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Abstract The finance world, relying more and more on mathematical models, also expects them to be fast, robust and cheap, specially for calibration purposes. The recent revolution in Graphical Processing Units (GPU) and Field-Programmable Gate Array (FPGA) has helped to reduce time and costs but it is the algorithms that ultimately prevail. In this respect, Model Order Reduction (MOR) seems to be specially suited to financial problems as it can reduce extremely computational costs [1]. We present how and when MOR can be extremely useful and how Proper Orthogonal Decomposition (POD) stands out as a valid MOR technique in finance [11]. We show the validity of its application to pricing of basket options, as well as to stochastic volatility models [7], through the solution of a reduced Black-Scholes PDE. Finally, its computational efficiency when compared with some extensively used numerical methods, as well as some of its limitations, are discussed.

1 Introduction

Model Order Reduction (MOR) emerged at the end of the twentieth century as an answer to the increasing complexity of models being developed. Higher and higher resolution schemes leads to bigger problems which, in turn, lead to the development of new accurate schemes (non-uniform and refined grids, higher-order schemes, sparse schemes, parallelization, problem-specific hardware, etc.). The goal of MOR is to generate smaller models, faster to solve and, if not with similar, with high enough precision with respect to the original Full Order Model (FOM). The Reduced Order Model (ROM) is then a cheaper and faster proxy of the FOM, making it ideal for multi-query problems: parameter studies, parameter optimization, inverse problems, control problems. In finance, and particularly option pricing, inverse

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problems arise when calibrating model parameters to market data, with volatility being one of the parameters, for example.

Among the different MOR techniques, Proper Orthogonal Decomposition (POD) stands out as a fairly robust technique as it is one of the few techniques able to tackle general non-linear problems. Due to its data-driven approach, it generates ROM in a tailored way.

The first FOM we want to reduce is an European-type basket option. Assuming a financial contract with n underlyings following geometrical Brownian motions (GBM) we obtain its price V as the solution of the PDE

$$\frac{\partial V}{\partial t}(t,\mathbf{s}) + \sum_{i}^{n} r s_i \frac{\partial V}{\partial s_i}(t,\mathbf{s}) + \frac{1}{2}\sum_{i,j}^{n} \rho^{\star}_{ij}\sigma_i\sigma_j s_i s_j \frac{\partial^2 V}{\partial s_i \partial s_j}(t,\mathbf{s}) - rV(t,\mathbf{s}) = 0 \tag{1}$$

with $s_i \in [0,\infty), t \in [0,T], \rho^{\star}_{ij} = 2\rho_{ij}, i \neq j$. ρ_{ij} is the correlation between stochastic processes S_i and S_j and σ_i is the annualized standard deviation of logarithmic returns of S_i.

As this parabolic PDE is, most of the time, supplied with a terminal condition at $t = T$, we will integrate it backwards in time. Depending on the characteristics of the financial contract, we supply (1) with appropriate boundary and terminal conditions.

The second model comes as a result of GBM being a very restrictive model in what concerns the paths of the underlying. Introducing a square root variance model with a mean reverting process for variance we obtain the *Heston PDE*

$$\begin{aligned}\frac{\partial V}{\partial t}(t,\nu,S) = &-\frac{1}{2}\nu S^2\frac{\partial^2 V}{\partial S^2}(t,\nu,S) - \rho\sigma\nu S\frac{\partial^2 V}{\partial \nu \partial S}(t,\nu,S) - \frac{1}{2}\sigma^2\nu\frac{\partial^2 V}{\partial \nu^2}(t,\nu,S)\\ &- rS\frac{\partial V}{\partial S}(t,\nu,S) - \kappa(\theta-\nu)\frac{\partial V}{\partial \nu} + rV(t,\nu,S),\end{aligned} \tag{2}$$

with ρ the correlation between Wiener processes, κ the rate of convergence of the volatility to its long-term mean, θ the long-term mean of variance and σ^2 the variance of variance.

2 Proper Orthogonal Decomposition

In practice, most reduced models are generated in a two step approach. In a first step, information from the full order model is retrieved and with that information a basis of a subspace is generated. In a second step, the original model is projected onto the same (different) subspace space spanned by this new basis, a procedure called Galerkin projection (Petrov-Galerkin projection). In that sense, POD is no different, the big difference being the basis generation, as it is generated solely from data.

The POD is a mathematical procedure that, given an ensemble of data, constructs a basis for the ensemble that is optimal in the following sense. Let X be a real Hilbert space, with inner product $(\cdot,\cdot)_X$, and $Y = \left[y_1 \; y_2 \; \dots \; y_n \right]$ an ensemble of n snapshots $y_i \in X$. Then, a POD basis is an orthonormal basis ψ_j such that the square error between the elements y_i and its l-partial sum of the decomposition of y_i in the space spanned by ψ_j, is minimized, i.e.

$$\min_{\{\psi_k\}_{k=1}^{l}} \mathscr{J}(\psi) = \min_{\{\psi_k\}_{k=1}^{l}} \sum_{i=1}^{n} \left\| y_i - \sum_{j=1}^{l} (y_i, \psi_j)_X \, \psi_j \right\|_X^2 , \tag{3}$$

subject to $(\psi_i, \psi_j)_X = \delta_{ij}$. It can be proved that the above minimization problem is equivalent to the eigenvalue problem

$$YY^{\top}\psi = \lambda\,\psi.$$

If we factorize Y using a Singular Value decomposition (SVD), we can see that the resulting left-singular vectors form a POD basis, where $\lambda = \sigma^2$, with σ the singular values of Y. For the POD basis, $\mathscr{J}(\psi) = \sum_{i=l+1}^{n} \lambda_i = \sum_{i=l+1}^{n} \sigma_i^2$. The size of the basis l necessary for a good approximation is problem dependent, although the following rule of thumb is usually very efficient [2]

$$\mathscr{E}(l) = \frac{\sum_{i=1}^{l} \sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2}. \tag{4}$$

As the singular values are ordered and reflect the relevance of each dimension in the state space, it is sometimes called *relative information measure*.

The second step in constructing a ROM is to project the PDE onto the space spanned by the POD basis. Rewriting our PDEs as

$$\frac{\partial}{\partial t} V = \mathscr{L} V, \tag{5}$$

where $\mathscr{L}$ is a linear operator, we project in a Galerkin fashion, i.e.

$$\left(\psi_i, \frac{\partial V}{\partial t} \right)_X = (\psi_i, \mathscr{L} V)_X , \qquad i = 1,\dots,l.$$

Substituting V by its representation in the POD basis of size l , $V = \sum_j^l a_j(t)\, \psi_j(\mathbf{s})$ and bearing in mind the orthogonality of the basis, we obtain the explicit system of ODEs

$$\dot{a}_i = \sum_{j=1}^{l} a_j(t)\, (\psi_i, \mathscr{L} \psi_l)_X \qquad i = 1,\dots,l.$$

The inner product exhibits two roles in the construction of the ROM. First by defining the POD basis optimality and secondly in the projection step of the PDE. Besides, there are two ways in which we can treat our projection step, before or after semi-discretizing our original, continuous PDE. In our numerical results we will use the former, where we use the method of lines (MOL) to discretize our PDE in space.

3 Numerical Results

We used the MOL to discretize equations (1) (for $n = 2$) and (2) obtaining a system of ODEs, with second order approximations for both first and second derivatives. We took n_i discretization points in direction x_i, resulting in a grid of size $N = \prod_i n_i$. Our PDE became then a system of ODEs with size equal to the total number of (interior) discretization points $N_{int} = \prod_i (n_i - 2)$, which can easily be written in a state-space formulation, common to most MOR techniques,

$$\dot{v} = Av + b \qquad v, b \in \mathbb{R}^{N_{int}}, \quad A \in \mathbb{R}^{N_{int} \times N_{int}} \tag{6}$$

where A has a sparse structure.

In this setting, we have $X = \mathbb{R}^{N_{int}}$ with the euclidean inner product, $(x_1, x_2)_X = x_1^T x_2$. Setting $v = \psi a$ and projecting the equation we obtained the reduced ODE system

$$\underbrace{I_N}_{\psi^T \psi} \dot{a} = \underbrace{\tilde{A}}_{\psi^T A \psi} a + \underbrace{\tilde{b}}_{\psi^T b} \tag{7}$$

We first solve (6), whose solution we will call truth solution, and then proceed to solve (7) using the same integration scheme as in (6). There is no need to ensure we use the same integration scheme however that will be generally the case either in third party software or for ease of implementation.

3.1 2D Basket Option

First we solved (1) for two underlyings (2D PDE) in a uniform grid with n_1 points in S_1-direction and n_2 points in S_2-direction for the spatial domain $\Omega = [0, 6K] \times [0, 6K]$, with a put option payoff as terminal condition, i.e.

$$V(T, S_1, S_2) = \phi(S_1, S_2) = \max(K - \omega_1 S_1 - \omega_2 S_2, 0), \quad \omega_1 + \omega_2 = 1, \omega_i > 0$$

and following boundary conditions (V^* is a 1D put option with a rescaled strike price, $K^* = K/\omega_1$)

$$\begin{aligned} V(t, S_{1min} = 0, S_2) = \omega_2 V^\star(t, S_2) \quad V(t, S_{1max} = 6K, S_2) = 0 \\ V(t, S_1, S_{2min} = 0) = \omega_1 V^\star(t, S_1) \quad V(t, S_1, S_{2max} = 6K) = 0. \end{aligned}$$

We used the following set of parameters

ρ	σ_1	σ_2	r	K	T	ω_1	n_1	n_2
0.5	0.1	0.2	0.025	100	1	0.25	20	40

Table 1 2D Parameters

We proceeded to solve the FOM with a trapezoidal integration in time with 100 time steps, retrieve our snapshots, generate the basis, project and solve the ROM with the same trapezoidal integration in time. We used all equally time spaced snapshots available to generate our basis. In Figure 1, we display the maximum absolute error between the FOM and the ROM at $t = 0$ for increasing number of basis elements and the corresponding squared singular values, σ^2.

First of all, we can observe an exponential decay in the singular values, a condition necessary for our FOM to possess the so-called *sparse representation property* [4]. Secondly, we can observe that only 20 basis vectors are enough to achieve a 10^{-12} precision.

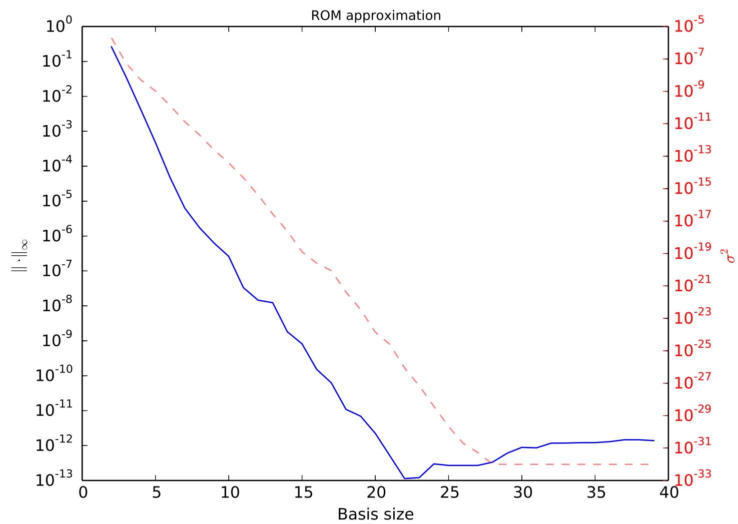


Fig. 1 Absolute Error $\|V_{FOM}(0,S_1,S_2) - V_{ROM}(0,S_1,S_2)\|_\infty$ at time $t = 0$ for reduced 2D Basket Option

3.2 Heston Model

In our second case, following [8], we applied an ADI-type scheme, modified Craig-Sneyd (MCS), to solve the ODE system resulting from the spatial discretization of (2). Contrary to the Basket Option case, we used a non-uniform spatial grid based on the hyperbolic sine with focus around $v = 0$ and $S = K$. We used a spatial domain $\Omega = [0,15] \times [0,30K]$ with a discretization consisting of 25 points in v direction and 50 in S direction. All the following results are for $n_t = 1000$.

To supply our PDE with the appropriate conditions, we define for our terminal condition as a *call* option payoff $\phi(v,S) = \max(S-K,0)$ and for the boundary conditions

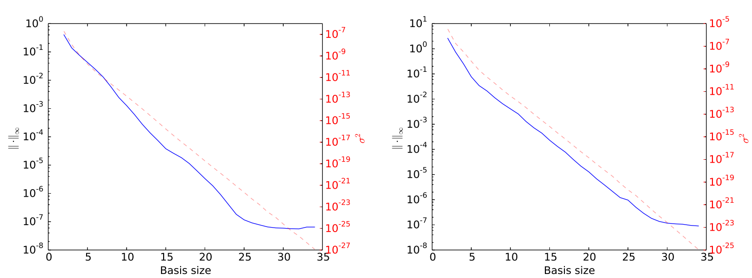
$$V(t,15,S) = Se^{-r_f t}, \quad V(t,\nu,0) = 0, \quad \frac{\partial V}{\partial S}(t,\nu,30K) = e^{-r_f t}.$$

Note that we do not impose any boundary condition at $\nu = 0$, a degenerate point of our PDE, as numerically we just use the degenerate PDE along $\nu = 0$, cf. [5]. As in [8], we tested our reduced models with four different set of parameters, originally taken from [3].

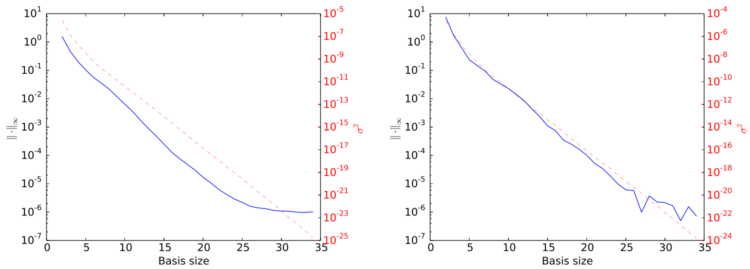
	ρ	σ	r_d	r_f	θ	κ	K	T
Case 1	-0.13	0.49	0.02	0.04	0.02	6.02	100	0.25
Case 2	-0.67	0.62	0.01	0.02	0.02	1.50	100	1
Case 3	-0.55	1.26	0.01	0.06	0.09	0.38	100	4
Case 4	0.78	0.15	0.1	0.02	0.06	0.3	100	5

Table 2 Heston Model parameters [8]

Figure 1 presents the results for the absolute error for the solution at time $t = 0$ for each of the four cases. Note that we decided to evaluate the error in a region of interest $[0,1] \times [0,6K]$ instead of the original grid span $[0,15] \times [0,30K]$ as those range of values would be of little or no use in practice. Even though we are in the realm of numerical analysis, we should note one thing about applying these methods in finance. With some exceptions (American-type options), we are mostly interested in the solution of our PDE at the initial time $t = 0$ or at a few selected times. This provides an opportunity to optimize our choice of snapshots in order to minimize the error at these selected times, procedure that was not taken into account in this case.



(a) Case 1 (b) Case 2



(c) Case 3 (d) Case 4

Fig. 1 Absolute Error $\|V_{FOM}(0,v,S) - V_{ROM}(0,v,S)\|_\infty$ at time $t = 0$ for reduced Heston Model and $(v,S) \in [0,1] \times [0,6K]$

Comparing these results with the FOM in [8], we can see that we have a very good approximation with an error of similar magnitude to the temporal discretization error in the FOM. We also would like to observe that in cases 1,2 and 4, the maximum error is attained near the focus of the grid while on case 3 it happens at the corner of the analysis domain, $(1,6K)$. The error of case 3 might then be even smaller if a more slim region of interest is considered.

3.3 ADI and MOR for higher-dimensional problems

Splitting methods, and ADI in particular, have recently been used in finance [6] as it lightens the weight of the curse of dimensionality by having to solve, at each time step, only tridiagonal systems implicitly, some of them time-independent. Following conventions in [10], we evaluate and compare the computational cost of using an ADI MCS scheme to solve a full model and the same cost for of solving a reduced one. In what follows, we take d as the number of dimensions in our problem, n_t the number of time-steps in our time-stepping scheme and n_d the number of discretization points in each direction. We will assume n_d is the same in all dimensions just to simplify the exposition as the general case just involves more calculations.

The MCS scheme consists at each time step of

- two explicit integrations of the mixed derivative terms
- two implicit integrations on each direction/dimension

Dealing with PDE of up to second order and with second order approximation for the derivative operators will result in tridiagonal systems for each A_j. For the mixed derivatives, the PDE may contain up to $\frac{d(d-1)}{2}$ mixed derivatives and each mixed derivative discretization will take 4 new points. So in total, we have $2d\,(d-1)$ diagonals in A_0. This situation occurs in financial PDEs as unless the correlation between each stochastic process is zero, we will always have all mixed derivatives terms. Assuming time independence of our matrices, the computational cost will then be:

1. Only once
$$d\frac{3^2 n_d^d}{2}$$
as for the LU decomposition for the tridiagonal matrices A_j
2. At each time step
   a. 2 Explicit steps
   $$2\left(2d\left(d-1\right)\right)n_d^d = 4d\left(d-1\right)n_d^d$$
   b. 2 Implicit steps per dimension
   $$2d5n_d^d = 10dn_d^d$$

So the total cost is

$$\begin{aligned} f\left(\cdot\right) &= d\frac{3^2 n_d^d}{2} + n_t\left(4d\left(d-1\right)n_d^d + 10dn_d^d\right) \\ &= \frac{9}{2}dn_d^d + \left(4d^2+6d\right)n_t n_d^d + dn_d^d\left(\frac{9}{2} + 6n_t + 4dn_t\right) \end{aligned} \tag{8}$$

We now represent graphically the computational cost of ADI vs a reduced model generated with basis of different sizes

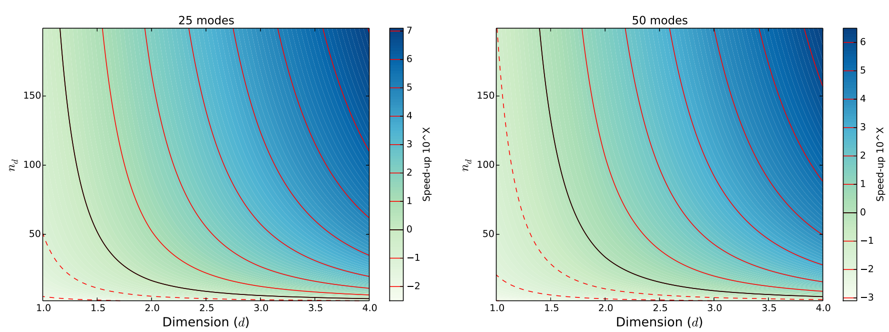


Fig. 2 Computational Advantage of MOR

We can see that we can achieve significant reduction in the number of operations (speed-up) already for a two-dimensional problem even in the 50 modes case. Due to the exponential dependence on dimension for the ADI method and the respective independence for the reduced model, we can theoretically obtain better and better results the higher the dimension of the problem. Although higher dimensional ADI methods still lack some rigorous proof of its properties (stability, consistency), in practice they have been applied with success [6] and so, at least for up to 4 dimensional problems, we can regard figure 2 as showing realistic cases of application.

4 Conclusion

We generated reduced models using POD for two of the most common mathematical models in finance: Basket Options and Heston Model. In both cases it was shown that 25 basis elements at most are needed to obtain the best approximation. We also showed that even for numerical schemes regarded as computationally efficient (ADI) we can obtain significant gains already on 3 and 4 dimensional problems [6]. The advantage is even more clear in a multi-query problem as the cost of SVD is dilluted over each online calculation. We expect that to be the case in parametric ROM, which will be subject to future work.

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