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Minimal failure probability for ceramic design via shape control

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Abstract

We consider the probability of failure for components made of brittle materials under one time application of a load as introduced by Weibull and Batdorf – Crosse and more recently studied by NASA and the STAU cooperation [19, 23] as an objective functional in shape optimization and prove the existence of optimal shapes in the class of shapes with a uniform cone property. The corresponding integrand of the objective functional has convexity properties that allow to derive lower-semicontinuity according to [12]. These properties require less restrictive regularity assumptions for the boundaries and state functions compared to [13]. Thereby, the weak formulation of linear elasticity can be kept for the abstract setting for shape optimization as presented in [16].

Key words: Probabilistic failure of ceramic structures, shape optimization, optimal reliability.

MSC (2010): 49Q10, 60G55

1 Introduction

This article introduces some objective functionals $J(\Omega, u)$ that have been known to mechanical engineers in the design of ceramic components for quite some time

[3, 17, 19, 23, 25, 27] to the field of shape optimization [1, 8, 9, 16, 21]. We will prove that such functionals fulfil the requirements of [12] and thus are lower semicontinuous in the weak topology of the Sobolev space $H^1(\hat{\Omega}, \mathbb{R}^3)$, where $\hat{\Omega} \subseteq \mathbb{R}^3$ is some bounded domain (constructed space) and the admissible shapes $\Omega \subseteq \hat{\Omega}$ share parts of their boundary with $\hat{\Omega}$, cf. Figure 1, and fulfil the uniform cone property. Denoting the admissible shapes with $\mathscr{O}^{\text{ad}}$, we will conclude that

$$\exists \Omega^* \in \mathscr{O}^{\text{ad}} \text{ such that } \mathscr{J}(\Omega^*) = J(\Omega^*, u(\Omega^*)) \leq J(\Omega, u(\Omega)) = \mathscr{J}(\Omega) \quad \forall \Omega \in \mathscr{O}^{\text{ad}}. \tag{1}$$

Here $u(\Omega) \in H^1_{\partial\Omega_D}(\Omega, \mathbb{R}^3) = \{u \in H^1(\Omega, \mathbb{R}^3) : u \restriction_{\partial\Omega_D} = 0\}$ is the solution to the linear elasticity PDE on Ω (or, more precisely, its extension to $\hat{\Omega}$) with given loads $g \in H^1(\hat{\Omega}, \mathbb{R}^3)$, $f \in L^2(\Omega, \mathbb{R}^3)$ in the weak sense

$$\mathscr{B}_\Omega(u(\Omega), v) = \int_\Omega f \cdot v \, dx + \int_{\partial\Omega_{N_{\text{fixed}}}} g \cdot v \, ds, \quad \forall v \in H^1_{\partial\Omega_D}(\Omega, \mathbb{R}^3) \tag{2}$$

where the boundary of Ω is the union of Dirichlet and Neumann boundaries

$$\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_{N_{\text{fixed}}}} \cup \overline{\partial\Omega_{N_{\text{free}}}}$$

with pairwise disjoint sets $\partial\Omega_D$, $\partial\Omega_{N_{\text{fixed}}}$ and $\partial\Omega_{N_{\text{free}}}$, where $\partial\Omega_D$ has positive Lebesgue surface measure and the derivative of v vanishes in normal direction on the free Neumann boundary $\partial\Omega_{N_{\text{free}}}$, sample domains Ω and $\hat{\Omega}$ are sketched in 2d in Figure 1. The left hand side of (2) is given by

$$\mathscr{B}_\Omega(u, v) = \int_\Omega \operatorname{tr}(\varepsilon(Du)\sigma(Dv)) \, dx \tag{3}$$

with $\varepsilon(Du) = \frac{1}{2}(Du + Du^*)$ the elastic strain field, $\sigma(Du) = \lambda \operatorname{tr}(\varepsilon(Du))I + 2\mu \, \varepsilon(Du)$ the elastic stress field and $\mu, \lambda > 0$ Lamé's constants. Du stands for the Jacobi matrix of u and tr denotes the trace. As a consequence the elasticity tensor $C = (c_{i,j,k,l})^3_{i,j,k,k=1}$ is defined by using

$$\sigma_{i,j} = c_{i,j,k,l}\varepsilon(Du)_{k,l}$$

and

$$\sigma_{i,j} = \lambda \operatorname{tr}(\varepsilon(Du))\delta_{i,j} + 2\mu\varepsilon(Du)_{i,j}$$

and thus it is symmetric in the sense that $c_{i,j,k,l} = c_{j,i,k,l} = c_{k,l,i,j}$. Furthermore, the elasticity tensor fulfils the following ellipticity condition: There exists a constant $q > 0$ such that for all symmetric 3×3 matrices $(\xi_{i,j})^3_{i,j=1}$ we have

$$c_{i,j,k,l}\xi_{i,j}\xi_{k,l} \geq q \; \xi_{i,j}\xi_{i,j}. \tag{4}$$

Let us now turn to the definition and motivation of the objective functionals. Ceramics frequently is chosen to construct mechanical components. Ceramics is temperature resistant and does not react with oxygen, sulphur or hydrogen even at high temperatures. On the negative side, the brittleness exposes ceramic structures

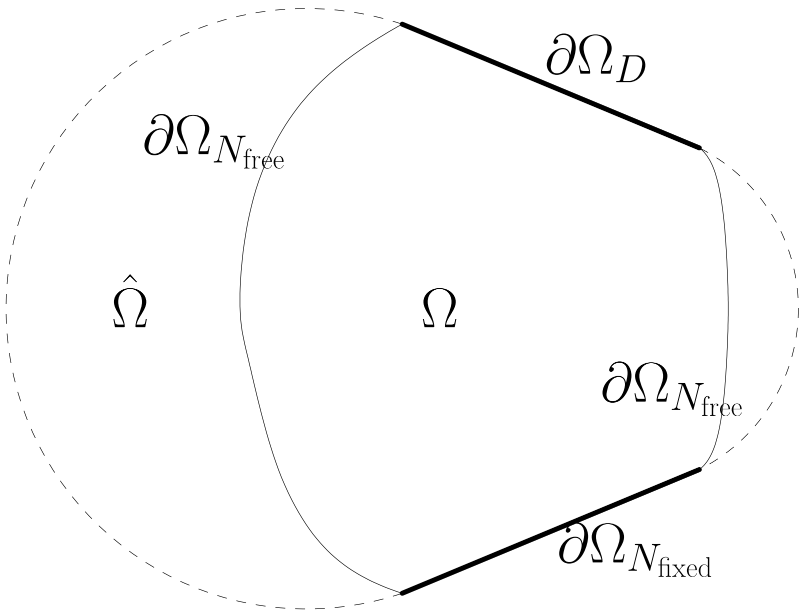

Figure 1: Domains Ω and $\hat{\Omega}$ represented in 2D, for simplicity

to the risk of spontaneous failure due to stress concentration at prefabricated voids or inclusions. As the formation of such microcracks is unavoidable in the sintering process and is stochastic by nature, the failure under or the resistance to a given mechanical load is a random event that occurs with a given failure probability. This was the insight by E. W. Weibull in his classical paper [25]. From the 1960ies on, this approach was taken up in a number of papers, see e.g. [3, 11, 26]. Here, we follow the approach of [3] that is also supported experimentally [4] in the case of small flaw sizes. The failure probability, i.e. the probability of spontaneous macroscopic rupture, for a given displacement field u on Ω is defined as

$$J(\Omega, u) = p_f = 1 - \exp\left\{-\frac{1}{4\pi}\int_\Omega\int_{S^2}\left(\frac{(n\cdot\sigma(Du)n)^+}{\sigma_0}\right)^m dn\,dx\right\} \tag{5}$$

where considerations from the large sample limit of extreme value theory are applied along with some approximations that can be controlled numerically to a reasonable extent [3, 17, 27]. We use the notation $x^+ = \max\{x, 0\}$ for the positive part of x. Here $S^2 \subseteq \mathbb{R}^3$ is the embedded two sphere and dn the induced measure on it.

The engineering task is to solve the optimal reliability criterion (1) under suitable constraints.

This approach gained new attention with the wide distribution of finite element software which allows the efficient calculation of an approximation to $u(\Omega)$ and thus the calculation of (5) and related functionals as a post processing step, cf. in particular the work related to the STAU postprocessor developed at Karlsruhe Institute of Technology [24, 5, 17, 18, 27].

From a shape optimization prospective, the objective functional (5) has attractive properties as

a) It has a clear material science derivation and a proven record of industrial application [5, 18, 27];

b) It permits to show the existence of optimal shapes by its convexity properties [12];

c) One can prove the existence of the shape derivatives $d\mathscr{J}(\Omega, V)$ under infinitisial transformations generated by a vector field V in the sense of [21], confer the forthcoming work [14].

The paper is organised as follows. In Section 2 we give some background material from linear fracture mechanics and the Poisson point process in order to motivate and generalize (5). Section 3 proves convexity of the resulting objective functionals. In section 4 we apply the strategy of [12] to conclude that optimally reliable designs exist.

Although the existence result in this article is less general in terms of the objective functionals than [13], it requires much less restrictive boundary regularity assumptions and technically follows a rather independent route. For other work on optimal design with the linear elasticity PDE as state equation, confer e.g. [1, 2, 9, 16] and references therein. These works however use objective functionals which considerably differ in their design intention and mathematical properties from what we consider here. This in particular applies to the compliance functional, which is not directly related to the failure of the component.

2 Survival probabilities from linear fracture mechanics

Let us first recall some elements of the classical engineering analysis of spontaneous failure of mechanical components from brittle material under given mechanical loads. In linear fracture mechanics, the three dimensional stress field close to a crack in a two dimensional plane close to the tip of the crack is of the form

$$\sigma = \frac{1}{\sqrt{2\pi r}}\{K_I\tilde{\sigma}^I(\varphi) + K_{II}\tilde{\sigma}^{II}(\varphi) + K_{III}\tilde{\sigma}^{III}(\varphi)\} + \text{regular Terms}, \tag{6}$$

where the detailed form of the shape functions $\tilde{\sigma}^{\#}(\varphi)$ is determined by complex analysis, [15, chapter 4]. Here r is the distance to the crack front and φ the angle of the shortest connection point considered to the crack front with the crack plane. The K - factors – also called stress intensity factors – depend on the amount and the mode of the loading, cf. Figure 2, and the geometry of the crack. Considering e.g. the tensile loading σ_n in a normal direction of the stress plane and the crack

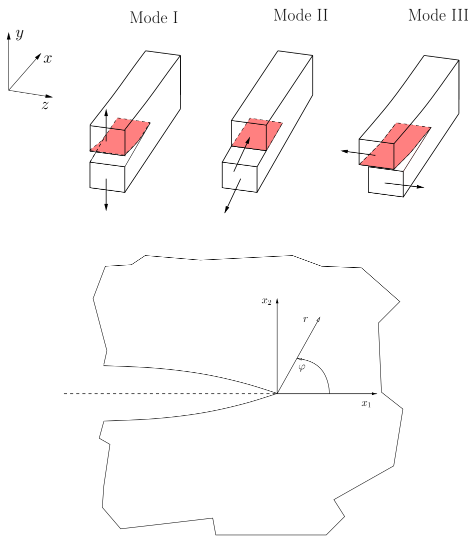

Figure 2: Different modes of the loading (top). r-φ coordinate system at the tip of the crack (bottom)

geometry circular with radius a ('penny crack') one obtains

$$K_I = \frac{2}{\pi}\sigma_n\sqrt{\pi a}. \tag{7}$$

Failure occurs in the ceramic component, if σ_n is positive and is large enough such that the stress intensity K_I exceeds a critical value K_{Ic}. Typical K_{Ic} values for ceramics that are measured in mechanical tests are $(4 \text{ to } 100) \times 10^3$ [MPa$\sqrt{\text{m}}$]. Apparently, in the case of compressive loads, i.e. $\sigma_n < 0$, no failure will occur no matter what the size a of the crack is. We note that it would be straight forward to incorporate more complex flaw geometries in the framework of this article, eg. for elliptic shapes K_I is modified with a factor $1-\sqrt{1-c^2}$, where $0 < c \leq 1$ the quotient between the length of the principal axes. The consideration of surface cracks (eg. due to manufacturing) will require the more involved analysis of [13].

The next step is the passage to stress fields with arbitrary orientation w.r.t. the crack plane, see (6). A large number of solutions has been proposed to the extension of the concept of critical K factors to the multi axial case [3, 11, 15, 25, 27]. Experimental evidence [4] indicates that for microscopic or mesoscopic initial flaws the shear stress influence to the strength of a ceramic component is negligible. We therefore follow [17, 27] and set

$$\sigma_n = (n \cdot \sigma(Du)n)^+ = \max\{n \cdot \sigma(Du)n, 0\}, \tag{8}$$

retaining the failure criterion $K_I(a, \sigma_n(x)) > K_{Ic}$ at the location $x \in \Omega$ of a crack with radius a.

The probabilistic model of flaw distributions is a marked Poisson point process (PPP) with the mark space given by $S^2 \times \mathbb{R}_+$. Here S^2 stands for the flaw orientation described by the normal n and the flaw radius a. Assumptions that lead to the PPP model are

- Flaws are uniformly distributed over the volume Ω of the component with an average number $z > 0$ of flaws per unit volume;
- Two flaws can always be distinguished either by their orientation, size or by their location;
- Orientations are uniformly distributed over S^2 and are independent of the flaw location;
- The distribution of the flaw radius is independent of location and orientation of the crack;
- The number of flaws in n given, non intersecting volumes $A_1, \dots, A_n \subseteq \Omega$ are statistically independent of each other.

If these assumptions are a good approximation to reality, the following mathematical model is adequate and essentially fixed by these assumptions, confer [22, Corollary 4.7]:

Definition 2.1. Let $\mathscr{M} = \Omega \times S^2 \times \mathbb{R}_+$ be the crack configuration space endowed with the sigma algebra $\mathscr{A}(\mathscr{M})$ defined as the Borel sigma algebra on $\mathscr{M}$. Let furthermore ν be the Radon measure on $\mathscr{A}(\mathscr{M})$ which is given by

$$\nu = dx \restriction_\Omega \otimes \frac{dn}{4\pi} \otimes \rho. \tag{9}$$

Here dx is the Lebesgue measure on $\mathbb{R}^3$, dn the surface measure on S^2 and ρ a positive Radon measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\rho([c, d])$ fixes the density (number per volume) of cracks with radius a, $c \leq a \leq b$. A natural assumption is that only finitely many cracks with a radius a above some finite limit can be contained in a given volume, i.e. $\rho([c, \infty)) < \infty \ \forall c > 0$.

The Poisson point process on the crack configuration space $\mathscr{M}$ with intensity measure ν is a mapping $N : \mathscr{E} \times \mathscr{A}(\mathscr{M}) \to \mathbb{N}_0$ where $\mathscr{E}$ is the set of some probability space $(\mathscr{E}, \mathscr{A}, P)$ such that the following conditions hold:

(i) $\forall A \in \mathscr{A}(\mathscr{M})$, $N(A) = N(., A) : (\mathscr{E}, \mathscr{A}, P) \to (\mathbb{N}_0, \mathscr{P}(\mathbb{N}_0))$ is a (counting) random variable;

(ii) $\forall \omega \in \mathscr{E}$, $N(\omega, .) : \mathscr{A}(\mathscr{M}) \to N_0 \subseteq \bar{\mathbb{R}}_+$ is a sigma finite measure;

(iii) $\forall n \in \mathbb{N}, A_1, \dots, A_n \in \mathscr{A}(\mathscr{M})$ mutually disjoint, the random variables $N(A_1)$, $\dots, N(A_n)$ are independent;

(iv) $\forall A \in \mathscr{A}(\mathscr{M})$ such that $\nu(A) < \infty$, $N(A)$ is Poisson distributed with mean $\nu(A)$, $N(A) \sim \mathrm{Po}(\nu(A))$, i.e.

$$P(N(A) = n) = e^{-\nu(A)} \frac{\nu(A)^n}{n!}. \tag{10}$$

Items i) and ii) are the definition of a general point process, iii) is needed for a general PPP on $\mathscr{M}$ and iv) fixes its distribution [22].

Lemma 2.2. *Let $u \in H^1(\Omega, \mathbb{R}^3)$ be given, then*

$$A_c = A_c(\Omega, Du) = \left\{(x, n, a) \in \mathscr{M} : K_I\left(a, (n \cdot \sigma(Du(x))n)^+\right) > K_{Ic}\right\} \in \mathscr{A}(\mathscr{M}). \tag{11}$$

Proof. $Du \in L^2(\Omega, \mathbb{R}^{3\times 3})$ is Borel measurable and so is $\sigma_n = (n \cdot \sigma(Du)n)^+$. Thus the set of critical crack configurations given $\sigma(Du)$ is measurable as the pre-image of the interval $[K_{Ic}, \infty)$ under the Borel measurable function $\mathscr{M} \ni (x, n, a) \to K_I\left(a, (n \cdot \sigma(Du(x))n)^+\right) \in \mathbb{R}_+$. □

Adopting the point of view (neglecting mechanical interactions between cracks) that the component fails if there is any crack with configuration in the critical set A_c, we obtain the following definition:

Definition 2.3. Let the survival probability of the component Ω given the displacement field $u \in H^1(\Omega, \mathbb{R}^3)$ is

$$p_s(\Omega|Du) = P(N(A_c(\Omega|Du)) = 0) = \exp\{-\nu(A_c(\Omega, Du))\}. \tag{12}$$

Apparently we want to maximise the survival probability $p_s(\Omega, Du(\Omega))$ in the shape control variable $\Omega \in \mathscr{O}^{\mathrm{ad}}$, which can equivalently be expressed as a minimization problem for the failure probability $p_f(\Omega, Du) = 1 - p_s(\Omega, Du)$ under the PDE constraint (2). This in turn is equivalent to the following PDE constraint minimization problem:

$$\bar{J}(\Omega, u) = \nu(A_c(\Omega, Du)) \longrightarrow \min, \quad u = u(\Omega) \text{ solves (2)}\ , \quad \Omega \in \mathscr{O}^{\mathrm{ad.}}. \tag{13}$$

A more explicit representation of $\nu(A_c(\Omega, Du))$ can be found with the help of the cumulative crack size function $\Phi(s) = \rho((s, \infty])$ of the crack radius, see also [17, 27]:

Lemma 2.4. *Let $u \in H^1(\Omega, \mathbb{R}^3)$, then $\bar{J}(\Omega, u) = \int_\Omega h(Du)\, dx$ with*

$$h(q) = \frac{1}{4\pi} \int_{S^2} \Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n \cdot q\, n)^+}\right)^2\right) dn \quad q \in \mathbb{R}^{3\times 3}. \tag{14}$$

Proof. By Fubni's theorem for positive functions, with $1_B(\xi) = 1$ if $\xi \in B$ and 0 otherwise defining the characteristic function of the set B,

$$\begin{aligned}\nu(A_c(\Omega, Du)) &= \frac{1}{4\pi}\int_\Omega\int_{S^2}\int_{\mathbb{R}_+} 1_{\{K_I(a,(n\cdot Du(x)\,n)^+)>K_{Ic}\}}(x,n,a)\,d\rho(a)\,dn\,dx \\ &= \frac{1}{4\pi}\int_\Omega\int_{S^2}\rho\left(a > \frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot Du\,n)^+}\right)^2\right)dn\,dx \\ &= \frac{1}{4\pi}\int_\Omega\int_{S^2}\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot Du\,n)^+}\right)^2\right)dn\,dx\end{aligned}$$

□

For later use, we prove the following:

Lemma 2.5. *The function $h(q)$ introduced in (14) depends continuously of q.*

Proof. We first note that for $q_l \to q$ with $(n\cdot qn)^+ > 0$,

$$\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot q_l n)^+}\right)^2\right) \to \Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot qn)^+}\right)^2\right)$$

by upper and lower continuity of the radon measure ρ on sets of finite measure. Let us now assume that $(n\cdot qn)^+ = 0$. In this case $\frac{\pi}{4}(\frac{K_{Ic}}{(n\cdot q_l n)^+})^2 \to \infty$, and thus

$$\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot q_l n)^+}\right)^2\right) \to 0 = \Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot qn)^+}\right)^2\right)$$

again by upper continuity and additivity of ρ. Furthermore, the integrand in the S^2 integral defining $h(q)$ by additivity of ρ is uniformly bounded by

$$\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{\sup_{n\in S^2, l\in\mathbb{N}}(n\cdot q_l n)^+}\right)^2\right) < \infty.$$

The assertion of the lemma thus follows from Lebesgue's theorem of dominated convergence. □

3 Convexity of the objective functional

Fuji showed [12] for scaler $u \in H^1(\Omega,\mathbb{R})$ that any objective functional $\bar{J}(\Omega, u) = \int_\Omega h(\nabla u)\,dx$ with convex, positive function $h : \mathbb{R}^3 \to \mathbb{R}_+$ is lower semcontinuous in the weak $H^1(\Omega,\mathbb{R})$ topology. As lower semicontinuity is an essential ingredient to existence proofs for optimal shapes, we now look for conditions on the crack radius distribution that will ensure the convexity of the function $h(q)$.

Proposition 3.1. *Suppose that the crack size density measure ρ is absolutely continuous w.r.t. the Lebsgue measure da, $d\rho(a) = \varrho(a)da$, $\varrho(a) > 0$ for $a \in \mathbb{R}_+$. We furthermore assume that $\alpha(a) = -\log \varrho(a)$ is differentiable on $\mathbb{R}^+$ and*

$$\alpha'(a) \geq \frac{3}{2}\frac{1}{a} \quad \forall a > 0. \tag{15}$$

Then $h(q)$ as defined in (14) is convex.

Proof. Let us define the auxiliary function $\tilde{h}(\kappa) = \Phi\left(\frac{1}{(\kappa^+)^2}\right)$ where we use the natural extension $\tilde{h}(\kappa) = 0$ for $\kappa \leq 0$ corresponding to $\lim_{a\to\infty} \Phi(a) = \rho([a,\infty)) = 0$ by upper continuity of the Radon measure ρ for sequences of decreasing sets with finite measure.

Note that $\tilde{h}(\kappa)$ is continuous and second order differentiable for $\kappa \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$. Thus, to show convexity, it suffices that $\tilde{h}''(\kappa) \geq 0 \ \forall \kappa \in \mathbb{R}\setminus\{0\}$. This is trivially true for $\kappa < 0$ as then $\tilde{h}''(\kappa) = 0$. Let us now investigate the case $\kappa > 0$. We get

$$\tilde{h}''(\kappa) = -4\varrho'\left(\frac{1}{\kappa^2}\right)\frac{1}{\kappa^6} - 6\varrho\left(\frac{1}{\kappa^2}\right)\frac{1}{\kappa^4} \overset{!}{>} 0.$$

This is equivalent to

$$-\frac{\varrho'\left(\frac{1}{\kappa^2}\right)}{\varrho\left(\frac{1}{\kappa^2}\right)} = \alpha'\left(\frac{1}{\kappa^2}\right) \overset{!}{>} \frac{3}{2}\kappa^2$$

which holds by the assumption (15) using the substitution $a = \frac{1}{\kappa^2}$.

Let now $q_1, q_2 \in \mathbb{R}^{3\times3}$ and $t \in (0,1)$. With $\kappa_j = \frac{2c(n\cdot q_j\, n)}{\sqrt{\pi}K_{Ic}} \neq 0$, $j = 1,2$, we get from the convexity of $\tilde{h}$

$$\begin{aligned}
\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot(tq_1 + (1-t)q_2)\,n)^+}\right)^2\right) &= \tilde{h}((t\kappa_1 + (1-t)\kappa_2)) \\
&\leq t\tilde{h}(\kappa_1) + (1-t)\tilde{h}(\kappa_2) \\
= t\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot q_1\, n)^+}\right)^2\right) &+ (1-t)\Phi\left(\frac{\pi}{4}\left(\frac{K_{Ic}}{(n\cdot q_2\, n)^+}\right)^2\right).
\end{aligned}$$

The case that involves one or two $\kappa_j = 0$ is trivial as the right hand side is infinite. Integration of this inequality in n over S^2 then yields convexity of h. □

Condition (15) restricts the tail behaviour of the a-density $\varrho(a)$ to a decrease at least as fast as const. $\times\, a^{-\beta}$ for $a \to \infty$ with $\beta \geq 3/2$ as $\alpha \geq$ const. $+ \beta\log(a)$.

Assuming an algebraic scaling for $\varrho(a)$, we can make contact with the classical Weibull type objective functionals (5).

Proposition 3.2. *Let $u \in H^1(\Omega, R^3)$ and $\beta \geq \frac{3}{2}$ be given such that*

$$\alpha(a) = \alpha_0 + \beta\log(a), \ \textit{i.e.}\ \varrho(a) = e^{-\alpha_0}a^{-\beta} \quad \forall a > 0. \tag{16}$$

Then

$$\bar{J}(\Omega, u) = \frac{1}{4\pi} \int_\Omega \int_{S^2} \left(\frac{n \cdot \sigma(Du)n}{\sigma_0} \right)^m dn\, dx, \tag{17}$$

with $m = 2(\beta - 1) \geq 1$ *and*

$$\sigma_0 = e^{-\alpha_0/2(\beta-1)} (\beta - 1)^{1/2(\beta-1)} \sqrt{\frac{4}{\pi}} K_{Ic} \tag{18}$$

Proof. We have $\Phi(a) = \frac{e^{\alpha_0}}{(\beta-1)} a^{-(\beta-1)}$. One obtains

$$\Phi\left(\frac{\pi}{4} \left(\frac{K_{Ic}}{(n \cdot \sigma(Du)n)^+} \right)^2 \right) = \left(\frac{(n \cdot \sigma(Du)n)^+}{e^{-\alpha_0/2(\beta-1)} (\beta - 1)^{1/(2(\beta-1)} \sqrt{\frac{4}{\pi}} K_{Ic}} \right)^{2(\beta-1)}.$$

□

Remark 3.3. Typical experimental values of m range from 5 to 25, see [25]. In particular the assumptions of Proposition 3.1 do not rule out the cases of physical interest. Note that the large m limit is deterministic.

Remark 3.4. The dimensional mismatch between σ_0 and the stress intensity K_{Ic} in equation (18) is explained by the fact that $\Phi(a)$ is a functional of a dimensional quantity a. Understanding Φ as a function of a numerical value, we need to introduce a length scale $a_0 = [\mathrm{m}]$ and consider $\Phi(a/a_0)$, which divides K_{1c} by $\sqrt{a_0}$.

Corollary 3.5. *Let the Weibull local failure intensity function* $h_W : \mathbb{R}^{3\times3}_s \to \mathbb{R}_+$ *be defined as*

$$h_W(q) = \frac{1}{4\pi} \int_{S^2} \left(\frac{(n \cdot q\, n)^+}{\sigma_0} \right)^m dn. \tag{19}$$

Then h_W *is convex for* $m \geq 1$ *and continuous.*

4 Shapes with optimal survival probability

Having the results of the previous section at hand, we can now show the existence of shapes with optimal survival property. We use the notation

$$C(\zeta, \theta, l) = \{x \in \mathbb{R}^3 : |x| < l, x \cdot \zeta > |x| \cos(\theta)\} \tag{20}$$

for the cone with height l, direction ζ and opening angle θ. We need the following definition:

Definition 4.1 (Ref. [12, 6])**.** *Let* Ω_0 *be a bounded open set in* $\mathbb{R}^3$. *For* $\theta \in (0, \pi/2), l > 0, r > 0, 2r \leq l$ *by* $\Pi(\theta, l, r)$ *we denote the set of all subsets* Ω *of* Ω_0 *satisfying the cone property, i.e., for any* $x \in \partial\Omega$ *there exists a cone* $C_x = C_x(\zeta_x, \theta, l)$, *where* ζ_x *denotes a unit vector in* $\mathbb{R}^3$, *s.t.*

$$y + C_x \subset \Omega, \quad y \in B(x, y) \cap \Omega,$$

where $B(x, r)$ *is the open ball in* $\mathbb{R}^n$ *with radius* r *centred at* x.

We now fix Ω_0 and as admissible shapes we accept all subsets of Ω_0 fulfiling the cone property. As we will deal with problems involving mixed boundary conditions, we need an appropriate extension operator.

Theorem 4.2 (Theorem II.1 in [6]). *Let $\theta, l, r \in \mathbb{R}$ s.t. $\theta \in]0, \pi/2[$ and $2r \leq l$ and let $n \in \mathbb{N}$. There exists a constant $K(\theta, l, r)$ depending on $\Omega \in \Pi(\theta, l, r)$ through θ, h, r, only, and s.t. for all $\Omega \in \Pi(\theta, l, r)$ there exists a linear and continuous extension operator $p_\Omega : H^n(\Omega) \to H^n(\mathbb{R}^3)$, s.t. $p_\Omega u(x) = u(x)$ for all $x \in \Omega$, with*

$$\|p_\Omega\| \leq K(\theta, l, r)$$

Proof. See proof of Theorem II.1 in [6]. □

Further, we need the following result.

Lemma 4.3 (Ref. [12]). *The class $\Pi(\theta, l, r)$ of domains is relatively compact and is closed with respect to the strong $L^2(\Omega_0)$ topology.*

Proof. Theorem III.1 in [6] states that $\Pi(\theta, l, r)$ is relative compact, Theorem III.2 in [6] shows that it is closed. □

Given a maximal domain $\widehat{\Omega}$ that fulfils the uniform cone property for given θ, l, r we define the set of admissible domains as

$$\mathscr{O}^{\mathrm{ad}} = \{\Omega \in \Pi(\theta, l, r) \ : \ \Omega \subset \widehat{\Omega}\}.$$

The main tool for showing the existence of optimal shape is the following theorem.

Theorem 4.4. *Let h be continuous, non negative, and convex. Assume that for $\{\Omega_n\} \subset \Pi(\theta, l, r)$ we have*

$$\Omega_n \to \Omega, \quad a.e. \ in \ \Omega_0,$$

i.e., the characteristic functions of Ω_n converge to the characteristic function of Ω, and that for the extension $\tilde{u}_n = p_\Omega(u_n)$ of $u_n \in H^1(\Omega_n)$ we have

$$\tilde{u}_n \rightharpoonup \tilde{u}, \quad in \ H^1(\Omega_0),$$

where $\tilde{u} = p_\Omega(u)$. Then, the following inequality holds:

$$\int_\Omega h(Du(x))dx \leq \liminf_{m\to\infty} \int_{\Omega_m} h(Du_m(x))dx.$$

Proof. The proof of Theorem 2.1 in [12] extends without modifications from scalar u to vector valued u. □

In order to apply this theorem, we have have to show that an arbitrary sequence of $\{\Omega_n, u_n\}$ of admissible domains and solutions has a subsequence that converges.

Lemma 4.5. *Let $\partial\Omega_D$ and $\partial\Omega_N$ be defined as above and let $\{\Omega_n, u_n\}_{n=1}^{\infty}$ an arbitrary sequence of admissible domains and their corresponding solutions, i.e., $\Omega_n \in \Pi(\theta, l, r)$ and $u_n = u(\Omega_n)$ solves* (2) *in the domain Ω_n. Then one can find its subsequence also denoted by the pair (Ω_n, u_n) and elements $\Omega \in \Pi(\theta, l, r)$ and $u \in (H^1(\mathbb{R}^3))^3$ such that*

$$\begin{cases} \Omega_n \to \Omega, \\ \tilde{u}_n \rightharpoonup \tilde{u}, \end{cases}$$

where $\tilde{u}_n$ and $\tilde{u}$ are the extensions of u_n and u to $\mathbb{R}^n$ and u solves (2) *in Ω.*

Proof. We define the set of admissible displacements as

$$\mathbb{V}(\Omega) = \{v \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \partial\Omega_D\}.$$

For the bilinear form $\mathscr{B}_\Omega$ in (2) using the ellipticity condition (4) we get for all $v \in \mathbb{V}(\Omega)$:

$$\begin{aligned} \mathscr{B}_\Omega(v, v) &= \int_\Omega \operatorname{tr}(\varepsilon(Dv)\sigma(Dv))dx \\ &= \int_\Omega \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} c_{i,j,k,l}\varepsilon(Dv)_{k,l}\varepsilon(Dv)_{i,j}dx \\ &\geq q \int_\Omega \sum_{i,j=1}^{3} \varepsilon(Dv)_{i,j}\varepsilon(Dv)_{i,j}dx = q\|\varepsilon(Dv)\|_{0,\Omega}^2. \end{aligned}$$

Using this, we obtain

$$q\|\varepsilon(Du_n)\|_{0,\Omega_n}^2 \leq \mathscr{B}_{\Omega_n}(u_n, u_n) = \int_{\Omega_n} f \cdot u_n dx + \int_{\partial(\Omega_n)_{N_{\text{fixed}}}} g \cdot u_n ds \leq (c_1 + c_2) \cdot \|u_n\|_{1,\Omega_n},$$

where the constant c_1 accounts for the bound of the integral over $f \cdot u_n$ and c_2 originates from the application of the trace theorem over the fixed Neumann boundary of the domain. Note that while c_2 depends on the domain under consideration obviously we can use the extension $\bar{u}_n$ of u_n to $\widehat{\Omega}$. As we have

$$\|\bar{u}_n\|_{1,\widehat{\Omega}} \leq K(\theta, l, r)\|\bar{u}_n\|_{1,\Omega_n} = K(\theta, l, r)\|u_n\|_{1,\Omega_n}$$

the estimate holds by including the factor $K(\theta, l, r)$ in c_2. From Korn's inequality we can follow that there exists a $\beta > 0$ such that for all $v \in \mathbb{V}(\Omega)$ we have

$$\|\varepsilon(v)\|_{0,\Omega}^2 \geq \beta\|v\|_{1,\Omega},$$

further a result from [20] guarantees that this β can be uniformly bounded for all domains under consideration and we obtain that there exists a constant c for all Ω_n such that

$$\|u_n\|_{1,\Omega_n} \leq c.$$

Due to Theorem 4.2 the extension $\tilde{u}_n$ of u_n to $\mathbb{R}^3$ is bounded and so is the extension to $\widehat{\Omega}$. Using this and Lemma 4.3 we obtain that there exists a subsequence of $\{\Omega_n, \tilde{u}_n\}_{n=1}^{\infty}$ where $\Omega_n \to \Omega$ due to Lemma 4.3 and where $\tilde{u}_n$ converges weakly to some function $u \in (H^1(\widetilde{\Omega}))^3$.

It remains to show that this u solves (2), for this purpose we proceed as in the proof of Proposition IV.1 in [6]. We have that

$$\mathscr{B}_{\Omega_n}(u_n, v) = \int_{\Omega_n} f \cdot v \, dx + \int_{\partial(\Omega_n)_N} g \cdot v \, ds, \quad \forall v \in H^1_{\partial(\Omega_n)_D}(\Omega_n).$$

This is equivalent to

$$\int_{\widehat{\Omega}} \chi(\Omega_n) \mathrm{tr}(\varepsilon(Du)\sigma(Dv)) \, dx = \int_{\widehat{\Omega}} \chi(\Omega_n) f \cdot v \, dx + \int_{\widehat{\Omega}} \chi(\partial(\Omega_n)_N) g \cdot v \, ds, \quad \forall v \in H^1_{\partial\Omega_D}(\Omega),$$

where $\chi(\Omega_n)$ denotes the characteristic function of Ω_n and $\chi(\partial(\Omega_n)_N)$ the one of $\partial(\Omega_n)_N$ as usual. We show the convergence of each of the integrals. For the first integral of the right hand side we obviously have that for each $v \in L^2(\widehat{\Omega})$ we have

$$|\chi(\Omega_n) v| \leq |v|$$

and as the characteristic function converges a.e., we obtain that $\chi(\Omega_n) v \to \chi(\Omega) v$ and so we get

$$\int_{\widehat{\Omega}} \chi(\Omega_n) f \cdot v \, dx \quad \to \int_{\widehat{\Omega}} \chi(\Omega) f \cdot v \, dx.$$

As this is true for $v \in L^2(\widehat{\Omega})$ it holds for $H^1_{\partial\Omega_D}(\Omega)$, as well. For the partial derivatives in the integral on the left hand side the same argument holds true. For the second integral on the right hand side we can argue in the same manner, as the convergence of $\chi(\Omega_n)$ implies the convergence of the characteristic function of the boundary $\partial(\Omega_n)_N$. □

We are now in the position to prove the main result of this work:

Theorem 4.6. *Let $\partial\Omega_D$ and $\partial\Omega_N$ be defined as above and let $\{\Omega_n, u(\Omega_n)\}_{n=1}^{\infty}$ a minimizing sequence of admissible domains and their corresponding solutions, i.e., $\Omega_n \in \Pi(\theta, l, r)$, $u(\Omega_n)$ solves (2) in the domain Ω_n and*

$$\lim_{n\to\infty} \bar{J}(\Omega_n, u(\Omega_n)) = \inf_{\Omega \in \Pi(\theta,l,r)} \bar{J}(\Omega, u(\Omega)), \tag{21}$$

Where $\bar{J}$ is defined as in (17). *Further assume that the requirements of Propositions 3.1 are fulfiled. Let $\tilde{u}^*$ and $\Omega^* \in \Pi(\theta, l, r)$ be the limit points of a subsequence as defined in Lemma 4.5. Then the restriction $u^* = u(\Omega^*)$ of the weak limit $\tilde{u}^*$ and Ω^* solve the shape optimization problem* (13). *Thus there exist shapes $\Omega^* \in \Pi(\theta, l, r)$ that minimize the probability of failure* (5). *This in particular applies to the Weibull model for $m > 0$.*

Proof. The function h given by (14) is obviously non negative, as the integrand is non negative. Proposition 3.1 gives convexity of h and Lemma 2.5 its continuity. Further, the strong convergence of the domains and the weak convergence of the corresponding solution is guaranteed by Lemma 4.5, so all requirements of Theorem 4.4 are fulfiled and the assertion follows. By Proposition 3.2 and Corollary 3.5, the Weibull model is a special case. □

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