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Abstract The Fichera theory was first proposed in 1960 by Gaetano Fichera and later developed by Olejnik and Radkevič in 1973. It turned out to be very useful for establishing the well-posedness of initial boundary value problems for parabolic partial differential equations degenerating to hyperbolic ones at the boundary.
In this paper we outline the application of the Fichera theory to interest rates models of Cox-Ingersoll-Ross (CIR) and Chan-Karolyi-Longstaff-Sanders (CKLS) type. For the one-factor CIR model the obtained results are consistent with the corresponding Feller condition.

1 Introduction

The *Fichera theory* focus on the question of appropriate *boundary conditions* (BCs) for parabolic partial differential equations (PDEs) degenerating at the boundary. According to the sign of the *Fichera function* one can separate the outflow or inflow part of the solution at the boundary. Thus, this classical theory indicates whether one has to supply a BC at the degenerating boundary.

In this paper we illustrate the application of the Fichera theory to the Cox-Ingersoll-Ross (CIR) interest rate model and its generalisation, the *Chan-Karolyi-Longstaff-Sanders* (CKLS) model [2]. Here, at the left boundary the interest rate tends to zero and thus the parabolic PDE degenerates to a hyperbolic one. For further applications of Fichera theory to other current models in financial mathematics we refer the interested reader to [4].

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2 The Boundary Value Problem for the Elliptic PDE

We consider an elliptic second order linear differential operator

$$Lu = a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + cu, \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric and induces a semi-definite quadratic form $\xi^\top A\xi \geq 0$ for all $\xi \in \mathbb{R}^n$. Σ denotes a piecewise smooth boundary of the domain Ω. The subset of Σ where the quadratic form vanishes, $\xi^\top A\xi = 0$, will be denoted as Σ_h (hyperbolic part) and the set of points of Σ where the quadratic form remains positive, $\xi^\top A\xi > 0$, is denoted as a Σ_p (parabolic) part. For Σ_h, the hyperbolic part of the boundary Σ_h, we introduce the *Fichera function*

$$b = \sum_{i=1}^{n}\left(b_i - \sum_{k=1}^{n}\frac{\partial a_{ik}}{\partial x_k}\right)\nu_i, \tag{2}$$

where ν_i is the direction cosine of the inner normal to Σ, i.e. it is $\nu_i = \cos(x_i, \mathbf{n_i})$, where $\mathbf{n_i}$ is the inward normal vector at the boundary.

On the *hyperbolic part* of the boundary Σ_h we define according to the sign of the Fichera function the three subsets Σ_0 ($b = 0$ tangential flow), Σ_+ ($b > 0$, outflow) and Σ_- ($b < 0$, inflow), i.e. the boundary $\Sigma = \Sigma_p \cup \Sigma_h$ can be written as a unification of four boundary parts: $\Sigma = \Sigma_p \cup \Sigma_0 \cup \Sigma_+ \cup \Sigma_-$.

Olejnik and Radkevič [7, Lemma 1.1.1] showed that the sign of the Fichera function b at the single points Σ_h does not change under smooth nondegenerate changes of independent variables in a given elliptic operator (1). In [7, Theorem 1.1.1] it is stated that the subsets Σ_0, Σ_+, Σ_- remain invariant under a smooth nonsingular changes of independent variables in the elliptic operator (1).

The *parabolic boundary* Σ_p can be rewritten as a unification of two sets Σ_p^D (Dirichlet BC) and Σ_p^N (Neumann BC). Let us state one simple example.

Example 1 ([6]). The boundary value problem for an elliptic PDE reads

$$\begin{aligned} Lu = f \qquad & \text{on} \quad \Omega \subset \mathbb{R}^n, \\ u = g \qquad & \text{on} \quad \Sigma_- \cup \Sigma_p^D \\ a_{ij}\frac{\partial u}{\partial x_i} n_j = h \qquad & \text{on} \quad \Sigma_p^N \end{aligned}$$

If Σ_p^N is an empty set, we obtain a Dirichlet problem; if Σ_p^D is an empty set, a Neumann problem; if Σ_p^D and Σ_p^N are not empty, the problem is of mixed Dirichlet-Neumann type. Recall that for hyperbolic PDEs one must not supply BCs for outflow boundaries (Σ_+) or boundaries where the characteristics are tangential to the boundary (Σ_0), since this may violate the information that is transported from the interior of the domain.

3 Application to one-factor interest rate Models of CKLS type

We start with an interest rate model in the form of a stochastic differential equation

$$dr = \kappa(\theta - r)\,dt + \sigma r^{\gamma}\,dW, \tag{3}$$

where κ, θ are positive constants, and γ non-negative. This CKLS model [2] is a mean-reversion process with non-constant volatility σr^{γ}. Using the Itô formula for a duplicating portfolio in a risk neutral world one can derive a PDE for the zero-coupon bond price $P(r,\tau)$:

$$\frac{\partial P}{\partial \tau} = \alpha(r,\tau)\frac{\partial^2 P}{\partial r^2} + \beta(r,\tau)\frac{\partial P}{\partial r} - rP, \qquad r>0,\ \tau>0, \tag{4}$$

where $\alpha(r,\tau) = \frac{1}{2}\sigma^2 r^{2\gamma}$, $\beta(r,\tau) = \kappa(\theta - r)$. A closed form formula for this model can be given in special cases, cf. [1]:

a) if $\gamma = 0$, this is the classical Vašíček model with constant volatility.
b) for $\gamma = 0.5$, we get the Cox-Ingersoll-Ross (CIR) model (CIR), [3].

For general γ (CKLS model) there is no closed form formula for the bond price $P(r,\tau)$ and the PDE (4) has to be solved numerically.

The volatility term in (4), for a short rate r tending to zero, is $\alpha(0,\tau) = 0$. Thus the parabolic PDE (4) reduces at $r = 0$ to the hyperbolic PDE

$$\frac{\partial P}{\partial \tau} = \kappa\theta\frac{\partial P}{\partial r}, \qquad \tau > 0. \tag{5}$$

Next, the Fichera function (2) for our model reads

$$b(r) = \beta(r,\tau) - \frac{\partial \alpha(r,\tau)}{\partial r}, \tag{6}$$

and we check the sign of (6) for $r \to 0+$:

- if $\lim_{r\to 0^+} b(r) \geq 0$ (outflow boundary) we must not supply any BCs at $r = 0$.
- if $\lim_{r\to 0^+} b(r) < 0$ (inflow boundary) we have to define BCs at $r = 0$.

Especially for the proposed model we get $b(r) = \kappa(\theta - r) - \sigma^2\gamma r^{2\gamma-1}$ and we we can distinguish the following situations:

a) for $\gamma = 0.5$ (CIR model) $\Rightarrow$ if $\kappa\theta - \sigma^2/2 \geq 0$, we do not need any BCs.
b) for $\gamma > 0.5 \Rightarrow$ if $\kappa\theta \geq 0$, we do not need any BCs.
c) for $\gamma \in (0, 0.5) \Rightarrow$ if $\lim_{r\to 0^+} b(r) = -\infty$, we need BCs.

Remark 1 (Feller condition). The Feller condition guaranteeing a positive interest rate defined by (3) for the one-factor CIR model is $2\kappa\theta > \sigma^2$ and is equivalent with the condition derived from the Fichera theory. If the Feller condition holds, then the Fichera theory states that one must not supply any BC at $r = 0$.

4 A two-factor interest rate Model

We consider a general two-factor model given by the set of two SDEs

$$\mathrm{d}x_1 = (a_1 + a_2 x_1 + a_3 x_2)\,\mathrm{d}t + \sigma_1 x_1^{\gamma_1}\,\mathrm{d}W_1, \tag{7}$$

$$\mathrm{d}x_2 = (b_1 + b_2 x_1 + b_3 x_2)\,\mathrm{d}t + \sigma_2 x_2^{\gamma_2}\,\mathrm{d}W_2, \tag{8}$$

$$Cov[\mathrm{d}W_1, \mathrm{d}W_2] = \rho\,\mathrm{d}t, \tag{9}$$

containing as special cases the Vašíček model ($\gamma_1 = \gamma_2 = 0$) and the CIR model ($\gamma_1 = \gamma_2 = 0.5$). The drift functions are defined as linear functions of the two variables x_1 and x_2. Choosing $a_1 = b_1 = b_2 = 0$ we get two-factor convergence model of CKLS type (in case of general $\gamma_1, \gamma_2 \geq 0$). The variable x_1 models the interest rate of a small country (e.g. Slovakia) before entering the monetary EURO union and the variable x_2 represents the interest rate of the union of the countries (such as the EU).

Applying the standard Itô formula one can easily derive a parabolic PDE

$$\frac{\partial P}{\partial \tau} = \tilde{a}_{11}\frac{\partial^2 P}{\partial x_1^2} + \tilde{a}_{22}\frac{\partial^2 P}{\partial x_2^2} + \tilde{a}_{12}\frac{\partial^2 P}{\partial x_1 \partial x_2} + \tilde{a}_{21}\frac{\partial^2 P}{\partial x_2 \partial x_1} + \tilde{b}_1\frac{\partial P}{\partial x_1} + \tilde{b}_2\frac{\partial P}{\partial x_2} + \tilde{c}P, \tag{10}$$

where $P(x,y,\tau)$ represents the bond price at time τ for interest rates x and y, and

$$\tilde{a}_{11} = \frac{\sigma_1^2 x_1^{2\gamma_1}}{2}, \qquad \tilde{a}_{22} = \frac{\sigma_2^2 x_2^{2\gamma_2}}{2}, \qquad \tilde{a}_{12} = \tilde{a}_{21} = \frac{1}{2}\rho\sigma_1 x_1^{\gamma_1}\sigma_2 x_2^{\gamma_2}$$
$$\tilde{b}_1 = a_1 + a_2 x_1 + a_3 x_2, \qquad \tilde{b}_2 = b_1 + b_2 x_1 + b_3 x_2, \qquad \tilde{c} = -x_1,$$

for $x_1, x_2 \geq 0$, $\tau \in (0,T)$, with initial condition $P(x_1,x_2,0) = 1$ for $x_1, x_2 \neq 0$.

Now, the Fichera function (2) in general reads

$$b(x_1,x_2) = \Big[a_1 + a_2 x_1 + a_3 x_2 - \Big(\sigma_1^2\gamma_1 x_1^{2\gamma_1-1} + \frac{1}{2}\rho\sigma_1 x_1^{\gamma_1}\sigma_2\gamma_2 x_2^{\gamma_2-1}\Big)\Big]\frac{x_1}{\sqrt{1+x_1^2}}$$
$$+ \Big[b_1 + b_2 x_1 + b_3 x_2 - \Big(\frac{1}{2}\rho\sigma_1\gamma_1 x_1^{\gamma_1-1}\sigma_2 x_2^{\gamma_2} + \sigma_2^2\gamma_2 x_2^{2\gamma_2-1}\Big)\Big]\frac{x_2}{\sqrt{1+x_2^2}}.$$

Depending on γ_1 and γ_2, we get the following results:

- For $\gamma_1 = \gamma_2 = 0$ (classical Vašíček model), the Fichera function simplifies to

  $$b(x_1,x_2) = (a_1 + b_1) + (a_2 + b_2)x_1 + (a_3 + b_3)x_2,$$

  and boundary conditions must be supplied, if

  $$\begin{cases} x_1 \leq -\frac{a_1+b_1+(a_3+b_3)x_2}{a_2+b_2} & \text{for} \quad a_2+b_2 \neq 0 \\ x_2 \leq -\frac{a_1+b_1}{a_3+b_3} & \text{for} \quad a_2+b_2=0, a_3+b_3 \neq 0 \\ a_1+b_1 \leq 0 & \text{for} \quad a_2+b_2=0, a_3+b_3=0 \end{cases}.$$

- For $\gamma_1 = \gamma_2 = 0.5$ (CIR model), the Fichera function simplifies to

$$b(x_1,x_2) = \left[a_1 + a_2x_1 + a_3x_2 - \left(\sigma_1^2\gamma_1 + \frac{1}{4}\rho\sigma_1\sigma_2\sqrt{\frac{x_1}{x_2}}\right)\right]\frac{x_1}{\sqrt{1+x_1^2}} + \left[b_1 + b_2x_1 + b_3x_2 - \left(\frac{1}{4}\rho\sigma_1\sigma_2\sqrt{\frac{x_2}{x_1}} + \sigma_2^2\gamma_2\right)\right]\frac{x_2}{\sqrt{1+x_2^2}}$$

  We must supply boundary conditions for $\rho > 0$, and must not for $\rho < 0$. For $\rho = 0$, BCs at $x_2 = 0$ must be posed if $x_1 \le \sigma_1^2\gamma_1/(2a_2) - a_1/a_2$ (assuming $a_2 > 0$, and for $x_1 = 0$, if $x_2 \le \sigma_2^2\gamma_2/(2b_2) - b_1/b_2$ (assuming $b_2 > 0$), otherwise not.
- For the general case $\gamma_1,\ \gamma_2 > 0$, we discuss the boundary $x_2 = 0, x_1 > 0$; due to symmetry, the case $x_2 = 0, x_1 > 0$ follows then by changing the roles of x_1 and x_2, as well as γ_1 and γ_2. For $x_2 = 0$ the Fichera function simplifies to

$$\lim_{x_2\to 0^+} b(x_1,x_2) = \left[a_1 + a_2x_1 - \sigma_1^2\gamma_1x_1^{2\gamma_1-1} - \frac{1}{2}\rho\sigma_1x_1^{\gamma_1}\sigma_2\gamma_2 0^{\gamma_2-1}\right]\frac{x_1}{\sqrt{1+x_1^2}}$$

$$= \begin{cases} \left[a_1 + a_2x_1 - \sigma_1^2\gamma_1x_1^{2\gamma_1-1}\right]\frac{x_1}{\sqrt{1+x_1^2}} & \rho = 0 \\ -\infty & 0 < \gamma_2 < 1, \rho \ne 0 \\ \left[a_1 + a_2x_1 - \sigma_1^2\gamma_1x_1^{2\gamma_1-1} - \frac{1}{2}\rho\sigma_1x_1^{\gamma_1}\sigma_2\right]\frac{x_1}{\sqrt{1+x_1^2}} & \gamma_2 = 1, \rho \ne 0 \\ \left[a_1 + a_2x_1 - \sigma_1^2\gamma_1x_1^{2\gamma_1-1}\right]\frac{x_1}{\sqrt{1+x_1^2}} & \gamma_2 > 1, \rho \ne 0 \end{cases}$$

  For $0 < \gamma_2 < 1$ and $\rho \ne 0$, BCs are needed, if ρ is positive, and BCs must not be posed, if ρ is negative. In all other cases, the sign of b, which defines whether BCs must be supplied or not, depends on a_1, a_2, σ_1, σ_2 and γ_1, see Fig. 1.

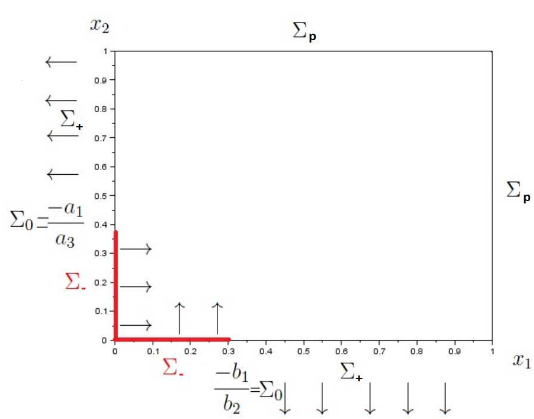

Fig. 1 Boundary decomposition in two-factor CIR model.

5 Numerical Results

Choosing set of parameters $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\gamma = 0.5$ (CIR), we get at $r = 0$ a positive Fichera function $b = \kappa\theta - \sigma^2/2 = 0.02 > 0$. This is equivalent with the statement that the Feller condition is satisfied. According to the Fichera theory, as soon as it is outflow part of boundary, we must not supply BCs. In this example in Fig. 2 and Fig. 4 and Table 1, we intentionally supplied BCs in an 'outflow' situation when we should not in order to illustrate what might happen if one disregards the Fichera theory. In the evolution of the solution we can observe a peak and oscillations close to the boundary. In Fig. 4 we plot with the relative error, which is reported also in the Table 1.

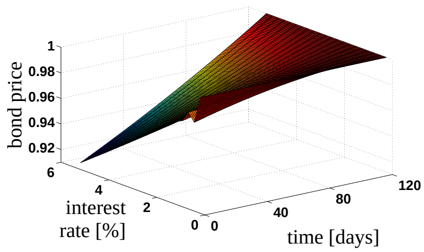

Fig. 2 Numerical solution, Dirichlet BC

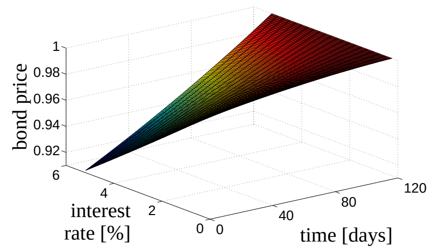

Fig. 3 Numerical solution, without BC

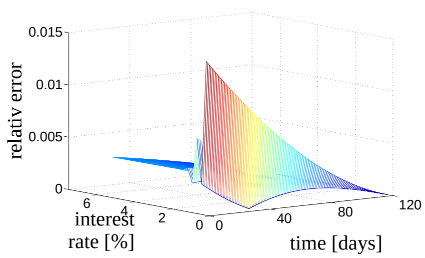

Fig. 4 Relative error, case wit Dirichlet BC

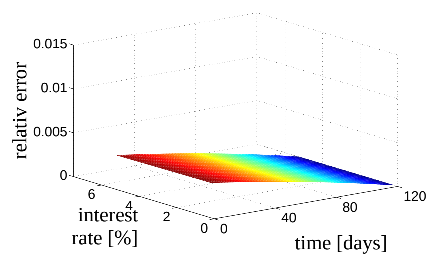

Fig. 5 Relative error, case without BC

Table 1 Relative error, case with BC

time[days]	relative error
1	0.0147
40	0.0079
80	0.0029
120 (maturity)	0

Table 2 Relative error, case without BC

time[days]	relative error
1	0.0039
40	0.0029
80	0.0015
120 (maturity)	0

In our example we used the same parameters, but with or without defining Dirichlet BC. Here, "without BC" means that we used for the numerical BC the limit of the interior PDE for $r \to 0$. The corresponding results are shown on the right hand side, in Fig. 3, Fig. 5 and the relative errors are recorded in Table 1.

For the numerical solution we used the implicit finite difference method from [5]. The reference solution is obtained either as the analytic solution for the CIR model ($\gamma = 0.5$, if Feller condition is satisfied), cf. [1] or in all other cases using a very fine resolution (and suitable BCs). The conditions at outflow boundaries are obtained by studying the limiting behaviour of the interior PDE or simply by horizontal extrapolation of appropriate order. Recall that negative values of the Fichera function (i.e. an inflow boundary) corresponds to a not satisfied Feller condition and may destroy the uniqueness of solutions to the PDE.

6 Conclusion

We discussed one and two factor interest rate models and applied the classical Fichera theory to the resulting degenerate parabolic PDEs. This theory provides highly relevant information how to supply BCs in these applications.

As a next step, we will investigate multi-factor models, which are coupled only via the correlation of the Brownian motion:

$$\begin{aligned} \mathrm{d}x_i &= (a_i + b_i x_i)\mathrm{d}t + \sigma_i x_i^{\gamma_i}\,\mathrm{d}W_i, \\ Cov[\mathrm{d}W_i, \mathrm{d}W_j] &= \rho_{ij}\,\mathrm{d}t, \qquad\qquad i,j = 1,\dots,n. \end{aligned}$$

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