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Modelling Stochastic Correlation

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Abstract This work deals with the stochastic modelling of correlation in finance. It is well known that the correlation between financial products, financial institutions, e.g., plays an essential role in pricing and evaluation of financial derivatives. Using simply a constant or deterministic correlation may lead to correlation risk, since market observations give evidence that the correlation is hardly a deterministic quantity.

The approach of modelling the correlation as a hyperbolic function of a stochastic process was proposed recently by Teng et al. in [14]. Here, we review this novel concept and generalize this approach to derive stochastic correlation processes (SCP) from a hyperbolic transformation of the modified Ornstein-Uhlenbeck process.
We determine a transition density function of this SCP in closed form which could be used easily to calibrate SCP models to historical data.

As an illustrating example of our new approach, we compute the price of a quantity adjusting option (Quanto) and discuss concisely the effect of considering stochastic correlation on pricing the Quanto.

1 Introduction

Correlation is a well established concept for quantifying the relationship between financial assets. It plays an essential role in several financial applications, e.g. the arbitrage pricing model [3] is based on correlation as a measure for the dependence of assets. Also in portfolio credit models, the default correlation is one fundamental factor of risk evaluation, see e.g. [2] and [13].

For two random variables X_1 and X_2 with finite *variances*, the *correlation* of them is defined as

$$\rho_{1,2} = \mathrm{Corr}(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}\,, \tag{1}$$

with the *covariance*

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}\big[(X_1 - \mu_1)(X_2 - \mu_2)\big], \tag{2}$$

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where μ_i and σ_i are the *expectation* and *standard deviation* of X_i, $i = 1,2$. Here $\rho_{1,2}$ denotes a coefficient number in the interval $[-1,1]$. The boundaries -1 and 1 will be reached if and only if X_1 and X_2 are indeed linearly related. The greater the absolute value of $\rho_{1,2}$ the stronger the dependence between X_1 and X_2 is.

Generally, there are several disadvantages or fallacies of the correlation concept (1), we state only some of them:

- If the random variables X_1 and X_2 are independent, then it follows $\rho_{1,2} = 0$. However, the converse implication does not hold, since in (1) only the two first moments are included. For example, we compute $\rho_{1,2} = 0$ for $X_2 = X_1^2$. Indeed, X_1 and X_2 depend even almost perfectly on each other. This illustrates that the correlation coefficient only recognizes linear dependencies between random variables.
- Correlation is invariant under strictly increasing linear transformations, but, in contrast to Copula methods, not invariant under nonlinear strictly increasing transformations. For example, in general the correlation of the random variables X_1 and X_2 does not equal the correlation of the random variables $\ln X_1$ and $\ln X_2$, i.e. after a transformation of the financial data the correlation may change.
- Usually, the given marginal distributions and pairwise correlations of a random vector cannot determine its joint distribution.
- Finally, as stated above, the variances of the two random variables X_1 and X_2 has to be finite. This assumptions is not fulfilled for every standard distribution, e.g. the Student's t-distribution with $\nu \leq 2$ possess an infinite variance.

For more detailed information about the disadvantages or fallacies we refer to [7]. Although this concept of correlation (1) to measure dependence inherits a couple of limitations, it has been widely applied in financial applications.

In financial markets, the first problem of using a correlation concept is the *observability*. Unlike price, exchange rate and so on, the correlation cannot be observed directly in the market and can only be measured in the context of a model. The easiest estimator of the correlation is the sample correlation coefficient. Given a series of N measurements of X_1 and X_2, which are observable quantities in the market, and denoting the measurements by $x_{1,j}$ and $x_{2,j}$, $j = 1,2,\dots,N$, the *sample coefficient correlation* reads

$$\hat{\rho}_{12} = \frac{\sum_{j=1}^{N}(x_{1,j}-\bar{\mu}_1)(x_{2,j}-\bar{\mu}_2)}{\sqrt{\sum_{j=1}^{N}(x_{1,j}-\bar{\mu}_1)^2\sum_{j=1}^{N}(x_{2,j}-\bar{\mu}_2)^2}}\,, \tag{3}$$

where $\bar{\mu}_1$ and $\bar{\mu}_2$ are the *sample means* of X_1 and X_2.

In financial models, stochastic processes are used quite often to model data series, like price, interest rate and exchange rate. The dependence between the series is given by the *correlated Brownian motions*. Two Brownian motions W_1 and W_2 are correlated by the symbolic notion

$$dW_{1,t}\,dW_{2,t} = \rho_{1,2}\,dt. \tag{4}$$

For example, in the multivariate Black-Scholes model, the correlation of the log-returns is used as a measure of the dependence between asset processes. A further example of coupled stochastic processes is the *quantity adjusting option (Quanto)* pricing in the Black-Scholes model:

$$\begin{cases} dS_t = \mu_S S_t\,dt + \sigma_S S_t\,dW_t^S \\ dR_t = \mu_R R_t\,dt + \sigma_R R_t\,dW_t^R, \end{cases} \tag{5}$$

with positive constants μ_S, μ_R, σ_S and σ_R. The first *stochastic differential equation (SDE)* describes the price of the traded asset in a currency A. The second SDE is used to model the exchange rate between currency A and another currency B. Besides, the Brownian motions are assumed to be correlated by a constant correlation $\rho \in [-1,1]$ which is a measure of co-movements between S_t and R_t.

As we explained above, the constant correlation coefficient defined by (1) only captures linear relationships between X_1 and X_2. Therefore, in the model (5) a linear dependence between S_t and R_t is assumed. However, from the market we realize that there is often a non-linear dependence between returns. Specifically, a constant correlation means that the two return processes are jointly stationary which is generally not true in the real world. Thus, the dependence can be hardly modelled by a fixed constant, i.e. the constant correlation may not be an appropriate measure of co-dependence. Using constant ("wrong") correlation may result some 'correlation risk'. There exist already some works which show that the correlation should not be constant and even changes over a small time interval as the volatility, see e.g. [12]. Several approaches generalize the constant correlation to a time-varying and stochastic concept, like *Dynamic Conditional Correlation model* in [11], *Local correlation models* see e.g. [6] and the *Wishart autoregressive process* proposed by Gourieroux [4] that guarantees the positive definiteness of the variance-covariance matrix.

In fact, either implied correlation in the context of a model or historical correlation from the market data show us that the correlation should be time-varying and behave like a stochastic process. To illustrate this statement, we make an example of historical correlations between *S&P 500 index* and *Euro/US-Dollar exchange rate* on a daily basis. We use $\bar{s}$ and $\bar{r}$ to denote the daily return series of S&P 500 and Euro/US-Dollar exchange rate and fix a time window n_T, e.g. $n_T = 60$ for 60-day historical correlation. At time t, using the n_T times most recent daily returns, the correlation at time t is given by the following estimator

$$\hat{\rho}_t = \frac{\sum_{j=1}^{n_T}(\hat{s}_{t-j} - \frac{1}{n_T}\sum_{j=1}^{n_T}\hat{s}_{t-j})(\hat{r}_{t-j} - \frac{1}{n_T}\sum_{j=1}^{n_T}\hat{r}_{t-j})}{\sqrt{\sum_{j=1}^{n_T}(\hat{s}_{t-j} - \frac{1}{n_T}\sum_{j=1}^{n_T}\hat{s}_{t-j})^2 \sum_{j=1}^{n_T}(\hat{r}_{t-j} - \frac{1}{n_T}\sum_{j=1}^{n_T}\hat{r}_{t-j})^2}} \,. \tag{6}$$

Then we roll it to the time $t+1$, and so on to obtain a series of correlations through the time. The 15-day, 30-day and 60-day historical correlations are displayed in Figure 1. We observe that the longer a time window

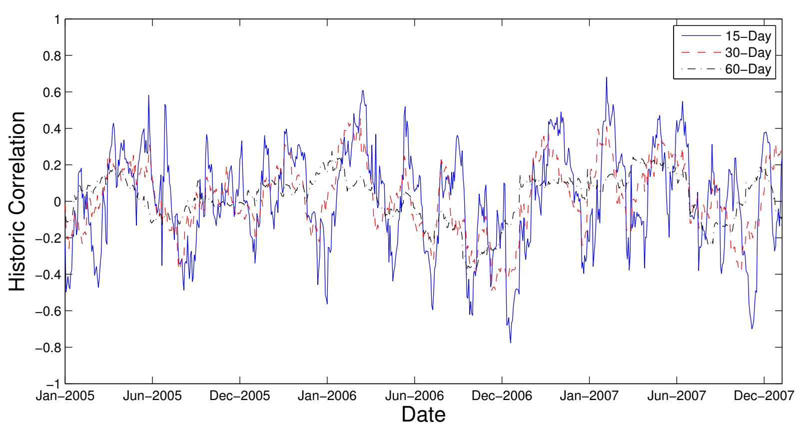
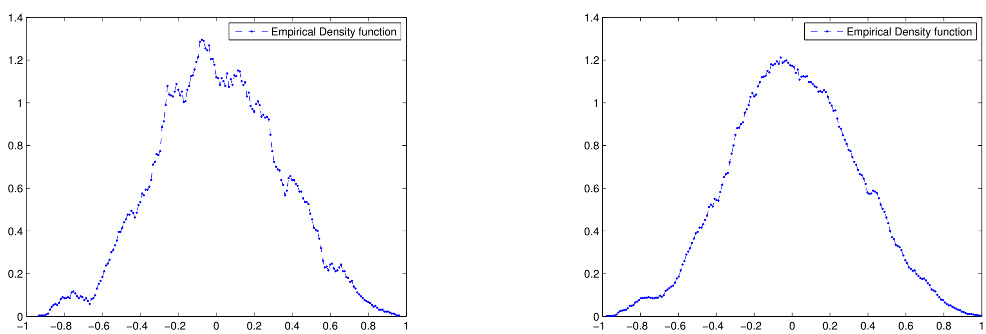


Fig. 1: Historical Correlation between S&P 500 and Euro/US-Dollar exchange rate.[1]

(the value of n_T) the less volatile a historical correlation is. In Figure 1, the 15-day historical correlation is more variable than the 30-day historical correlation which is again more variable than the 60-day correlation. With a longer averaging period a *long-term correlation* is calculated. If we choose $n_T = 10$ or 15 days, the estimated estimated correlation for each time t using (6), could be seen as a *short-term correlation* of the current market phenomena whose immediate past returns are used for the estimation. It is worthwhile noting that the events, especially, some extreme events in a time window will affect the correlation which would be estimated in the following time windows, even has a delayed effect on the long-term correlation.

If one assumes that the phenomena in the past could be a reflection of the future, one would like to use the historical correlation as a forecast for the future. It could be a better way for correlation forecasting, if one describes the correlation using a *mean-reverting stochastic process*. Besides, modelling correlation as a stochastic process, not only the variation of the short-term correlation can be reflected, also the attributes of long-term correlation is determined by the long-term parameter values, like long-term mean value and mean reversion speed.

To see more properties, which a mean-reverting stochastic process should have to be a SCP, we plot its empirical density functions in Figure 2 using different bandwidths. We refer to [1] for details about the estimation of density function from historical data.



(a) bandwidth $1/40$ (b) bandwidth $1/30$

Fig. 2: Empirical Density function of the historical correlation between S&P 500 and Euro/US-Dollar exchange rate.

From studying the empirical density functions we require that the *stochastic correlation process* should satisfy the following properties:

(i) only takes values in the interval $(-1, 1)$,
(ii) varies around a mean value,
(iii) the probability mass tends to zero at the boundaries -1, $+1$.

One stochastic correlation process was proposed by van Emmerich [16], including a restriction on the parameter range to ensure that the boundaries -1 and 1 of the correlation process are not attractive and unattainable. A modified *Jacobi process* is suggested in [8] modelling stochastic correlation. A more general stochastic correlation process was proposed by Teng et al. [14], which relies on the hyperbolic transformation

[1] Source of data: www.yahoo.com

with the function *tangens hyperbolicus* of any mean-reverting process with positive and negative values, the properties (i)–(iii) above can be thus directly satisfied without facing any additional parameter restrictions. Hence, the subsequent calibration process is much simpler.

In this work, we study the general SCP by Teng et al. [14]. We show that the correlation process by van Emmerich can be obtained by this general method, i.e. the correlation process by van Emmerich turns out to be a special case of the hyperbolic transformation of a stochastic process. Furthermore, we apply this general approach to find a new SCP which has a transition density function in closed form. Finally, as an illustrating example, we compute the price of Quanto under stochastic correlation by our new SCP and investigate the effect of considering stochastic correlation on pricing the Quanto.

2 A General Stochastic Correlation Model

Here we study the hyperbolic transformation proposed in [14] of a mean-reverting process to be a correlation process. We show that the correlation process model of van Emmerich [16] can be obtained by transforming a mean-reverting process with the tangens hyperbolicus function. We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an information filtration $(\mathscr{F}_t)_{t\in\mathbb{R}^+}$ satisfying the standard conditions, see e.g. [9].

2.1 The transformed Mean-reverting Process

For the motivations and the properties (i)–(iii) in Section 1, Teng et al. [14] proposed the tangens hyperbolicus function of a mean-reverting stochastic process X_t, like the Ornstein-Uhlenbeck process [15] or the square root diffusion processes (with positive and negative values)

$$dX_t = a(t,X_t)dt + b(t,X_t)\,dW_t, \quad t \geq 0, X_0 = x_0, \tag{7}$$

to model the correlations as

$$\rho_t = \tanh(X_t), \quad \rho_0 = \tanh(x_0) \in (-1,1). \tag{8}$$

Obviously, the properties (i)–(iii) are fulfilled due to the range of values of the tangens hyperbolicus and mean reversion of the process. Besides, the function tanh is symmetrical and measurable. Although the function tanh can not really attain -1 and 1 which presents perfect negative and perfect positive dependence, respectively. It should make no difference to use this function for modelling correlations, because the correlation equal to -1 or 1 is indeed an extreme event which happens very rarely in the real market, see e.g. Figure 1. Besides, the function tanh tends to the boundaries -1 and 1 at infinity.

Applying *Itô's Lemma* with (8)

$$d\rho_t = d\tanh(X_t) = \frac{\partial \tanh(X_t)}{\partial t}\,dt + \frac{\partial \tanh(X_t)}{\partial x}\,dX_t + \frac{1}{2}\frac{\partial^2 \tanh(X_t)}{\partial x^2}(dX_t)^2\,, \tag{9}$$

we obtain the *stochastic correlation process (SCP)*

$$d\rho_t = (1-\rho_t^2)\left((\tilde{a} - \rho_t\tilde{b}^2)dt + \tilde{b}dW_t\right), \quad t \geq 0, \tag{10}$$

where $\rho_0 \in (-1,1)$, $\tilde{a} = a(t, \text{artanh}(\rho_t))$ and $\tilde{b} = b(t, \text{artanh}(\rho_t))$. From (10) we see that there is a suitable number of free parameters to calibrate the model to market data. Besides, it is obvious, in this approach any mean-reverting process (with positive and negative values) can be considered without facing any additional parameter restrictions. The free parameters are hidden in the functions a and b, see the example (13) in Section 2.3 and (21) in Section 3.1.

2.2 Transformation with other functions

Although we could intuitively observe that the function $\tanh(x)$ is eminently suitable for correlation modelling, one can still ask whether other functions having values inside the interval $(-1,1)$, like trigonometric functions or $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$, $x \in \mathbb{R}$ can also be applied for this purpose? In theory, such functions could be used for the SCP model above. However, the trigonometric function is a periodic function, the arising complex number will complicate further calculations. For the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$, its Itô's formula for (7) is given by

$$\begin{aligned} d\rho_t = d\frac{2}{\pi}\arctan\Big(\frac{\pi}{2}X_t\Big) &= \left(\frac{\tilde{a}}{\big(1+\tan^2(\frac{\rho_t\pi}{2})\big)} - \frac{\pi\tilde{b}^2\tan(\frac{\rho_t\pi}{2})}{2\big(1+\tan^2(\frac{\rho_t\pi}{2})\big)^2}\right)dt \\ &\quad + \frac{\tilde{b}}{\big(1+\tan^2(\frac{\rho_t\pi}{2})\big)}\,dW_t\,, \end{aligned} \tag{11}$$

which is rather complicate such that the further computation will turn out to be tedious. Nevertheless, we will additionally consider the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$ which is, like $\tanh(x)$ close to the identity in the neighbourhood of $x = 0$, see Figure 3. However, compared with $\tanh(x)$, the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$ grows much slower up to 1 and down to -1, the estimation of the correlation will thus be worsened, similar to the estimation for the heavy tailed distributions.

2.3 The Correlation Model of van Emmerich

As an example, we show that the correlation model of van Emmerich can be obtained by transforming a special mean-reverting process (12), i.e. the van Emmerich's correlation process is just a special case of the general transformation [14]. To do so, we define the following mean-reverting process

$$dX_t = \frac{\kappa\big(\mu - \tanh(X_t)\big)}{1-\tanh^2(X_t)}dt + \frac{\sigma}{\sqrt{1-\tanh^2(X_t)}}dW_t, \quad t \ge 0,\ X_0 = x_0, \tag{12}$$

where κ and σ are positive, $\mu \in (-1,1)$. Next, we transform (12) with $\rho_t = \tanh(X_t)$. Again, applying Itô's Lemma we obtain in a tedious calculation

$$d\rho_t = \big[(\kappa(\mu - \rho_t)) - \sigma^2\rho_t\big]\,dt + \sigma\sqrt{1-\rho_t^2}\,dW_t. \tag{13}$$

Next, if we define

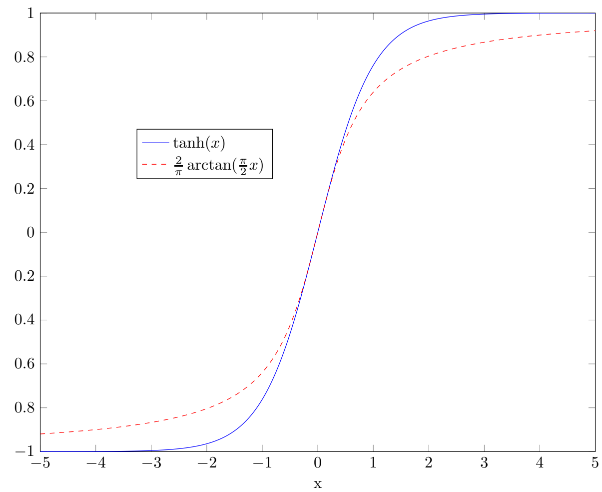


Fig. 3: Comparison of $\tanh(x)$ and $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$: the later is less steep having larger tails.

$$\kappa^* = \kappa + \sigma^2, \quad \mu^* = \frac{\kappa\mu}{\kappa+\sigma^2}, \quad \sigma^* = \sigma \tag{14}$$

the correlation process (13) can be rewritten as

$$d\rho_t = \kappa^*(\mu^* - \rho_t)dt + \sigma^*\sqrt{1-\rho_t^2}\,dW_t, \tag{15}$$

which is exactly the van Emmerich's correlation process in [16]. Due to the transformation with the function tanh, the correlations provided by (15), with coefficients (14), are obviously located in the interval $(-1,1)$. We can check this important property in another way: We recall that van Emmerich [16] derived the analytic condition

$$\kappa^* \geq \frac{\sigma^*}{1 \pm \mu*} \tag{16}$$

to ensure that the boundaries -1 and 1 are unattainable. We see that the correlation process (15) must have already satisfied the condition (16): Substituting (14) in (16) we obtain

$$\frac{\sigma^2}{\kappa(1\pm\mu)+\sigma^2} \leq 1, \tag{17}$$

which always holds whilst κ is positive and $\mu \in (-1,1)$.

3 Stochastic Correlation with a modified Ornstein-Uhlenbeck process

In this section, we specify a SCP by a hyperbolic transformation of the modified Ornstein-Uhlenbeck process. The derivation of the transition density function of this SCP is provided in a closed form. Then, we analyse this density function and show how to fit the correlation process to the historical market data.

3.1 The Transformed modified Ornstein-Uhlenbeck process

The *Ornstein-Uhlenbeck process* is defined by the SDE

$$dX_t = \kappa(\mu - X_t)\,dt + \sigma\,dW_t, \tag{18}$$

where κ, $\sigma > 0$ and X_0, $\mu \in \mathbb{R}$. If we want to restrict the mean value μ to be only in $(-1,1)$, it is reasonable to modify the Ornstein-Uhlenbeck process (18) as

$$dX_t = \kappa\big(\mu - \tanh(X_t)\big)\,dt + \sigma\,dW_t, \tag{19}$$

where κ, $\sigma > 0$ and X_0, $\mu \in (-1,1)$.

Lemma 1. *Applying Itô's Lemma with $\rho_t = \tanh(X_t)$,*

$$d\rho_t = \frac{\partial \tanh(X_t)}{\partial x}\,dX_t + \frac{1}{2}\frac{\partial^2 \tanh(X_t)}{\partial x^2}\sigma^2\,dt \tag{20}$$

yields the stochastic correlation process

$$d\rho_t = (1-\rho_t^2)\big(\kappa(\mu - \rho_t) - \sigma^2\rho_t\big)\,dt + (1-\rho_t^2)\sigma\,dW_t, \tag{21}$$

where $t \geq 0$, $\rho_0 \in (-1,1)$, κ, $\sigma > 0$ and $\mu \in (-1,1)$.

Proof.

$$\begin{aligned}(20) &= \operatorname{sech}^2(X_t)\kappa\big(\mu - \tanh(X_t)\big)dt - \operatorname{sech}^2(X_t)\frac{\sinh(X_t)}{\cosh(X_t)}\sigma^2 dt + \operatorname{sech}^2(X_t)\sigma^2 dW_t \\ &= (1-\rho_t^2)\kappa(\mu-\rho_t)dt - (1-\rho_t^2)\rho_t\sigma^2 dt + (1-\rho_t^2)\sigma^2 dW_t = (21). \qquad \square\end{aligned}$$

Next, we modify the notation and rewrite (21) as

$$\kappa^* = \kappa + \sigma^2, \quad \mu^* = \frac{\kappa\mu}{\kappa+\sigma^2}, \quad \sigma^* = \sigma, \tag{22}$$

$$\frac{d\rho_t}{1-\rho_t^2} = \kappa^*(\mu^* - \rho_t)dt + \sigma^*\,dW_t, \tag{23}$$

where $t \geq 0$, $\rho_0 \in (-1,1)$, κ^*, $\sigma^* > 0$ and $\mu^* \in (-1,1)$.

3.2 Transition density function

For calibration purposes, we first determine the *transition density function* of (23) with the aid of the *Fokker-Planck equation* [10]. Then, we obtain the parameters of the correlation process (23) by fitting the density function to the market data.

Let us assume that the SCP (23) possesses a transition density $f(t,\tilde{\rho}|\rho_0)$ which satisfies the following Fokker-Planck equation

$$\frac{\partial}{\partial t} f(t,\tilde{\rho}) + \frac{\partial}{\partial \tilde{\rho}}\big(\hat{a}(t,\tilde{\rho}) f(t,\tilde{\rho})\big) - \frac{1}{2}\frac{\partial^2}{\partial \tilde{\rho}^2}\big(\hat{b}(t,\tilde{\rho})^2 f(t,\tilde{\rho})\big) = 0, \tag{24}$$

with

$$\hat{a}(t,\tilde{\rho}) = \kappa^*(\mu^* - \rho_t)(1-\tilde{\rho}^2), \quad \hat{b}(t,\tilde{\rho}) = (1-\tilde{\rho}^2)\sigma^*. \tag{25}$$

For the calibration purpose we consider the stationary density (for $t \to \infty$)

$$f(\tilde{\rho}) := \lim_{t\to\infty} f(t,\tilde{\rho}|\rho_0). \tag{26}$$

With the above construction the SCP (23) is also a mean-reverting process. Thus one can show that every two solutions of (24) are the same for $t \to \infty$, i.e. a unique stationary solution $f(\tilde{\rho})$ exists, c.f. [10].

In the sequel, we show how to determine the analytical stationary density function $f(\tilde{\rho})$ of the SCP (23). First, the stationary density function $f(\tilde{\rho})$ obviously satisfies

$$\frac{\partial}{\partial \tilde{\rho}}\Big((1-\tilde{\rho}^2)\big(\kappa^*(\mu^* - \rho_t)\big) f(\tilde{\rho})\Big) = \frac{1}{2}\frac{\partial^2}{\partial \tilde{\rho}^2}\big((1-\tilde{\rho}^2)\sigma^*\big)^2 f(\tilde{\rho}). \tag{27}$$

By solving the elliptic equation (27) we obtain the stationary density $f(\tilde{\rho})$ as

$$\begin{aligned} f(\tilde{\rho}) = &\frac{m}{2^{\frac{\kappa^*}{\sigma^*}}}(1+\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}+\frac{\kappa^*\mu^*}{\sigma^{*2}}}(1-\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}-\frac{\kappa^*\mu^*}{\sigma^{*2}}} \\ &+ \frac{n}{\tilde{\rho}^2-1}\left(\frac{1}{2}\right)^{\frac{2\sigma^{*2}-\kappa^*}{\sigma^{*2}}} F\left(1, \frac{2\sigma^{*2}-2\kappa^*}{\sigma^{*2}}, \frac{(-\mu^*-1)\kappa^*+2\sigma^{*2}}{\sigma^{*2}}, \frac{\tilde{\rho}}{2}+\frac{1}{2}\right) \end{aligned} \tag{28}$$

with the constants m, $n \in \mathbb{R}$ and the *hypergeometric function* F which is defined as

$$F(a,b,c,x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{(a)_k(b)_k}{(c)_k}, \qquad |x| < 1, \tag{29}$$

where $(\cdot)_k$ denotes the *Pochhammer symbol*,

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1), \quad (a)_0 = 1. \tag{30}$$

Next we need to fix the constants m and n in (28) to obtain the stationary density. Due to the mean reversion the stationary density $f(\tilde{\rho})$ must satisfy

$$\int_{-1}^{1} \tilde{\rho} f(\tilde{\rho})\, d\tilde{\rho} = \mu^*.$$

If we choose $\mu^* = 0$, we observe that the first term in (28) becomes

$$\frac{m}{2^{\frac{\kappa^*}{\sigma^{*2}}}}(1+\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}}(1-\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}}, \tag{31}$$

which is obviously symmetric around $\tilde{\rho} = 0$, i.e. the condition (31) will be fulfilled for $n = 0$. In the sequel we assume that $n \equiv 0$ for all general $\mu^* \in (-1,1)$ such that the transition density function (28) can be rewritten as

$$f(\tilde{\rho}) = \frac{m}{2^{\frac{\kappa^*}{\sigma^{*2}}}}(1+\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}+\frac{\kappa^*\mu^*}{\sigma^{*2}}}(1-\tilde{\rho})^{\frac{\kappa^*-2\sigma^{*2}}{\sigma^{*2}}-\frac{\kappa^*\mu^*}{\sigma^{*2}}}. \tag{32}$$

To determine the value of m we can employ the basic property of a density function

$$\int_{-1}^{1} f(\tilde{\rho})\, d\tilde{\rho} = 1. \tag{33}$$

The constant m in (32) must be chosen such that the normalization condition (33) is always fulfilled. We set

$$a = \frac{\kappa^* - 2\sigma^{*2}}{\sigma^{*2}}, \quad b = \frac{\kappa^*\mu^*}{\sigma^{*2}}, \tag{34}$$

and substitute it into (32) to obtain

$$f(\tilde{\rho}) = \frac{m}{2^{\frac{\kappa^*}{\sigma^{*2}}}}(1+\tilde{\rho})^{a+b}(1-\tilde{\rho})^{a-b}. \tag{35}$$

The fact, as long as

$$a \pm b > -1, \tag{36}$$

the integral

$$\int_{-1}^{1} (1+\tilde{\rho})^{a+b}(1-\tilde{\rho})^{a-b}\, d\tilde{\rho}$$

has the solution

$$\begin{aligned} M :=& \frac{\Gamma(1+a-b)F(1,-a-b,2+a-b,-1)}{\Gamma(2+a-b)} \\ &+ \frac{\Gamma(1+a+b)F(1,-a+b,2+a+b,-1)}{\Gamma(2+a+b)}, \end{aligned} \tag{37}$$

with the hypergeometric function F defined in (29) and the *gamma function* Γ.

Using the definitions of κ^*, μ^* and σ^* in (22) we want to check the condition (36). Therefore, together with (34) we obtain

$$a = \frac{\kappa^* - 2\sigma^{*2}}{\sigma^{*2}} = \frac{\kappa - \sigma^2}{\sigma^2}, \quad b = \frac{\kappa^*\mu^*}{\sigma^{*2}} = \frac{\kappa\mu}{\sigma^2}. \tag{38}$$

We consider the following simple calculations

$$a+b > -1 \Leftarrow \tfrac{\kappa-\sigma^2}{\sigma^2} + \tfrac{\kappa\mu}{\sigma^2} > -1 \Leftarrow \kappa(1+\mu) > 0 \Leftarrow \mu > -1,$$

$$a-b > -1 \Leftarrow \tfrac{\kappa-\sigma^2}{\sigma^2} - \tfrac{\kappa\mu}{\sigma^2} > -1 \Leftarrow \kappa(1-\mu) > 0 \Leftarrow \mu < 1$$

and realize that the condition (36) will always hold due to $\mu \in (-1,1)$. Thus, the constant m can be determined as

$$m = \frac{2^{\frac{\kappa^*}{\sigma^{*2}}}}{M}. \tag{39}$$

Finally, we obtain the transition density function in a closed form as

$$f(\tilde{\rho}) = \frac{(1+\tilde{\rho})^{a+b}(1-\tilde{\rho})^{a-b}}{M}, \tag{40}$$

with a, b defined in (34) and M in (37). The parameters κ^*, μ^* and σ^*, or rather, κ, μ and σ can be obtained by fitting the expression (40) to the historical correlation from market data, see Section 3.3.

We could generalize the correlation process (23) with the same definition but directly with the arbitrary parameter coefficients $\kappa > 0$, $\mu \in (-1,1)$ and $\sigma > 0$, like

$$\frac{d\rho_t}{1-\rho_t^2} = \kappa(\mu - \rho_t)dt + \sigma\, dW_t. \tag{41}$$

For this case, we have for a and b, as defined in (34), as

$$a = \frac{\kappa - 2\sigma^2}{\sigma^2}, \quad b = \frac{\kappa\mu}{\sigma^2}. \tag{42}$$

We perform a similar calculation for checking the condition (36) as above:

$$a+b > -1 \Leftarrow \tfrac{\kappa-2\sigma^2}{\sigma^2} + \tfrac{\kappa\mu}{\sigma^2} > -1 \Leftarrow \kappa(1+\mu) > \sigma^2 \Leftarrow \kappa > \tfrac{\sigma^2}{1+\mu},$$

$$a-b > -1 \Leftarrow \tfrac{\kappa-2\sigma^2}{\sigma^2} - \tfrac{\kappa\mu}{\sigma^2} > -1 \Leftarrow \kappa(1-\mu) > \sigma^2 \Leftarrow \kappa > \tfrac{\sigma^2}{1-\mu}.$$

Thus, the process (41) could be employed for the stochastic correlation if the condition

$$\kappa > \frac{\sigma^2}{1 \pm \mu} \tag{43}$$

is fulfilled. We find that this condition dovetails nicely with that condition in [16], which ensures that the boundaries -1 and 1 are unattainable.

To further illustrate the transition density function $f(\tilde{\rho})$, we display in Figures 4, 5 and 6 the behaviour of $f(\tilde{\rho})$ for different values of each parameter. In Figure 4, we let $\kappa = 2$ and $\mu = 0$ and display $f(\tilde{\rho})$ ith different values of σ, which is equal to 0.3, 0.4 and 0.5, respectively. Obviously, σ shows the magnitude of variation from the mean value $\mu = 0$. Next, we fix $\kappa = 2$ and $\sigma = 0.3$, the behaviour of $f(\tilde{\rho})$ only with varying mean value $\mu = -0.5$, $\mu = 0$ and $\mu = 0.5$ can be found in Figure 5. However, whilst $\mu = -0.5$ and $\mu = 0.5$ we can observe that the peak of the corresponding $f(\tilde{\rho})$ does not locate exactly at the points $\tilde{\rho} = -0.5$ and $\tilde{\rho} = 0.5$, respectively. The reason is that, the value of κ, which is mean reversion rate, is not large enough. In order to illustrate the role of κ, we set $\mu = -0.5$, $\sigma = 0.5$ and vary the value of κ, see Figure 6. For $\kappa = 3$, the peak of the transition density function is far away from the mean value -0.5. However, in contrast the peak reaches already the point $\tilde{\rho} = -0.5$ when $\kappa = 12$.

3.3 Calibration

We assume that the correlation is itself observable. Under this assumption the transition density can be used for calibration purposes. One uses usually *maximum-likelihood estimation (MLE)* when the density function is known. Considering the density function (40), it will be tedious to determine its likelihood-function. Another approach to estimate the parameters is to fit the empirically observed density to the stationary

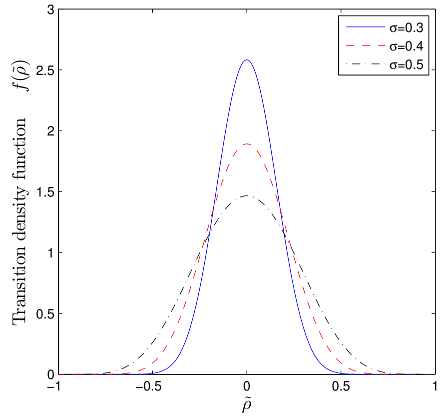


Fig. 4: Comparison of $f(\tilde{\rho})$ for different values of σ ($\kappa = 2$ and $\mu = 0$).

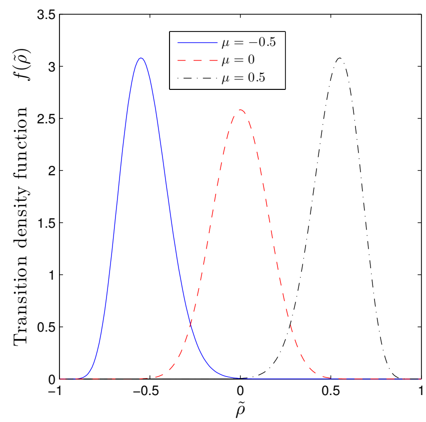


Fig. 5: Comparison of $f(\tilde{\rho})$ for different values of μ ($\kappa = 2$ and $\sigma = 0.3$).

density (40). As an example we fit the historical data from Figures 2a to (40). The fitting by nonlinear least-squares works well, see Figure 7.

4 Stochastically correlated Brownian motions

The remaining problem is how to incorporate the stochastic correlation process in the financial model, like option pricing. In Section 1, we mentioned that a widely used approach for dependence is the (constant) correlated Brownian motions. In order to consider a stochastic correlation, we need the concept of *stochastically*

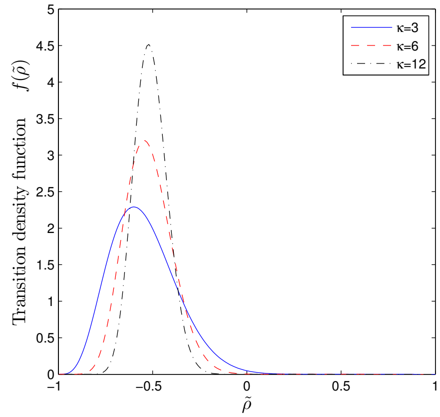


Fig. 6: Comparison of $f(\tilde\rho)$ for different values of κ ($\mu = -0.5$ and $\sigma = 0.5$).

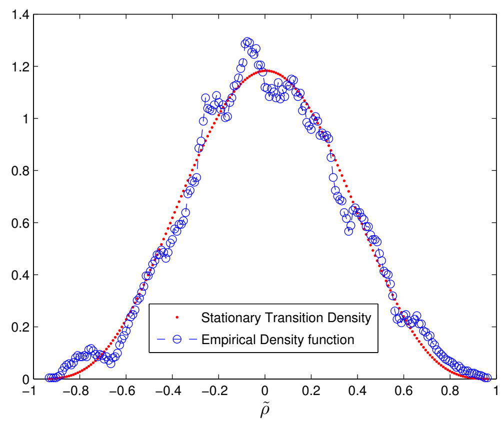


Fig. 7: Correlation between S&P 500 and Euro/US-Dollar exchange rate, empirical density compared to density (40) computed with $\kappa = 7.937$, $\mu = 0.003$ and $\sigma = 1.186$ (Mean Squared error: 2.46e-06).

correlated Brownian motions. In the following, we study the stochastically correlated Brownian motions following the work of van Emmerich [16].

Based on two independent Brownian motions $W_{2,t}$ and $W_{3,t}$ we define

$$W_{1,t} = \int_0^t \rho_s \, dW_{2,s} + \int_0^t \sqrt{1-\rho_s^2} \, dW_{3,s}, \tag{44}$$

where ρ_t is one SCP of type (10), and we assume that W_t in (10) is independent of each $W_{i,t}$, for $i = 2, 3$.

Lemma 2. *$W_{1,t}$ satisfies*

(1) $W_{1,0} = 0$,

(2) $\mathbb{E}\left[(W_{1,t})^2\right]=t$,
(3) $\mathbb{E}[W_{1,t}|\mathscr{F}_s]=W_{1,s}, \quad for \quad s\le t.$

Proof. (1) is obvious. We calculate the two expected values as follows:

$$\begin{aligned}\mathbb{E}\left[(W_{1,t})^2\right] &= \mathbb{E}\left[\left(\int_0^t \rho_s\, dW_{2,s}\right)^2+\left(\int_0^t \sqrt{1-\rho_s^2}\, dW_{3,s}\right)^2 \right.\\ &\qquad \left. +2\int_0^t \rho_s\, dW_{2,s}\int_0^t \sqrt{1-\rho_s^2}\, dW_{3,s}\right]\\ &= \mathbb{E}\left[\int_0^t \rho_s^2\, ds+\int_0^t (1-\rho_s^2)\, ds\right]+\underbrace{\mathbb{E}\left[2\int_0^t \rho_s\, dW_{2,s}\int_0^t \sqrt{1-\rho_s^2}\, dW_{3,s}\right]}_{=0,\text{ since } W_2\perp W_3.}\\ &= \int_0^t 1\, ds = t\end{aligned}$$

$$\mathbb{E}[W_{1,t}|\mathscr{F}_s]=W_{1,s}+\underbrace{\mathbb{E}\left[\int_s^t \rho_{s_1}\, dW_{2,s_1}+\int_s^t \sqrt{1-\rho_{s_1}^2}\, dW_{3,s_1}|\mathscr{F}_s\right]}_{:=0}. \qquad \square$$

This means that we have defined one new Brownian motion $W_{1,t}$ regarding the two independent Brownian motion $W_{2,t}$ and $W_{3,t}$. Besides, we can check that

$$\mathbb{E}\left[W_{1,t}\cdot W_{2,t}\right]=\mathbb{E}\left[\int_0^t \rho_s\, ds\right], \tag{45}$$

which is the definition for the case that the Brownian motions $W_{1,t}$ and $W_{2,t}$ are correlated by the SCP ρ_t. One can immediately see that (45) agrees for

$$\mathbb{E}\left[W_{1,t}\cdot W_{2,t}\right]=\rho t, \tag{46}$$

where $W_{1,t}$ and $W_{2,t}$ are correlated by the constant ρ. Indeed, (45) can be also seen as that $W_{1,t}$ and $W_{2,t}$ are correlated by the *average correlation*

$$\frac{1}{t}\int_0^t \mathbb{E}[\rho_s]\, ds \tag{47}$$

which is a constant.

5 Pricing Quantos with Stochastic Correlation

To illustrate the impact of using stochastic correlation on option pricing, we use quanto options as an example. These options hedge the exchange rate risk when investing in financial products not valued in the domestic currency. To price these options, one has to consider the correlation between the currency exchange rate R_t between domestic and foreign currencies, and the price S_t of the underlying. We assume that S_t and R_t follow the coupled stochastic process (5) by

$$\begin{cases} dS_t &= \mu_S S_t\, dt + \sigma_S S_t\, dW_t^S \\ dR_t &= \mu_R R_t\, dt + \sigma_R R_t\, dW_t^R, \end{cases} \tag{48}$$

where W_t^S and W_t^R are correlated using the SCP (41) as:

$$\frac{d\rho_t}{1-\rho_t^2} = \kappa(\mu - \rho_t)\, dt + \sigma\, dW_t. \tag{49}$$

W_t is assumed to be independent of W_t^S and W_t^R.

We consider as an example a Put-Option on the S&P 500 with payoff in Euro [14]

$$(\underbrace{\text{Strike}}_{:=K} - \underbrace{\text{S\&P500}_T}_{:=S_T})^+,$$

where $(\cdot)^+ = \max\,(0,\cdot)$. Then the payoff in US-Dollar can be written with the Euro/US-Dollar exchange rate as

$$\underbrace{\text{exchangeRate}_0}_{:=R_0} \cdot (\text{Strike} - \text{S\&P500}_T)^+.$$

We denote the risk-free interest rate of Euro and US-Dollar respectively by r_e and r_d. If W_t^S and W_t^R are correlated with a constant correlation, the price of a Quanto Put-Option in the Black-Scholes (BS) model with continuous dividend yield is [17]:

$$P_{\text{Quanto}}(S_0, K, r_e, r_d, D, \sigma_S, \sigma_R, T) = R_0 \left(K \exp^{-r_d T} \mathcal{N}(-d_2) - S_0 \exp^{-DT} \mathcal{N}(-d_1) \right),$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + ((r_d - D) + \frac{\sigma_S^2}{2})/T}{\sigma_S \sqrt{T}}, \quad d_2 = d_1 - \sigma_S \sqrt{T}, \quad D = r_d - r_e + \rho \sigma_S \sigma_R.$$

We follow the train of thoughts in [14] to incorporate the stochasticity of the correlation in the BS price. The no-arbitrage principle requires

$$\frac{1}{R_0} \exp(r_e T) \mathbb{E}[R_T] = \exp(r_d T) \tag{50}$$

and

$$\frac{1}{R_0} \frac{1}{S_0} \mathbb{E}[S_T R_T] = \exp(r_d T). \tag{51}$$

(50) can be interpreted as: The expected return of one unit of US-Dollar, exchanged to Euro, risk-free invested in the Euro countries and re-exchanged to US-Dollar must equal the risk-free return on one unit of US-Dollar in US-Dollar countries. The interpretation of (51) is analogous, the left side of (51) describes the re-exchanged expected value of an investment of one US-Dollar into the underlying with price S. Further computing (50) and (51) by aid of Itô's lemma we obtain

$$\mu_R = r_d - r_e \tag{52}$$

and

$$\mu_S = r_d - \mu_R - \frac{1}{T}\ln\mathbb{E}\left[\exp\left(\sigma_S\sigma_R\int_0^T \rho_t dt\right)\right]. \tag{53}$$

In the BS model, we interpret (53) as a return minus the continuous dividend

$$D(\rho_t) := \mu_R + \frac{1}{T}\ln\mathbb{E}\left[\exp\left(\sigma_S\sigma_R\int_0^T \rho_t dt\right)\right] = r_d - r_e + \frac{1}{T}\ln\mathbb{E}\left[\exp\left(\sigma_S\sigma_R\int_0^T \rho_t dt\right)\right].$$

The integral of the stochastic correlation ρ_t can be computed numerically using e.g. the Milstein scheme [5]. Finally, the price of a Quanto Put-Option in the extended BS model incorporating the SCP reads

$$\begin{aligned} P_{\text{Quanto}} &= P_{\text{Quanto}}(S_0, K, r_e, r_d, D(\rho_t), \sigma_S, \sigma_R, T) \\ &= R_0\big(K\exp^{-r_d T}\mathscr{N}(-d_2) - S_0\exp^{-D(\rho_t)T}\mathscr{N}(-d_1)\big) \end{aligned} \tag{54}$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + ((r_d - D(\rho_t)) + \frac{\sigma_S^2}{2})/T}{\sigma_S\sqrt{T}}, \qquad d_2 = d_1 - \sigma_S\sqrt{T}.$$

The price of a Quanto Call-Option is derived easily from the put-call parity [17].

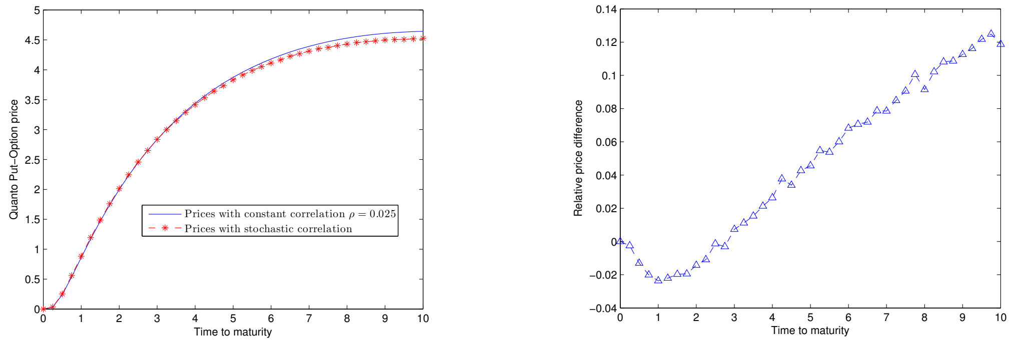
We use a conditional Monte-Carlo approach and first simulate all the paths of ρ_t^i, for $i \in \{1,2,\ldots,M\}$ and for each path we can compute a price P^i_{Quanto} by the pricing formula (54). Then the fair price $\mathbb{P}_0$ can be approximated by the mean value over all prices

$$\mathbb{P}_0 = \mathbb{E}[\mathbb{E}[P_{\text{Quanto}}|\mathscr{F}_t]] \approx \frac{\sum_{i=1}^M P^i_{\text{Quanto}}}{M}. \tag{55}$$

In Figure 8, we assume the parameter for the Black-Scholes model and use the estimated parameter for the SCP model (see Figure 7). Besides, we apply the sample coefficient correlation (3) to estimate a constant correlation using the whole historical data (Jan 2003 – Mar 2013) of S&P 500 and Euro/US-Dollar exchange rate, which is 0.025. At the same time, we can let the SCP starting from the first correlation in the historical correlations. In Figure 8b we present the relative difference between the price with constant correlation and stochastic correlation. We can observe, whilst the maturity T is shorter than three years, the price with constant correlation is lower than the price with stochastic correlation. Then, from nearly $T = 3$, the price calculated with constant correlation becomes higher than the corresponding price calculated with stochastic correlation. The reason for this, before the time $T = 3$, the SCP provides the correlations which are closed to the initial correlation $\rho_0 = 0.3$ which is larger than the constant correlation $\rho = 0.025$. That's why is the price with stochastic correlation higher than the price with constant correlation before $T = 3$ due to the fact that the price of quanto increases direct proportional with that correlation. As the time increases, the generated correlations tend to the mean value μ.

If we give a lower initial correlation than the constant correlation $\rho = 0.025$, e.g. set $\rho_0 = -0.6$. At the same time we choose a greater value for μ, say 0.1. The desired results should be that the price with constant correlation is higher than the price with stochastic correlation within the short maturity, and then increases for the longer maturity, which can be observed in Figures 9. To illustrate the role of the parameters κ and σ in pricing the quanto, we calculate the prices using the parameter collection in which only κ and σ is varying. The results for varying κ is displayed in Figure 10a and for varying σ can be found in Figure 10b.

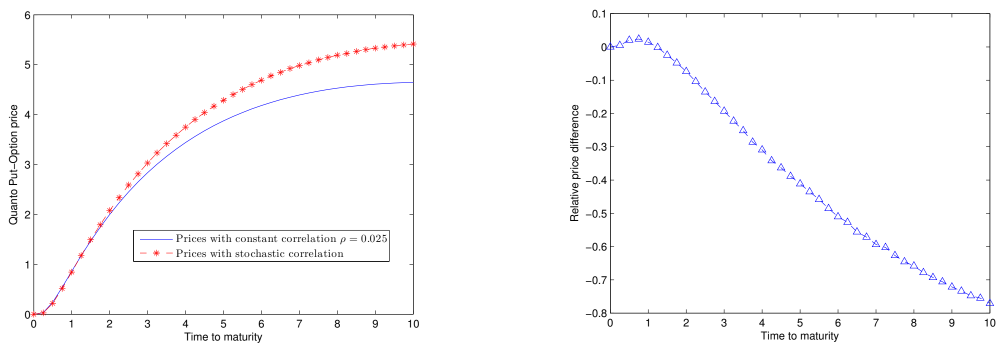
Figure 10a shows that for increasing κ the stochastic correlation rapidly tends to the mean value μ of the SCP process, which is set to be equal to the constant correlation value. The effect of stochastic correlation



(a) Comparison of prices between using stochastic and constant correlation

(b) Relative price difference

Fig. 8: BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.05$, $r_e = 0.03$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, SCP parameters: $\kappa = 7.937$, $\mu = 0.003$, $\sigma = 1.186$ and $\rho_0 = 0.3$.



(a) Comparison of prices between using stochastic and constant correlation

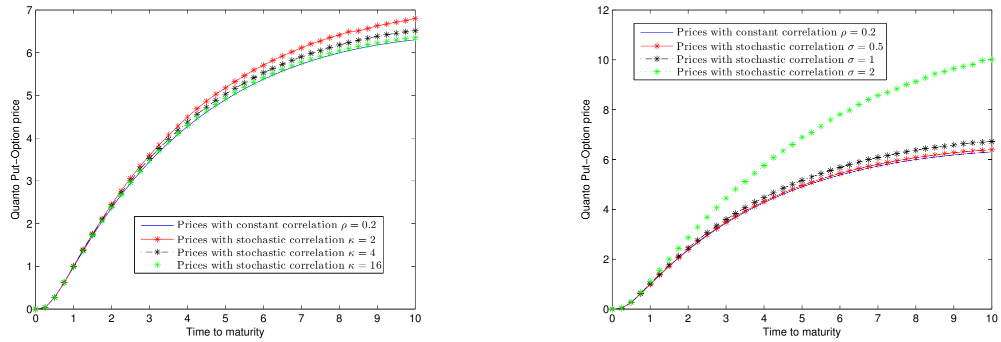
(b) Relative price difference

Fig. 9: BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.05$, $r_e = 0.03$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, SCP parameters: $\kappa = 7.937$, $\mu = 0.1$, $\sigma = 1.186$ and $\rho_0 = -0.6$.

is quite small in this case. In contrast, as shown in Figure 10b, an increase in the diffusion σ (and thus randomness in the SCP process) shows an increasing impact of the SCP model on the prices.

6 Conclusion

In this work we have revised concisely stochastic correlation models. Market observations give strong evidence that financial quantities are correlated in a strongly nonlinear, non-deterministic way. Instead of assuming a constant correlation, correlation has to be modelled as a stochastic process. We discussed first



(a) SCP parameters: $\mu = 0.2$, $\sigma = 0.6$, $\rho_0 = 0.2$ and varying $\kappa = 2, 4, 16$

(b) SCP parameter: $\kappa = 6$, $\sigma = 0.6$, $\rho_0 = 0.2$ and varying $\sigma = 0.5, 1, 2$

Fig. 10: BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.05$, $r_e = 0.03$, $\sigma_S = 0.2$, $\sigma_R = 0.4$.

the general stochastic correlation model proposed in [14] and proved that the stochastic correlation process in [16] can be obtained by applying this general approach.

We generalized our approach [14] to derive a stochastic correlation model from a hyperbolic transformation of the modified Ornstein-Uhlenbeck process allowing for a transition density function in a closed form and an easy-to-handle calibration to historical data. As an example, we computed the fair price of a Quanto Put-option with stochastic correlation. The numerical results showed that the correlation risk caused by using a wrong (constant) correlation model cannot be neglected.

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