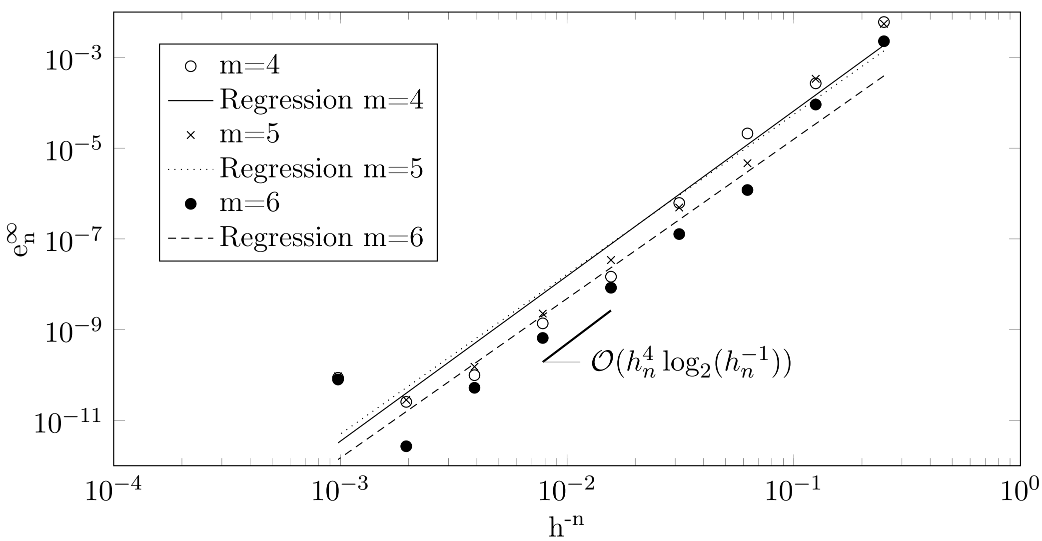

Figure 3: Convergence for different choices of m and $d = 2$ (Exp.1).

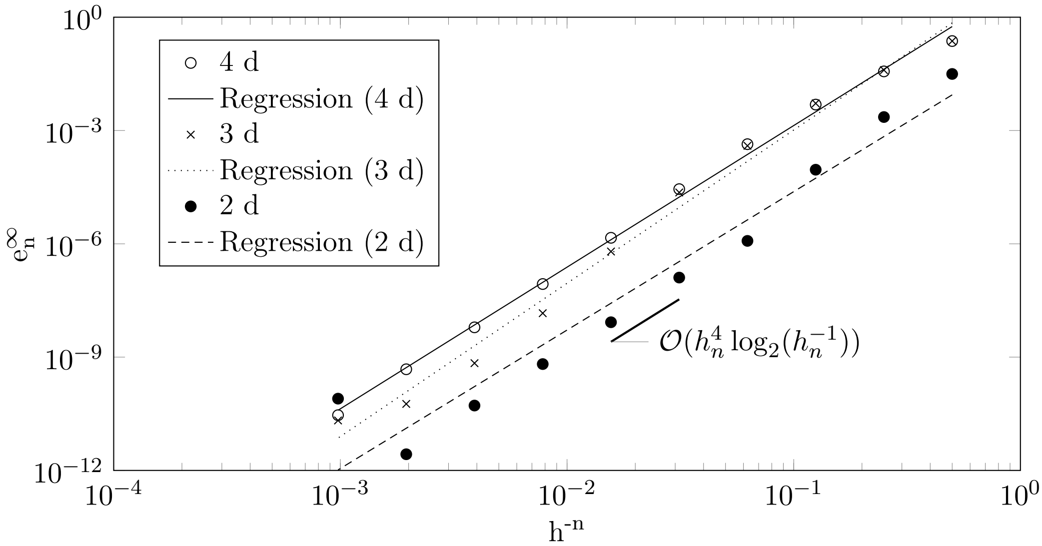

Figure 4: Convergence for $m = 6$ and $d = 2, 3, 4$ (Exp.1).

consider a function, which was used by Reisinger (2012):

$$u(x) = \exp\left(-\tfrac{1}{2}\sum_{i=1}^{d}(x_i - p_i)^2\right),$$

where $p_1 = 0.2208$, $p_2 = 0.2907$, $p_3 = 0.2805$ and $p_4 = 0.2703$. Hence we have $f(x) = \sum_{i=1}^{d}\left[(p_i - x_i)^2 - 1\right]u(x)$.

In Figure 5 the effect of extrapolation with polynomials of order four, five and six is compared. The higher order leads to a better rate of convergence. However the theoretical order cannot be reached. This observation can be explained by the usage of second order finite difference stencils if the level is equal to one in a coordinate direction and since we cannot apply high order extrapolation in the case of one level being equal to two. Therefore we neglect these grids in our combination technique, which leads to the following slightly modified version

$$u_s^{n,k} = \sum_{q=0}^{d-1}(-1)^q\binom{d-1}{q}\sum_{\substack{|l|_1=n+(d-1)-q\\ \min\{l_i, i=1,..,d\}>k}} u_l(x).$$

The case $k = 0$ coincides with the standard combination technique, which can be found in the literature, e.g. Bungartz & Griebel (2004),Griebel et al. (1992). Since for this technique also the lower order terms cancel out, one can derive the same order of pointwise convergence, expect for the leading coefficients, than for the standard combination technique.

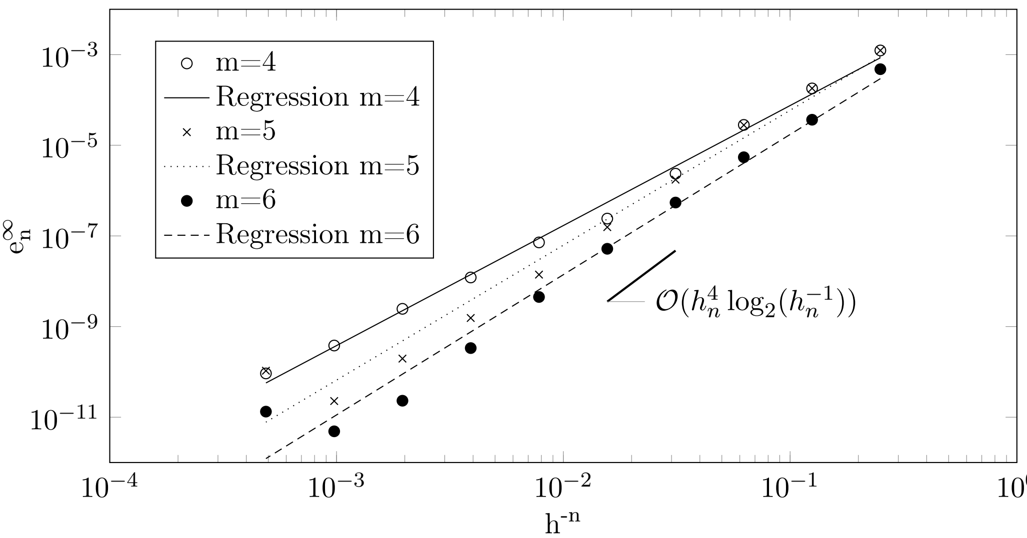

Figure 5: Convergence for different choices of m and $d = 2$ (Exp.2).

Figure 6 shows the improvement of the rate of convergence if coarse grids are dropped. In the case of $k = 1$ a fourth order extrapolation polynomial can be used at the boundaries on all sub-grids, whereas for $k = 2$ we can apply sixth order extrapolation on all grids.

In Figure 7 the error in the maximum-norm for $m = 6$, $k = 2$ for the two, three and four dimensional Poisson problem is plotted. The rate of convergence is in line with the theoretical considerations of the previous sections.

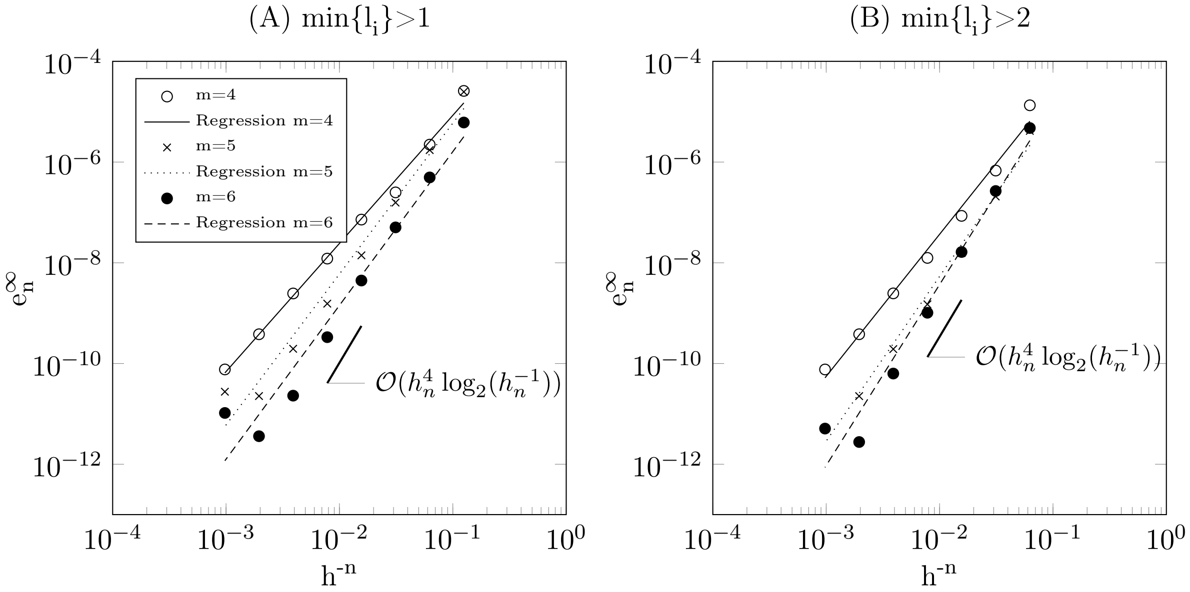

Figure 6: Convergence for $m = 4, 5, 6$ and $d = 2$.
(A): All grids with level one removed.
(B): All grids with level smaller or equal to two removed (Exp.2).

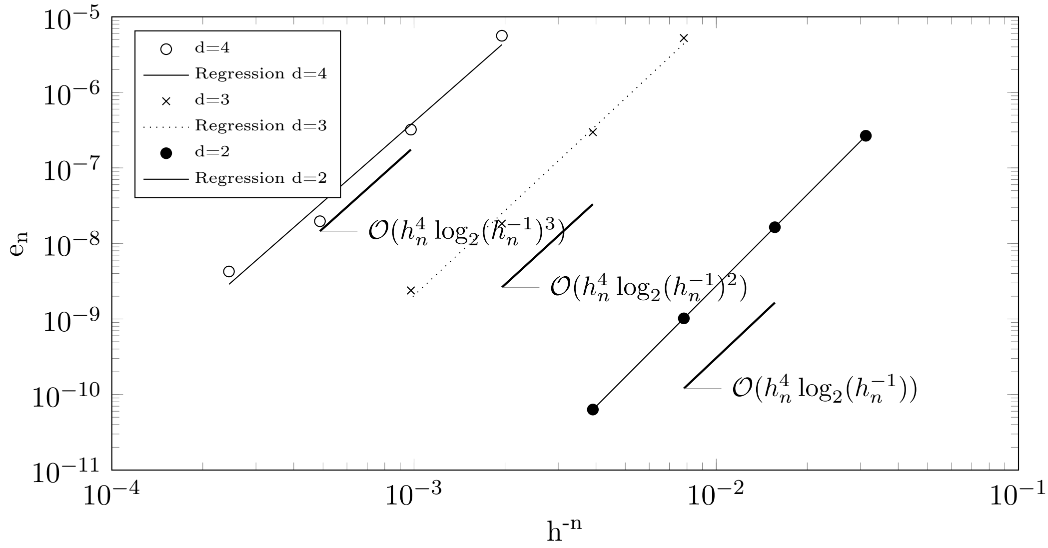

Figure 7: Convergence for sparse grid solution ($m = 6$, $\min\{l_i\} > 2$) in the two, three and four dimensional case (Exp.2).

7 Conclusion

In this paper we have introduced a tensor product based interpolation technique for the combination technique. The high order case with fourth order finite difference stencils and cubic spline interpolation has been investigated in detail for the Poisson problem. Therefore we derived the splitting structure of the interpolation of the pointwise error of the finite difference solution. It turned out that the high order of the discrete solution can be preserved if tensor product interpolation of the same order is used to combine the solutions within the combination technique.

In the case of sufficiently smooth solutions, which vanish at the boundary, the numerical experiments directly validated our theoretical results. For solutions with non homogeneous Dirichlet data the experiments revealed some deterioration coming from coarse grids, where no high order solutions can be be computed. These problems could be cured with the help of high order extrapolation at the boundary to some extent or by neglecting those coarse grids in the combination technique.

In a next step we want to extend this framework to parabolic equations from computational finance. Here a lot of high dimensional problems arise: such as basket options, fx options or carbon emission allowances (Hendricks & Ehrhardt (2014)). We want to investigate in how far the often non smooth payoff deteriorates the numerical order of convergence if high order schemes are applied.

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