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The Pricing of Quanto Options under Dynamic Correlation

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Abstract

The Quanto option is a cash-settled, cross-currency derivative in which the underlying asset has a payoff in one country, but the payoff is converted to another currency in which the option is settled. Thus, the correlation between the underlying asset and currency exchange rate plays an important role on pricing such options.
Market observations give clear evidence that financial quantities are correlated in a strongly nonlinear, non-deterministic way. In this work, instead of assuming a constant correlation, we develop a strategy for pricing the Quanto option under dynamic correlation in a closed formula, including the calibration to market data.
By comparing the pricing and hedging strategy with and without dynamic correlation, we study the effect of dynamic correlation on the option pricing and hedging. The numerical results show that the prices of Quanto option under dynamic correlation can be better fitted to the market prices than using simply a constant correlation.

Key words: Quanto options, dynamic correlation, Hyperbolic tangent, Black-Scholes equation, Correlation risk
MSC 2000: 39A50, 91B28, 91G20, 91G60, 97M30

1 Introduction

The *Quanto* is a cross-currency contract which has a payoff defined with respect to an asset or an index in one country, but then the payoff is converted to another currency for payment. Thus, for pricing Quanto options, the correlation between assets and currency exchange rate must be considered. In [8], a partial differential equation (PDE) for pricing

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any contract with underlying measured in one currency but paid in another has been derived by constructing a arbitrage-free and self-financing portfolio. However, a constant correlation has been assumed.

From the market observations we realize that financial quantities are correlated in a strongly nonlinear, non-deterministic way. Using a constant (wrong) correlation leads to correlation risk, since the constant correlation is not true in the real world. A concept of conditional correlation is proposed in [2]. A few stochastic correlation processes and its application for pricing option can be found in [1], [3], [4] and [5]. Because stochastic correlation models lack analytical tractability, Teng et al. [6] proposed a new dynamic correlation model which has been incorporated into the Heston model.

In this work, we investigate the pricing, calibration and hedging of Quanto options under dynamic correlation using the dynamic correlation model in [6]. In the next section we briefly review our dynamic correlation model [6]. Section 3 is devoted to the derivation of the closed pricing formula of Quanto option under dynamic correlation model. To recognize the effect of using a dynamic correlation, the price and the hedging strategy of Quanto option between using dynamic and constant correlation are compared in Section 4. In Section 5, we calibrate the Quanto option pricing model with dynamic and constant correlation to the market data and compare them.

2 The dynamic correlation model

From [4] and [5] one knows that a stochastic process modelling correlation must satisfy the following properties:

1. it only takes values in the interval $(-1, 1)$,
2. it varies around a mean value,
3. the probability mass tends to zero at the boundaries $-1, 1$.

In a similar way, a dynamic function (only depending on time t) must first take values only in the interval $(-1, 1)$ to model correlation. The dynamic correlation function should have a limit for increasing time, like the mean reversion in case of stochastic correlation. Therefore, Teng et al. [6] proposed to use

$$\rho_t := E\left[\tanh(X_t)\right] \tag{1}$$

for the dynamic correlation function, where X_t is any mean-reverting process with positive and negative values. For a fixed parameter of X_t, the correlation function ρ_t depends only on t. Furthermore, it is obvious that ρ_t takes values only in $(-1, 1)$ for all t and converges for $t \to \infty$. A further motivation for (1) could be that one may be more interested in forecasting future average correlation, namely $E[\rho_t]$ for the correlation process ρ_t.

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By choosing X_t in (1) to be the *Ornstein-Uhlenbeck process* [7]

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t, \quad t \geq 0, \tag{2}$$

the closed-form expression for ρ_t has been derived as, cf. [6]

$$\rho_t = 1 - \frac{\mathrm{e}^{-A-\frac{B}{2}}}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh(\frac{\pi u}{2})} \cdot \mathrm{e}^{iu(A+B)+u^2\frac{B}{2}} du, \tag{3}$$

with

$$A = \mathrm{e}^{-\kappa t} \tanh^{-1}(\rho_0) + \mu(1 - \mathrm{e}^{-\kappa t}), \tag{4}$$

$$B = -\frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa t}). \tag{5}$$

The proof can be found in [6]. In (3), the limit of ρ_t for $t \to \infty$ is not exact μ but depends mainly on μ. Furthermore, σ represents the magnitude of variation from the value around μ and κ represents the speed of ρ_t tending to its limit. For a detailed illustration of the role of each parameter in (3) we refer to [6].

In the multi-asset modelling, the correlated Brownian motion (BMs) has been often used to consider the relationship between assets, where a constant correlation has been assumed. In the following, we show how to construct dynamically correlated BMs. First, two BMs W_t^1 and W_t^2 are called *dynamically correlated* with correlation function ρ_t, if they satisfy

$$E[W_t^1 W_t^2] = \int_0^t \rho_s ds, \tag{6}$$

where $\rho_t : [0, t] \to (-1, 1)$.

For two independent BMs W_t^1 and W_t^3, W_t^2 defined by

$$W_t^2 = \int_0^t \rho_s dW_t^1 + \int_0^t \sqrt{1 - \rho_s^2}\, dW_t^3, \tag{7}$$

is a BM and correlated with W_t^1 dynamically by ρ_t.

3 Quanto options under dynamic correlation

In this Section, we derive the pricing formula of Quanto options with incorporated dynamic correlation. We define R to be the exchange rate between domestic and foreign currency and S is the level of an index traded in the foreign countries. We assume that they satisfy

$$\begin{cases} dS_t &= \mu_S S_t\, dt + \sigma_S S_t\, dW_t^S \\ dR_t &= \mu_R R_t\, dt + \sigma_R R_t\, dW_t^R, \end{cases} \tag{8}$$

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where W_t^S and W_t^R are correlated dynamically using the correlation function (3).

Following the train of thoughts in [8] we construct a portfolio consisting of the quanto in question, hedged with foreign currency and the asset S :

$$\Pi = V(R, S, \rho_t, t) - \Delta_R R - \Delta_S RS. \tag{9}$$

We remark that every term in this equation values in domestic currency. Δ_R is the number of foreign currency we hold short, so $-\Delta_R R$ is the value in domestic currency of that foreign currency. It is similar to understand $-\Delta_S RS$.

The change in the value of the portfolio due to the change in the value of its components and the interest rate of foreign currency (r_f) can be obtained with the aid of the *Itô lemma* as

$$\begin{aligned} d\Pi &= \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_R^2 R^2 \frac{\partial^2 V}{\partial R^2} + \rho_t \sigma_R \sigma_S RS \frac{\partial^2 V}{\partial S \partial R} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} \\ &- \rho_t \sigma_R \sigma_S \Delta_S RS - r_f \Delta_R R\Big) dt + \Big(\frac{\partial V}{\partial R} - \Delta_R - \Delta_S S\Big) dR \\ &+ \Big(\frac{\partial V}{\partial S} - \Delta_S R\Big) dS. \end{aligned} \tag{10}$$

We now choose

$$\Delta_R = \frac{\partial V}{\partial R} - \frac{S}{R}\frac{\partial V}{\partial S} \text{ and } \Delta_S = \frac{1}{R}\frac{\partial V}{\partial S} \tag{11}$$

to hedge the risk in the portfolio. Thus, the return on this risk-free portfolio must be equal to the domestic currency risk-free rate (r_d), which yields

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma_R^2 R^2 \frac{\partial^2 V}{\partial R^2} + \rho_t \sigma_R \sigma_S RS \frac{\partial^2 V}{\partial S \partial R} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} \\ &+ R\frac{\partial V}{\partial R}(r_d - r_f) + S\frac{\partial V}{\partial S}(r_f - \rho_t \sigma_R \sigma_S) - r_d V = 0. \end{aligned} \tag{12}$$

To fully specify a particular quanto we consider a Quanto Put-option with the payoff at maturity

$$W(S, t) = R_0 \max(K - S_T, 0) \tag{13}$$

where R_0 is the exchange rate at the time zero (today). This means, it is agreed upon at the inception of the contracts that the exchange rate at the time-zero will be used at maturity. So there is no currency risks to appear. By substituting (13) into (12) we obtain

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 W}{\partial S^2} + S\frac{\partial W}{\partial S}(r_f - \rho_t \sigma_R \sigma_S) - r_d W = 0, \tag{14}$$

which is just the simple one factor Black-Scholes equation with a time-dependent dividend yield of

$$D(\rho_t) = r_d - r_f + \sigma_R \sigma_S \frac{1}{t}\int_0^t \rho_s ds. \tag{15}$$

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Finally, the price of a Quanto Put-Option in the extended Black-Scholes model incorporating time-dependent dividend yield can be derived as

$$P = R_0\Big(K\exp^{-r_dT}\mathcal{N}(-d_2) - S_0\exp^{(r_f-r_d)T-\sigma_R\sigma_S\int_0^T\rho_t dt}\mathcal{N}(-d_1)\Big), \tag{16}$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r_f - \sigma_R\sigma_S\int_0^T\rho_t dt + \frac{\sigma_S^2}{2})/T}{\sigma_S\sqrt{T}}, \qquad d_2 = d_1 - \sigma_S\sqrt{T}, \tag{17}$$

where the correlation function ρ_t is defined in (3). The price of a Quanto Call-Option can be derived easily from the put-call parity.

4 Dynamic correlation vs. constant correlation

As an example, think of investing a Put-option on the Deutsche Bank stock traded in Euro (foreign currency) and converted to USD (domestic currency) at maturity. We assume that $S_0 = 36, R_0 = 1.3, r_d = 0.05, r_f = 0.03, \sigma_R = 0.3$ and $\sigma_S = 0.2$. For the dynamic correlation function we set $\rho_0 = 0, \kappa = 2, \mu = 0.2$ and $\sigma = 0.5$, and set the value of constant correlation to be 0.2. In Figure 1 we display the prices using constant and dynamic correlation for different strikes and maturities. We see that prices under dynamic correlation are higher

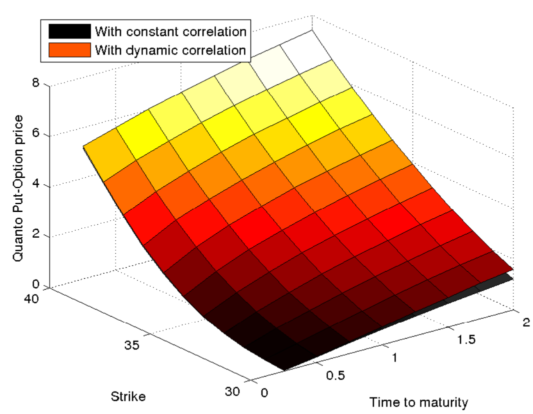

Figure 1: Comparison of prices between using constant and dynamic correlation with $\kappa = 2$, $\mu = 0.25$, $\sigma = 0.5$ and $\rho_0 = 0$ (correlation process parameters) and $\rho = 0$ (constant correlation).

than the price using constant correlation. To clarify the difference between them we show the difference in Figure 2.

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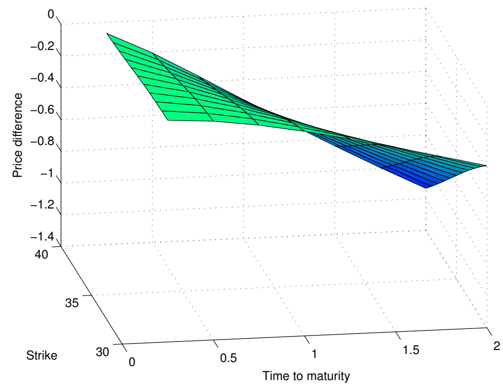

Figure 2: The difference of the prices in Figure 1.

We now keep all the value of parameters to be the same except for setting $\mu = 0$. From (3) we see that the dynamic correlation function takes value always around zero, which is the value of the constant correlation. This means that the price differences must be less than the last case. To see this, we plot the price differences for this case in Figure 3 and compare it to Figure 2.

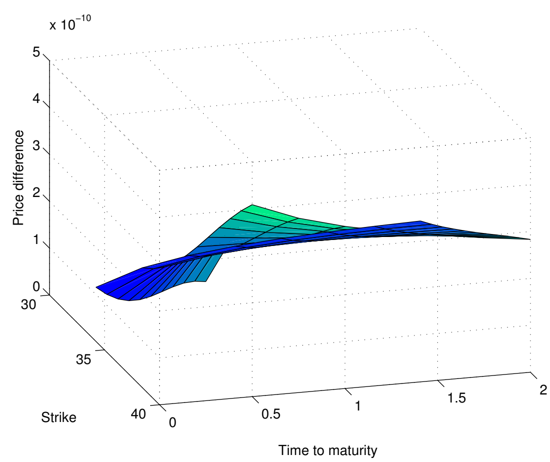

Figure 3: Price differences between using constant and dynamic correlation with $\kappa = 2$, $\mu = 0$, $\sigma = 0.5$ and $\rho_0 = 0$ (correlation process parameters) and $\rho = 0$ (constant Correlation).

Furthermore, we can set $\sigma = 0.5$ and $\kappa = 8$ so that dynamic correlation function reaches its limit (around zero) rapidly. For this case, the prices with and without dynamic

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correlation must be more closer to each other, see the price differences in Figure 4.

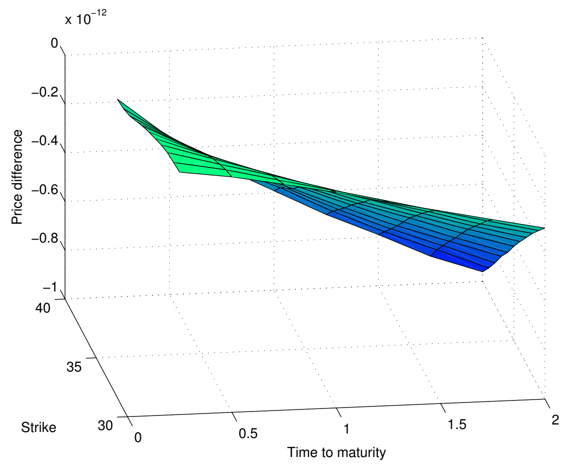

Figure 4: Price differences between using constant and dynamic correlation with $\kappa = 8$, $\mu = 0$, $\sigma = 0.1$ and $\rho_0 = 0$ (correlation process parameters) and $\rho = 0$ (constant correlation).

In the following, we discuss the effect of dynamic correlation on the hedging strategy. We consider the delta as an example. For using a dynamic correlation, the delta is given by

$$\Delta_d = \Phi(d_1) - 1 \tag{18}$$

where Φ is standard normal distribution function and d_1 is defined in (17). Similarly, the delta for using a constant correlation is given by

$$\Delta_c = \Phi(d_1) - 1 \tag{19}$$

where d_1 is defined in (17) for setting $\rho_t = \rho$. We take the same values for all BS parameters as in Figure 1 and set $\rho = \rho_0 = 0, \kappa = 2, \mu = 0.6$ and $\sigma = 0.5$. Then, we compare the delta of a Quanto Put-option (T=1) for different spot prices under dynamic correlation to the corresponding delta with constant correlation in Figure 5. We observe that the delta values using constant correlation are larger than the delta values under dynamic correlation.

5 Calibration to the market data

Here we illustrate the existing advantage of using dynamic correlation for the calibration to the market data. We take the Quanto puts on Deutsche Bank on July 30, 2013. The spot price is $S = 35.9$ Euro, the strike K_j ranges in [32, 33, 34, 35.9, 37, 38]. In the United States, if one invests these puts, the Euro-USD exchange rate is needed to convert the payoff

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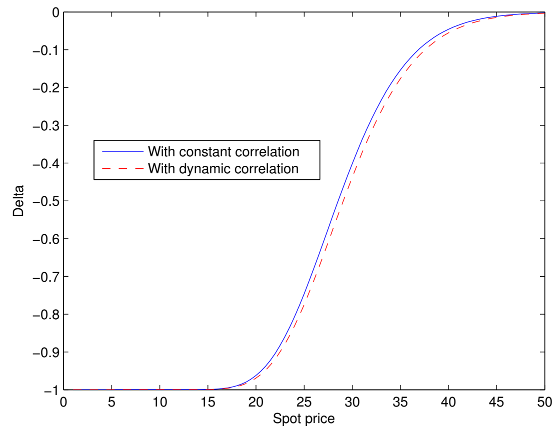

Figure 5: Comparison of the delta hedging with and without dynamic correlation.

σ_S	σ_R	ρ	RMSE
0.32	0.20	-0.04	27×10^{-4}

Table 1: Estimated model parameters using constant correlation.

in USD, which is $R_0 = 1.35$ on July 30, 2013. Furthermore, both interest rates r_f and r_d are 0.05 and the contract is considered for different maturities, $T_i \in [30, 90, 180, 240]$ days.

For each strike and maturity we denote the market price with $P^{Mkt}(\tau_i, K_j)$ and the corresponding model price with $P^{Mod}(\tau_i, K_j)$. We obtain the model parameters by minimizing e.g. the relative mean squares error (RMSE)

$$\frac{1}{N}\sum_{i,j} w_{ij}\frac{(P^{Mkt}(\tau_i, K_j) - P^{Mod}(\tau_i, K_j))^2}{P^{Mkt}(\tau_i, K_j)}, \tag{20}$$

where N is the number of prices and w_{ij} is an optional weight. We estimate the parameters of the model using constant and dynamic correlation, and report the estimated parameters and the errors in Table 1 and 2. We observe that the RMSE using constant correlation is almost three times larger than the RMSE using dynamic correlation. Furthermore, we present the plots of the market prices, the model prices with constant and dynamic correlation in Figure

σ_S	σ_R	ρ_0	κ	μ	σ	RMSE
0.34	0.42	-0.57	2.07	0.49	0.3	9.3×10^{-4}

Table 2: Estimated model parameters using dynamic correlation.

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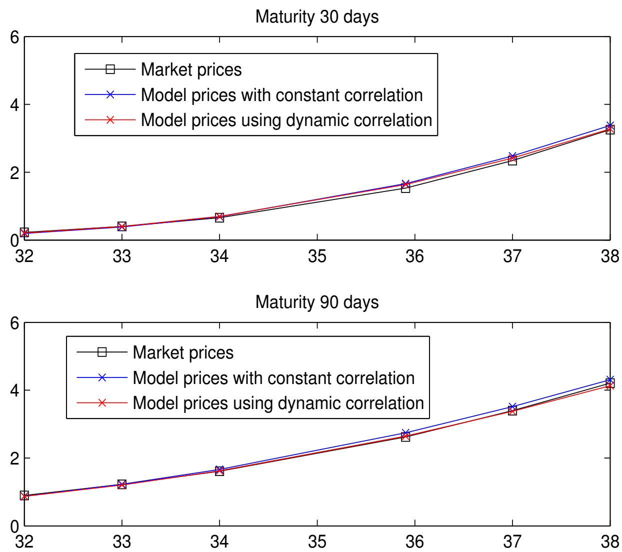

Figure 6: **T = 30 and 60 days**

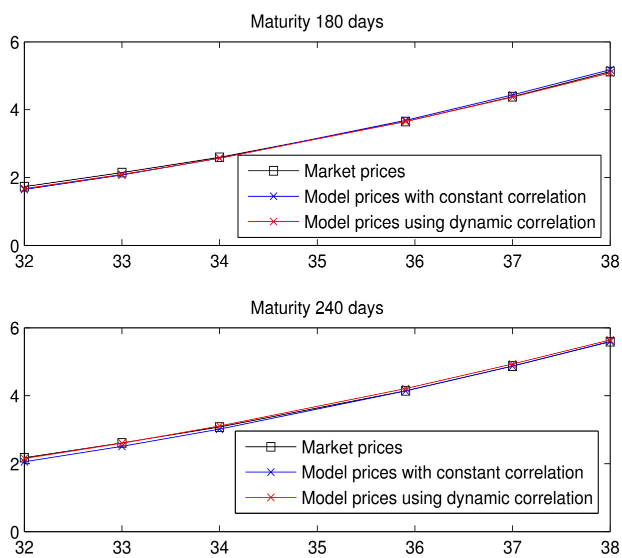

Figure 7: **T = 120 and 240 days**

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6 and 7. Either from the Table 1 and 2 or from the Figure 6 and 7, we directly conclude that the model under dynamic correlation can be better fitted to the market prices.

6 Conclusion

Since financial quantities are correlated in a strongly nonlinear, non-deterministic way, instead of assuming constant correlation we used the dynamic correlation function [6] for pricing Quanto options. We derived the price formula of a Quanto option in a closed form with a dynamic correlation. From the price differences between using constant and dynamic correlation we realize that assuming a constant correlation leads to correlation risk, because the correlation is hardly to be constant in the real world. The hedging strategy with dynamic correlation has also been analyzed. The calibration to market data illustrates that it is more realistic to model correlation dynamically than just using a constant correlation.

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