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The Dynamic Correlation Model and its Application to the Heston Model

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Abstract

Correlation plays an essential role in many problems of finance and economics, such as pricing financial products and hedging strategies, since it models the degree of relationship between, e.g., financial products and financial institutions. However, usually for simplicity the correlation coefficient is assumed to be a constant in many models, although financial quantities are correlated in a strongly nonlinear way in the real market.
This work provides a new time-dependent correlation function, which can be easily used to construct dynamically (time-dependent) correlated Brownian motions and flexibly incorporated in many financial models. The aim by using our time-dependent correlation function is to reasonably choose additional parameters to increase the fitting quality on the one hand but also add an economic concept on the other hand. As examples, we illustrate the applications of dynamic correlation in the Heston model. From our numerical results we conclude that the Heston model extended by incorporating time-dependent correlations can provide a better volatility smile than the pure Heston model.

Keywords *Correlation Risk, Dynamic Correlation, Ornstein-Uhlenbeck process, Heston model, Stochastic Correlation Process, Affine Process.*

1 Introduction

Correlation is a well established concept for quantifying interdependence. It plays an essential role in several problems of finance and economics, such as pricing financial products and hedging strategies. For example, in [3] the arbitrage pricing model is based on correlation as a measure for the dependence of assets as well as in portfolio credit models, the default correlation is one fundamental factor of risk evaluation, see [1], [2] and [12].

In most of the financial models, the correlation has been considered as a constant. However, this is not a realistic assumption due to the well-known fact that the correlation is hardly a fixed constant, see e.g. [7] and [13]. For example, in many situations the pure Heston model [9] can not provide enough skews or smiles in the implied volatility surface as market requires, especially for a short maturity. A reason for this might be that deterministically correlated Brownian motions (BMs) of the price process and the variance process is used, as the correlation affects mainly the slope of implied volatility smile. If the correlation is modelled with a time-dependent dynamic function, better skews or smiles will be provided in the implied volatility surface by reasonably choosing additional parameters. Furthermore, compared with the way to extend a model by using time-dependent parameter, e.g., [6, 10] for the Heston model, a time-dependent correlation function adds an economic concept (nonlinear relationship) and its application will be considerably simple.

The key of modelling correlation as a time-dependent function is able to ensure that the boundaries -1 and 1 of the correlation function are not attractive and unattainable for any time. In this work, we build up a reasonable and appropriate time-dependent correlation function, so that one can reasonably choose additional parameters to increase the fitting quality on the one hand but also add an economic concept on the other hand. Thus many problems of finance and economics can be treated under dynamic correlation which is much more realistic than with a constant correlation to model real world phenomena.

The outline of the remaining part is as follows. Section 2 is devoted to a specific dynamic correlation function and its (analytical) computation. In Section 3, we present the concept of dynamically (time-dependent) correlated Brownian motions and its construction. The incorporation of our new dynamic correlation model in the Heston model is illustrated in Section 4. Finally, in Section 5 we conclude this work.

2 The Dynamic Correlation Function

In this section we introduce an appropriate and reasonable dynamic correlation function. Actually, it is in high demand to find such a correlation function which must satisfy the correlation properties: it provides only the values in the interval $(-1, 1)$ for any time; it converges for increasing time. We find the following simple idea: we denote the dynamic correlation with $\bar{\rho}$ and propose simply using

$$\bar{\rho}_t := E\left[\tanh(X_t)\right], \quad t > 0 \tag{1}$$

for the *dynamic correlation function,* where X_t is any mean-reverting process with positive and negative values. For a fixed parameter of X_t, the correlation function $\bar{\rho}_t : [0, t] \to (-1, 1)$ depends only on t. We observe that the dynamic correlation model (1) satisfies the wished properties: first, it is obvious that $\bar{\rho}_t$ takes values only in $(-1, 1)$ for all t. Besides, it converges to a value for increasing time due to the mean reversion of the used process X_t.

X_t in (1) could be any mean-reverting process with positive and negative values. As an example, let X_t be the *Ornstein-Uhlenbeck process* [14]

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t, \quad t \geq 0. \tag{2}$$

We are interested in computing $E[\rho_t]$ for the known parameters in (2). We compute $\bar{\rho}_t = E[\tanh(X_t)]$ as

$$\bar{\rho}_t = E[\tanh(X_t)] = E\left[1 - \mathrm{e}^{-X_t} \cdot \frac{2}{\mathrm{e}^{-X_t} + \mathrm{e}^{X_t}}\right] = 1 - E\left[\mathrm{e}^{-X_t} \cdot \frac{1}{\cosh(X_t)}\right]. \tag{3}$$

We set $g(X_t) = 1/\cosh(X_t)$. Applying the results by Chen and Joslin [4], the expectation in (3) can be found in closed-form expression (up to an integral) as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(u) \cdot E[\mathrm{e}^{-X_t}\mathrm{e}^{iuX_t}]\, du, \tag{4}$$

where $i = \sqrt{-1}$ denotes the imaginary unit and $\hat{g}$ is the Fourier transform of g, which is known analytically by $\hat{g}(u) = \pi/\cosh(\frac{\pi u}{2})$. Denoting $CF(t, u|X_0, \kappa, \mu, \sigma)$ as the characteristic function of X_t, the expectation in (4) can be presented by $CF(t, i + u|X_0, \kappa, \mu, \sigma)$. Thus, we obtain the closed-form expression for $\bar{\rho}_t$:

$$\bar{\rho}_t = 1 - \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh(\frac{\pi u}{2})} \cdot CF(t, i + u|X_0, \kappa, \mu, \sigma) du. \tag{5}$$

The next step is to calculate the expression of $CF(t, i+u|X_0, \kappa, \mu, \sigma)$. X_t is the Ornstein-Uhlenbeck process and its characteristic function $CF(t, u|X_0, \kappa, \mu, \sigma)$ can be obtained analytically, e.g. using the framework of the affine process, see [5]. Then, we only need to substitute $u+i$ for u in the characteristic function of X_t to calculate $CF(t, i+u|X_0, \kappa, \mu, \sigma)$ which is given by

$$CF(t, i+u|X_0, \kappa, \mu, \sigma) = \mathrm{e}^{-A-\frac{B}{2}+iu(A+B)+u^2\frac{B}{2}}, \tag{6}$$

with

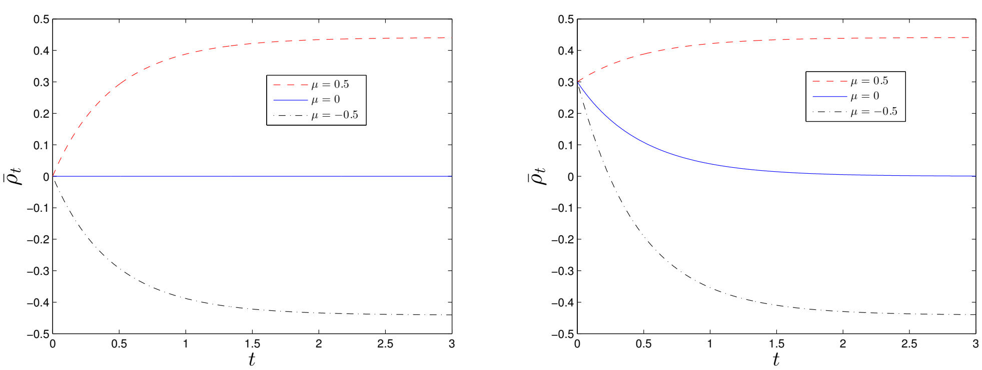
$$A = \mathrm{e}^{-\kappa t}X_0 + \mu(1-\mathrm{e}^{-\kappa t}), \quad B = -\frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa t}) \tag{7}$$

Finally, the dynamic correlation function $\bar{\rho}_t$ can be computed by

$$\bar{\rho}_t = 1 - \frac{\mathrm{e}^{-A-\frac{B}{2}}}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh(\frac{\pi u}{2})} \cdot \mathrm{e}^{iu(A+B)+u^2\frac{B}{2}}du, \tag{8}$$

where A and B are defined in (7). In fact, X_0 in A is $\operatorname{artanh}(\bar{\rho}_0)$.

To illustrate the role of each parameter in (8), we plot $\bar{\rho}_t$ for a couple of parameters. First in Figure 1, we let $\kappa = 2$ and $\sigma = 0.5$ and display $\bar{\rho}_t$ with different values of μ, which is set to be $0.5,\ 0$ and -0.5, respectively. Obviously, μ determines the limit of



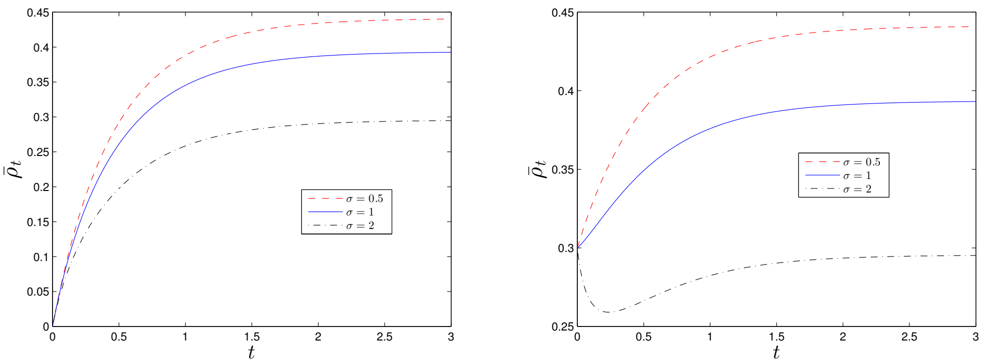
(a) $\bar{\rho}_0 = 0$

(b) $\bar{\rho}_0 = 0.3$

Figure 1: Dynamic correlation $\bar{\rho}_t$ for varying μ ($\kappa = 2$ and $\sigma = 0.5$).

$\bar{\rho}_t$ for a long time t. However, μ is not the exact limiting value. Considering Figure 1(a) where the initial value of the correlation function is 0, we see that $\bar{\rho}_t$ is increasing to a value around $\mu = 0.5$ and decreasing to a value around $\mu = 0.5$ as t goes on, when $\mu = 0.5$ and -0.5, respectively. Besides, for $\mu = \bar{\rho}_0 = 0$ we observe that the correlation function $\bar{\rho}_t$ yields always 0 which is the same as constant correlation $\rho = 0$. We change $\bar{\rho}_0$ to be 0.3 and keep the value of all other parameters, the curves of $\bar{\rho}_t$ can be found in Figure 1(b).

Next, we fix $\kappa = 2$ and $\mu = 0.5$ and show $\bar{\rho}_t$ for the varying $\sigma = 0.5,\ 1$ and 2 in Figure 2. Obviously, σ shows the magnitude of variation from the value around $\mu = 0.5$. In



(a) $\bar{\rho}_0 = 0$ (b) $\bar{\rho}_0 = 0.3$

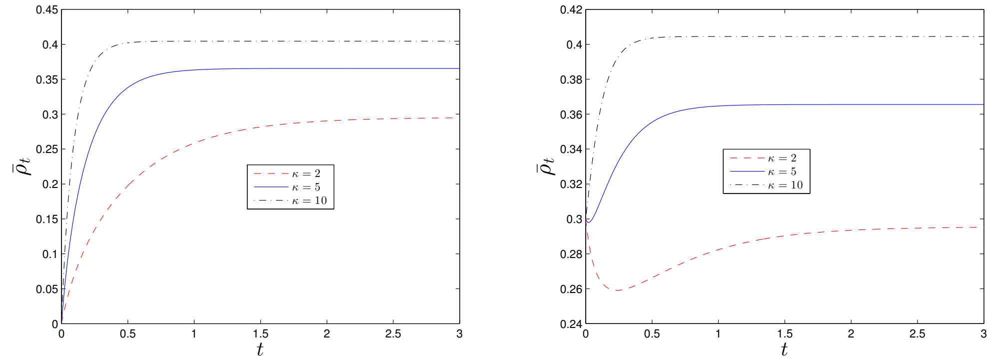
Figure 2: Dynamic correlation $\bar{\rho}_t$ for varying σ ($\kappa = 2$ and $\mu = 0.5$).

Figure 2(a) we see, the larger the value of σ is, the stronger the deviations of $\bar{\rho}_t$ is from the value around $\mu = 0.5$. More interesting is that $\bar{\rho}_t$ first decreases until $t \approx 0.25$, then increases and tends to a value, see Figure 2(b) where $\bar{\rho}_0 = 0.3$ and $\sigma = 2$.

Again, in order to illustrate the role of κ, we set $\mu = 0.5,\ \sigma = 2$ and vary the value of κ, see Figure 3. From Figure 3(a) it is easy to observe that κ represents the speed of $\bar{\rho}_t$ tending to its limit. Especially, as we have seen in Figure 2(b), the curve is more unstable for $\kappa = 2$ and $\sigma = 2$ in Figure 3(b). However, if σ remains with the same value while the value of κ is increased, we can see that curves of $\bar{\rho}_t$ become more stable and tend straightly to its limit. If one incorporates the dynamic correlation function (8) to a financial model, the parameter $\bar{\rho}_0,\ \kappa,\ \mu,$ and σ could be estimated by fitting the model to market data.

3 Dynamically correlated Brownian motions

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$, satisfying the usual conditions, see e.g. [11]. At a time $t > 0$, the correlation coefficient of two



(a) $\bar{\rho}_0 = 0$ (b) $\bar{\rho}_0 = 0.3$

Figure 3: Dynamic correlation $\bar{\rho}_t$ for varying κ ($\mu = 0.5$ and $\sigma = 2$).

Brownian motions (BMs) W_t^1 and W_t^2 is defined as

$$\rho_t^{1,2} = \frac{E\left[W_t^1 W_t^2\right]}{t}. \tag{9}$$

If we assume that $\rho_t^{1,2}$ is constant, $\rho_t^{1,2} = \rho^{1,2}$ for all $t > 0$, say W_t^1 and W_t^2 are correlated with the constant $\rho^{1,2}$.

3.1 Definition of dynamically correlated Brownian Motions

In the following, we show how to define the dynamically correlated Brownian motions, where the correlation is not same for each instant of time. Let $(\Delta_n)_{n\in\mathbb{N}} := \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ be a partition of $[0, t]$ with the mesh $\|(\Delta_n)\| := \max\limits_{1\le i\le n}(t_i - t_{i-1})$, we calculate

$$\begin{aligned} E\left[W_t^1 W_t^2\right] &= E\left[\left((W_{t_n}^1 - W_{t_{n-1}}^1) + (W_{t_{n-1}}^1 - W_{t_{n-2}}^1) + \cdots + (W_{t_1}^1 - W_{t_0}^1)\right) \cdot \left((W_{t_n}^2 - W_{t_{n-1}}^2)\right.\right. \\ &\quad \left.\left. + (W_{t_{n-1}}^2 - W_{t_{n-2}}^2) + \cdots + (W_{t_1}^2 - W_{t_0}^2)\right)\right] = E\left[\sum_{i=1}^{n}(W_{t_i}^1 - W_{t_{i-1}}^1)(W_{t_i}^2 - W_{t_{i-1}}^2)\right] \\ &= E\left[\sum_{i=1}^{n}\rho_{t_i - t_{i-1}}^{1,2}(t_i - t_{i-1})\right] \stackrel{\substack{\|\Delta_n\|\to 0\\ n\to\infty}}{=} E\left[\int_0^t \rho_s^{1,2} ds\right]. \end{aligned} \tag{10}$$

Therefore, we give the definition of dynamically correlated BMs.

Definition 3.1. *Two Brownian motions W_t^1 and W_t^2 are called dynamically correlated with correlation function ρ_t, if they satisfy*

$$E\left[W_t^1 W_t^2\right] = \int_0^t \rho_s ds, \tag{11}$$

where $\rho_t : [0,t] \to [-1,1]$. The average correlation of W_t^1 and W_t^2, ρ_{Av}, is given by $\rho_{Av} := \frac{1}{t}\int_0^t \rho_s ds$.

3.2 Construction of dynamically correlated Brownian Motions

We consider first the two-dimensional case and let ρ_t be a correlation function. For two independent BMs W_t^1 and W_t^3 we define

$$W_t^2 = \int_0^t \rho_s dW_s^1 + \int_0^t \sqrt{1-\rho_s^2}\, dW_s^3, \tag{12}$$

with the symbolic expression

$$dW_t^2 = \rho_t dW_t^1 + \sqrt{1-\rho_t^2}\, dW_t^3. \tag{13}$$

It can be easily verified that W_t^2 is a BM and correlated with W_t^1 dynamically by ρ_t. Besides, the covariance matrix and the average correlation matrix of $\mathbb{W}_t = (W_t^1, W_t^2)$ can be determined, given by

$$\begin{pmatrix} t & \int_0^t \rho_s ds \\ \int_0^t \rho_s ds & t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \frac{1}{t}\int_0^t \rho_s ds \\ \frac{1}{t}\int_0^t \rho_s ds & 1 \end{pmatrix},$$

respectively.

The construction above could be also generalized to n-dimensions. We denote a standard n-dimensional BM by $\mathbb{Z}_t = (Z_{1,t}, ..., Z_{n,t})$ and the matrix of dynamic correlations $\mathcal{R}_t = (\rho_t^{i,j})_{1<i,j<n}$ which has the Cholesky decomposition for each time t, $\mathcal{R}_t = \mathbb{A}_t \mathbb{A}_t^\top$ with $\mathbb{A}_t = (a_t^{i,j})_{1<i,j<n}$. We define a new n-dimensional process $\mathbb{W}_t = (W_{1,t}, ..., W_{n,t})$ by

$$W_{i,t} = \sum_{j=1}^n a_t^{ij}\, dZ_{j,t}, \quad i = 1, \cdots, n. \tag{14}$$

We can easily verify that $\mathbb{W}_t$ statisfies the following properties:

- $\mathbb{W}_0 = \mathbf{0}$ and the paths are continuous with probability 1.
- The increments $\mathbb{W}_{t_1} - \mathbb{W}_{t_0}$ and $\mathbb{W}_{t_2} - \mathbb{W}_{t_1}$ are independent for $0 \leq t_0 < t_1 < t_2 < t$.
- For $0 \leq s < t$, the increment $\mathbb{W}_t - \mathbb{W}_s$ is multivariate normally distributed with mean zero and covariance matrix $\Sigma : \mathbb{W}_t - \mathbb{W}_s \sim N(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} t-s & \int_s^t \rho_u^{1,2} du & \cdots & \int_s^t \rho_u^{1,n} du \\ \int_s^t \rho_u^{2,1} du & t-s & \cdots & \int_s^t \rho_u^{2,n} du \\ \vdots & \vdots & \ddots & \vdots \\ \int_s^t \rho_u^{n,1} du & \int_s^t \rho_u^{n,2} du & \cdots & t-s \end{pmatrix}.$$

We call the process $(\mathbb{W}_t)_{t\geq 0}$ *a n-dimensional dynamically correlated Brownian motion,* with the matrix $\mathcal{R}_t$.

4 Dynamic Correlation in the Heston Model

As mentioned before, in many situations the pure Heston model has a limitation on presenting properly a volatility smile. For this problem, several time-dependent Heston models have been proposed for a good fitting to implied volatilities, e.g. [10] and [6]. In this section, we show how to incorporate our time-dependent correlation function into the Heston model.

4.1 Incorporating dynamic correlations

Heston's stochastic volatility model is specified as

$$\begin{aligned} dS_t &= \mu_S S_t dt + \sqrt{\nu_t}\, S_t\, dW_t^S, & (15) \\ d\nu_t &= \kappa_\nu(\mu_\nu - \nu_t)dt + \sigma_\nu \sqrt{\nu_t}\, dW_t^\nu, & (16) \end{aligned}$$

where (15) is the price of the spot asset, (16) is the volatility (variance) and W_t^S and W_t^ν are correlated with a constant $\rho_{S\nu}$. To incorporate the time-dependent correlations, we assume that dS_t and $d\nu_t$ are correlated by the time-dependent correlation function $\bar{\rho}_t$ instead of the constant correlation $\rho_{S\nu}$. The extended Heston model with dynamic correlation $\bar{\rho}$ is specified as

$$\begin{aligned} dS_t &= \mu_S S_t dt + \sqrt{\nu_t}\, S_t\, dW_t^1, & (17) \\ d\nu_t &= \kappa_\nu(\mu_\nu - \nu_t)dt + \sigma_\nu \sqrt{\nu_t}(\bar{\rho}_t\, dW_t^1 + \sqrt{1-\bar{\rho}_t^2}\, dW_t^2), & (18) \end{aligned}$$

where W_t^1 and W_t^2 are independent. Applying Itô's lemma and no-arbitrage arguments yields [9]

$$\begin{aligned}\frac{1}{2}\nu^2 S^2\frac{\partial^2 U}{\partial S^2} + \bar{\rho}_t\sigma_\nu\nu S\frac{\partial^2 U}{\partial S\partial\nu} + \frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 U}{\partial\nu^2} + rS\frac{\partial U}{\partial S}\\ + [\kappa_\nu(\mu_\nu-\nu) - \tilde{\lambda}(S,\nu,\bar{\rho},t)\nu]\frac{\partial U}{\partial\nu} - rU + \frac{\partial U}{\partial t} = 0,\end{aligned} \tag{19}$$

where $\bar{\rho}_t$ is defined in (8) but with the parameter $\bar{\rho}_0,\ \kappa_\rho,\ \mu_\rho,$ and ν_ρ. It is worth mentioning that the market price of volatility risk depends also on the dynamic correlation, which could be written as $\tilde{\lambda}(S,\nu,\bar{\rho}_t,t)$. This means, the price of correlation risk embedding in the price of volatility risk has been considered.

We consider e.g. a European call option with strike price K and maturity T in the Heston model

$$C(S,\nu,t,\bar{\rho}_t) = SP_1 - KP(t,T)P_2, \quad \tau = T-t, \tag{20}$$

where $P(t,T)$ is the discount factor and both probabilities P_1, P_2 must satisfy the PDE (19) as well as their characteristic functions, $f_1(S,\nu,\bar{\rho}_t,\phi,t)$ and $f_2(S,\nu,\bar{\rho}_t,\phi,t)$

$$f_j(S,\nu,\bar{\rho}_t,\phi,t) = \mathrm{e}^{C_j(\tau,\phi)+D_j(\tau,\phi)\nu+i\phi\ln S}, \quad j=1,2. \tag{21}$$

By substituting this functional form (21) into the PDE (19) we can obtain the following ordinary differential equations (ODEs) for the unknown functions C and D :

$$-\frac{1}{2}\phi^2 + \bar{\rho}_t\sigma_\nu\phi i D_j + \frac{1}{2}\sigma_\nu^2 D_j^2 + u_j\phi i - b_j D_j + \frac{\partial D_j}{\partial t} = 0, \tag{22}$$

$$r\phi i + \kappa_\nu\mu_\nu D_j + \frac{\partial C_j}{\partial t} = 0, \tag{23}$$

with the initial conditions $C_j(0,\phi) = D_j(0,\phi) = 0$

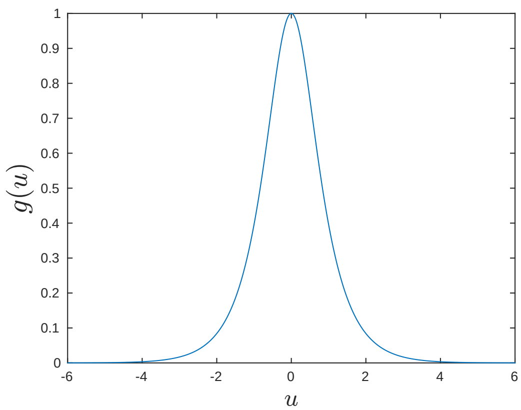
$$u_1 = 0.5, \quad u_2 = -0.5, \quad b_1 = \kappa_\nu + \lambda - \bar{\rho}_t\sigma_\nu \quad \text{and} \quad b_2 = \kappa_\nu + \lambda, \tag{24}$$

where

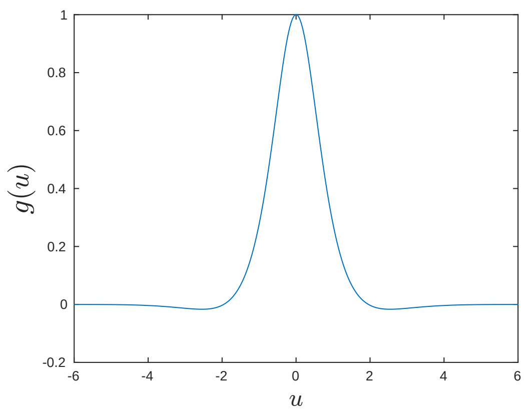
$$\bar{\rho}_t = 1 - \frac{\mathrm{e}^{-A(t)-\frac{B(t)}{2}}}{2}\int_{-\infty}^{\infty}\underbrace{\frac{1}{\cosh(\frac{\pi u}{2})}\cdot\mathrm{e}^{iu(A(t)+B(t))+u^2\frac{B(t)}{2}}}_{:=g(u)}\,du, \tag{25}$$

with $A(t) = \mathrm{e}^{-\kappa_\rho t}\mathrm{artanh}(\bar{\rho}_0) + \mu_\rho(1-\mathrm{e}^{-\kappa_\rho t}),\ B(t) = -\frac{\sigma_\rho^2}{2\kappa_\rho}(1-\mathrm{e}^{-2\kappa_\rho t})$.

Obviously, (22)-(23) can not be solved analytically. Therefore, we need to find an efficient way to compute the option price numerically. We firstly generate the dynamic correlations using (25). We observe that $g(u)$ is a symmetric function about $u = 0$ and vanishes (approaches zero) for a sufficiently large absolute value of u, see Figure 4. For these two reasons, the numerical integration in (25) can be done rapidly. Next we use an explicit Runge-Kutta method, the matlab routine `ode45`, to obtain



(a) $t = 0.1$



(b) $t = 10$

Figure 4: $g(u)$ under $\rho_0 = 0.3, \kappa_\rho = 2, \mu_\rho = -0.8, \sigma_\rho = 0.1$.

C and D in (22)-(23) and thus also the characteristic functions (21). Finally, we employ the COS method [8] to obtain the option price $C(S, \nu, t, \bar{\rho})$ in (20). Thanks to the COS method, although we solved that ODE system numerically, the time for obtaining European option prices is less than 0.1 seconds so that a calibration can be performed.

4.2 Calibration of the Heston Model under dynamic correlation

In this section we calibrate the Heston model extended by our time-dependent correlation function to the real market data (Nikk300 index Call-options on July 16, 2012) and compare these to the pure Heston model [9] and the time-dependent Heston model [10].

We consider a set of N maturities $T_i,\ i = 1, \dots, N$ and a set of M strikes $K_j,\ j = 1, \dots, M$. Then for each combination of maturity and strike we have a market price $V^M(T_i, K_j) = V_{ij}^M$ and a corresponding model price $V(T_i, K_j; \Theta) = V_{ij}^\Theta$ generated by using (20). We choose the relative mean error sum of squares (RMSE) for the loss function $\frac{1}{M\times N}\sum_{i,j}\frac{(V_{ij}^M-V_{ij}^\Theta)^2}{V_{ij}^M}$, which can be minimized to obtain the parameter estimates

$$\hat{\Theta} = \arg\min \frac{1}{M\times N}\sum_{i,j}\frac{(V_{ij}^M - V_{ij}^\Theta)^2}{V_{ij}^M}. \tag{26}$$

For the optimization we restrict $\bar{\rho}_0$ to the interval $(-1,1)$ but not the value of μ_ρ. Since it is not directly the limit of the correlation function but the mean reversion of the Ornstein-Uhlenbeck process, thus, it could take any value in $\mathbb{R}$. Our experiments showed us, that it is enough and appropriate to restrict μ_ρ to the interval $[-4,4]$.

We state our estimated parameters and the estimation error for the pure Heston model (abbr. PH), the Heston model under our time-dependent correlations (CH), the time-dependent Heston model by Mikhailov and Ngel [10] (MN) in Table 1, 2 and 3, respectively. We see that the estimation error using the CH model is distinctly less

The pure Heston model					
$\hat{\nu}_0$	$\hat{\kappa}_\nu$	$\hat{\mu}_\nu$	$\hat{\sigma}_\nu$	$\hat{\rho}$	Estimation Error
0.029	4.746	0.053	1.108	-0.355	1.10×10^{-3}

Table 1: The estimated parameters for the pure Heston model using Call-options on the Nikk300 index on July 16, 2012.

The extended Heston model by using our time-dependent correlation function								
$\hat{\nu}_0$	$\hat{\kappa}_\nu$	$\hat{\mu}_\nu$	$\hat{\sigma}_\nu$	$\hat{\bar{\rho}}_0$	$\hat{\kappa}_\rho$	$\hat{\mu}_\rho$	$\hat{\sigma}_\rho$	Estimation Error
0.027	5.542	0.055	1.224	-0.165	5.333	-0.752	0.434	2.38×10^{-4}

Table 2: The estimated parameters for the Heston model under time-dependent correlations using Call-options on the Nikk300 index on July 16, 2012.

The time-dependent Heston model by Mikhailov and Ngel						
Maturity	$\hat{\nu}_0$	$\hat{\kappa}_\nu$	$\hat{\mu}_\nu$	$\hat{\sigma}_\nu$	$\hat{\rho}$	Estimation Error
1/12	0.025	2.749	0.095	1.172	-0.201	1.78×10^{-4}
1/4	0.012	2.936	0.076	0.524	-0.411	2.45×10^{-5}
1/2	0.011	2.890	0.058	0.592	-0.430	1.14×10^{-5}
1	0.001	2.911	0.051	0.558	-0.389	4.28×10^{-6}

Table 3: The estimated parameters for the time-dependent Heston model by Mikhailov and Ngel using Call-options on the Nikk300 index on July 16, 2012.

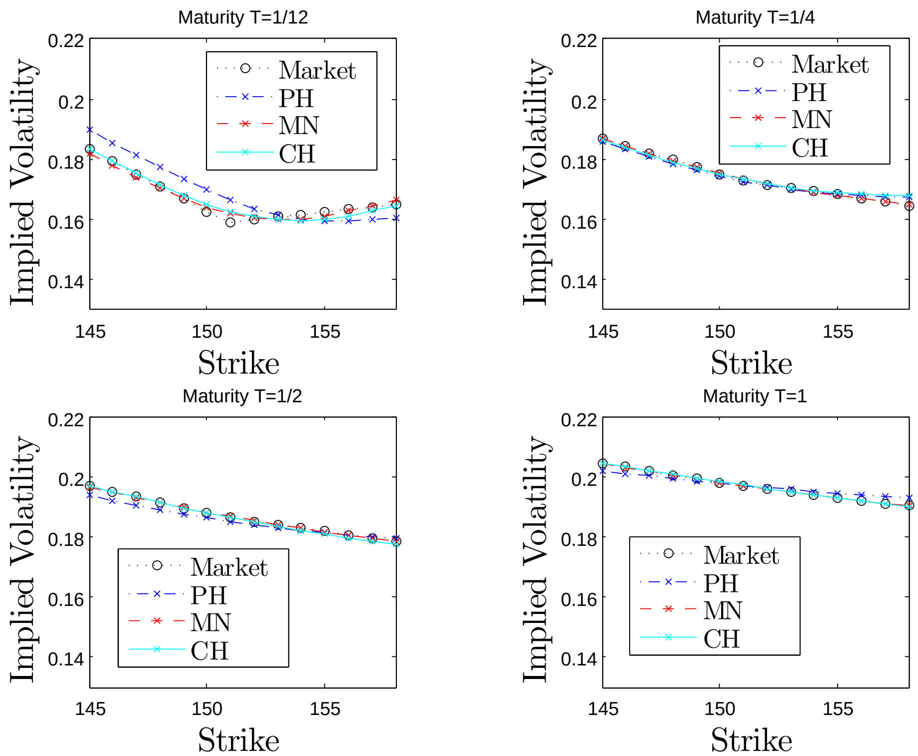


Figure 5: The comparison of implied volatilities for all the models to the market volatilities of the Call-options on the Nikk300 index on July 16, 2012, where the spot price is 150.9.

than the error using the PH model and almost the same to the error (sum of errors for each maturity) under the MN model. To illustrate more clearly, for each maturity we compare the implied volatilities for all the models to the market volatilities in Figure 5. We can observe that the implied volatilities for the CH model are much more closer to the market volatilities than the implied volatilities for the PH model, especially has the better volatility smile for the short maturity $T = 1/12$. Comparing to the MN model, the implied volatilities for our model are almost the same to them, however, our CH model has an economic interpretation, namely the correlation is nonlinear and time-dependent as market required. We conclude that the Heston model extended by incorporating our time-dependent correlations can provide better volatility smiles compared to the pure Heston model. The time-dependent correlation function can be easily and directly introduced into the financial models.

5 Conclusion

In this work, we first investigated the dynamically (time-dependent) correlated Brownian motions and its construction. Furthermore, we proposed a new dynamic correlation function which can be easily incorporated into another financial model. The aim by using our dynamic correlation function is to reasonably choose additional parameters to increase the fitting quality on the one hand side but also add an economic concept on the other hand side.

As an example for the application, we incorporated our time-dependent correlation function into the Heston model. An experiment on estimation of the models using real market data has been provided. The numerical results show that the Heston model extended by using our time-dependent correlation function provides better volatility smiles compared to the pure Heston model. Besides, this time-dependent correlation function could be easily and directly imposed to the financial models and thus it is preferred to use instead of a constant correlation.

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