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An unconditionally stable explicit finite difference scheme for nonlinear European option pricing problems

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Abstract. This paper considers the numerical solution of two nonlinear Black-Scholes equations, modelling the replication of contingent claims in illiquid markets. A monotone unconditionally stable explicit finite difference scheme, ensuring positive numerical solution and avoiding unstable oscillations, is proposed. Consistency and convergence of the scheme are studied. Numerical experiments validate these properties of the scheme.

1 Introduction and formulation of the differential problems

The interest in pricing financial derivatives - among them in pricing options - arises from the fact that financial derivatives, also called contingent claims, can be used to minimize losses caused by price fluctuations of the underlying assets. The process of protection is called *hedging*.

Option pricing theory has made a great leap forward since the development of the Black-Scholes option pricing model by Fisher Black and Myron Scholes in 1973 and previously by Robert Merton. The celebrated Black-Scholes model, a starting-point of modern finance,

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,\ (S,t) \in (0,\infty) \times (0,T),\\ V(S,T) = f(S),\ S \in (0,\infty)\end{aligned} \tag{1}$$

is based on several stylized assumptions which are too restrictive in practice. If transactions costs, feedback effects from the trading activity, market illiquidity are not neglected the linear Black-Scholes equation is replaced by a nonlinear one. Many of the nonlinear modifications of the Black-Scholes equation can be summarized in the following backward parabolic problem:

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}\left(S, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}\right)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, (S,t) \in (0,\infty) \times (0,T)\\ V(S,T) = f(S), S \in (0,\infty)\end{aligned} \tag{2}$$

where S denotes the price of underlying asset, t stands for time, T - for maturity, $V(S,t)$ is the option price, $\hat{\sigma}$ is the (modified) volatility function, r - the risk free interest rate and $f(S)$ is the payoff function.

There are a many results on numerical solution of (1) and its generalizations [8,24,26]. However, in the numerical literature, only few results can be found on the numerical solution of nonlinear Black-Sholes equations of type (2). In [5] Company et al. propose an explicit finite difference scheme that requires a restrictive stability condition on the time and spatial mesh sizes. Ankudinova and Ehrhardt [3] use a Crank-Nicolson method combined with a high order compact difference scheme to construct a numerical scheme for the linearized Black-Scholes equation using frozen values of nonlinear volatility. Implicit numerical schemes for nonlinear option pricing problems with uncertain volatility have been analyzed by Lesmana and Wang [17] as well as by Pooley et al. [21], where an iterative approach is required to solve the nonlinear algebraic equations resulting from the discretization.

This paper focuses on nonlinear models, pricing the replication of a European contingent claim in a market with imperfect liquidity. Market liquidity has become currently an issue of very high concern in financial risk management. Most of the option pricing models assume that an option trader cannot affect the underlying asset price in trading the underlying asset to replicate the option payoff, regardless of the trading size. This is reasonable only in a perfect liquid market. The market liquidity of assets affects their prices and expected returns. Theory and empirical evidence suggests that investors require higher return on assets with lower market liquidity to compensate them for the higher cost of trading these assets.

1.1 The Frey and Patie model

We first consider the approach of Frey and Patie [10] in modeling the hedge cost when replicating the option payoff in illiquid markets. It is assumed that stock price dynamics follows the stochastic differential equation (SDE)

$$dS_t = \sigma S_{t-} dW_t + \rho\lambda(S_{t-}) S_{t-} d\alpha_t^+, \tag{3}$$

where W is a standard Brownian motion, σ is the constant volatility, S_{t-} denotes the left limit $\lim_{S\to t, S<t} S_{t-}$ of the stock price S_{t-}, α_t denotes the number of shares in the portfolio at time t and $\lim_{\alpha_t^+\to t, S<t} \alpha_S$. The parameter ρ is a characteristic of the market that does not depend on the payoff of the derivatives. For $\rho = 0$ the model (3) reduces to the linear Black-Scholes model, but we consider the general nonlinear case $\rho \neq 0$. The liquidity profile of the market is described by the continuous and positive function $\lambda(S)$.

The problem of hedging a terminal value claim with maturity T and payoff $f(S)$ is modeled as a nonlinear version of the Black-Scholes partial differential equation (PDE) for the hedge cost $V(S,t)$

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1-\rho\lambda(S)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} = 0, \quad (S,t) \in \Omega_T := \Omega \times (0,T], \\ &\qquad\qquad\qquad\qquad\qquad V(S,T) = f(S), \quad S \in \Omega := (0,+\infty), \end{aligned} \tag{4}$$

where space derivative $\frac{\partial^2 V}{\partial S^2}(S,t)$ satisfies the assumption A4 of [10]

$$1 - \rho\lambda(S)\frac{\partial^2 V}{\partial S^2}(S,t) \geq \delta_0 > 0, \quad (S,t) \in \Omega_T. \tag{5}$$

The payoff function $f(S)$ is assumed to be a continuous piece-wise linear function and in the case of replicating a vanilla European option we have

$$V(S,T) = \begin{cases} \max(S-E,0), \text{ in the case of a call option,} \\ \max(E-S,0), \text{ in the case of a put option,} \end{cases} \tag{6}$$

where E is the strike price.

1.2 The Liu and Yong model

Liu and Yong [20] considered also the problem of hedging a terminal value claim with maturity T and payoff $f(S)$ for the stock price SDE for $t \geq 0$

$$dS(t) = \{\mu(t,S(t)) + \lambda(t,S(t))\eta(t)\}dt + \{\sigma(t,S(t)) + \lambda(t,S(t))\zeta(t)\}dW(t),$$

$\mu(t,S(t))$ and $\sigma(t,S(t))$ are the expected return and the volatility, respectively, $\lambda(t,S(t))$ is the price impact function of the trader for some processes $\eta(t)$ and $\zeta(t)$. They obtained nonlinear Black-Scholes PDE

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1-\lambda(S,t)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in \Omega_T, \\ &\qquad\qquad\qquad\qquad\qquad V(S,T) = f(S), \quad S \in \Omega := (0,+\infty), \end{aligned} \tag{7}$$

for the case of constant interest rate ($r \geq 0$) and reference volatility ($\sigma > 0$) and the terminal condition is chosen as in (6).

The existence and uniqueness of a classical (Hölder) solution of the problem (7) and comparison principle is studied [20]. Also, the solution satisfies (under some regularity assumptions):

$$1 - \lambda(S,t)S\frac{\partial^2 V}{\partial S^2} \geq \delta_0 > 0, \; (S,t) \in \Omega \times [0,T], \tag{8}$$

$f(e^x)$ is Lipschitz continuous and $e^{-\beta\sqrt{1+x^2}}f(e^x)$ is bounded for some $\beta \geq 0$.

The price impact function $\lambda(S,t)$, implemented in the nonlinear problem (7),

$$\lambda(S,t) = \begin{cases} \frac{\gamma}{S}(1-e^{-\beta(T-t)}), \quad \underline{S} \leq S \leq \overline{S}, \\ 0, \; otherwise. \end{cases} \tag{9}$$

reflects the assumption that as a trader buys, the stock price goes up and as the trader sells, the stock price goes down. The constant price impact coefficient $\gamma > 0$ measures the price impact per traded share and $\underline{S}$ and $\overline{S}$ represent, respectively, the lower and upper limit of the stock price within which there is a price impact.

The price impact function influences heavily both the differential problem and the numerical method. By inspecting (9) one observes that the PDE (7) *is linear outside the interval* $(\underline{S}, \overline{S})$.

1.3 Qualitative behaviour of the truncated nonlinear problem

Since, in general, grid based numerical methods are applied to a finite domain one must consider an artificial truncation of the domain when the differential problem is defined on the semi-real axis which is the common case in computational finance. The rule of the thumb "*three or four times the exercise price*" is often used when choosing the right boundary number $S_{max} = b$. Kangro and Nicolaides [16] provide a rigorous mathematical analysis of the location of the artificial boundary b in the general case of time- and space-dependent coefficients in linear one- and multidimensional Black-Scholes problems. Consistent with their analysis, we consider problems (4) and (7), equipped with the Dirichlet boundary conditions

$$V(0,T) = f(0) = 0,\ V(b,t) = f(b) = \max(b - E, 0). \tag{10}$$

on the truncated domain $\Omega_T^{tr} := [0,b] \times [0,T]$.

We discuss the qualitative behaviour of the nonlinear problems (4) and (7), in particular the well-posedness conditions (5) and (8), using the arguments of Agliardi et al. [2] on the comparison principle for nonlinear degenerate backward parabolic PDEs

$$u_t + x^2 G(x,t,p,q) = 0,\ (x,t) \in \Pi_T := (0,b) \times (0,T), \tag{11}$$

where $p \equiv \frac{\partial u}{\partial x}, q \equiv \frac{\partial^2 u}{\partial x^2}$ and $G \in C^2(\Pi_T \times \mathbb{R}^2)$. The parabolic part of the boundary is denoted by $\Gamma_T = I \cup II \cup III$, where $I = \{x = 0, 0 < t < T\}$, $II = \{0 < x < b, t = T\}$, $III = \{x = b, 0 < t < T\}$, $0 = x_{min} \le x_{max} = b$.

Suppose that u is a classical solution of (11) in Π_T and

$$G_q \equiv \frac{\partial G}{\partial q}(x,t,\psi^{'}(x),\psi^{''}(x)) > 0 \ \text{ for }\ 0 \le x \le b,\ ' \equiv \partial/\partial x, \tag{12}$$

where

$$u_{II} = u(x,T) = \psi(x) \in C^2[a,b],$$

i.e. $\partial u/\partial x = \psi^{'}(x), \partial^2 u/\partial x^2 = \psi^{''}(x)$ for $t = T$. It is proved in [2] that under the condition (12) and $G(x,t,0,0) = 0$ there exists a positive constant $\delta > 0$ such that $\max_{\overline{\Pi}_{T,T-\delta}} = \max_{\Gamma_{T,T-\delta}}$, where $\overline{\Pi}_{T,T-\delta} = \{\delta < t < T, a \le x \le b\}$, $\Gamma_{T,T-\delta}$ is the parabolic part of the boundary of $\Pi_{T,T-\delta}$. Similarly, $\min_{\overline{\Pi}_{T,T-\delta}} = \min_{\Gamma_{T,T-\delta}}$. As discussed in [2], one can obtain comparison principle in the whole rectangle Π_T by constructing barrier functions.

Remark 1. Since the payoff function $V(S,T) = f(S)$ (6) is non-smooth in order to apply the local comparison principle result one has to apply a smoothing technique, see, for example, Evans [9], in order to validate the comparison result. Also, in order to rewrite the Liu and Yong model (7) as given in (11) one has apply the Barles-Soner transformation to remove the convection term.

Let us now consider the Frey and Patie problem (4). We have

$$G_q(S,q) = \frac{\sigma^2}{2\left(1-\rho\lambda(S)Sq\right)^2} \frac{1+\rho\lambda(S)Sq}{1-\rho\lambda(S)Sq}$$

so that the condition (12) for the local comparison principle takes the form

$$\frac{\sigma^2}{2\left(1-\rho\lambda(S)Sf''\right)^2} \frac{1+\rho\lambda(S)Sf''}{1-\rho\lambda(S)Sf''} > 0 \ \text{ for } \ 0 = S_{min} \leq S \leq S_{max} = b.$$

This inequality will be fulfilled if, for example,

$$\left|f''(S)\right| < \frac{1}{\rho \max\left|\lambda(S)S\right|} \ \text{ in the interval } [0,b]. \tag{13}$$

Then also (5) holds.

By the comparison principle (that is equivalent to maximum principle for (11)) we have that *the solution of the initial-boundary value problems, corresponding to* (4) *and* (7) *is positive (short for non-negative) if* $f(S)$ *is positive.* This is an important property since the problems model the hedge cost of replication of a contingent claim.

The aim of this paper is the numerical solution of (4) and (7) by a stable positive explicit finite difference scheme. Section 2 analyzes the semi-discretization of the nonlinear problem (4) and the consequent linearization by the Picard iteration. In Section 3 we present and investigate the local Crank-Nicolson time-stepping method (LCN) for the problem (4). The next section, Section 4, is devoted to the application of the numerical method to the problem (7). Finally, numerical experiments are given in Section 5, illustrating the obtained properties of the numerical method and giving numerical evidence to our theoretical analysis.

2 Spatial semi-discretization and linearization

This section is devoted to the analysis of the semi-discretization of equations (4), equipped with the final condition (6) and boundary conditions (10). After performing a time-reversal we introduce the spatial grid Ω_h with step $h = \triangle S$ by the nodes $S_i = ih,\ i = 0,\ldots,M$ so that $Mh = b$, while we set $t_n = n\tau,\ n = 1,2,\ldots,N$ for the temporal step $\tau = T/N$.

Before we apply the spatial semi-discretization the following standard assumption on the regularity of the differential solution is made.

Assumption 1 *The solution of the problem* (4) *has continuous spatial derivatives up to fourth order.*

The corresponding *autonomous* ODEs system (by the method of vertical lines) for the semi-discrete solution $v(t) = [v_1(t), \ldots, v_{M-1}(t)]^\top$ by the centered-space approximation

$$\frac{\partial^2 V}{\partial S^2}(S_i, t) = \frac{V(S_{i+1}, t) - 2V(S_i, t) + V(S_{i-1}, t)}{h^2} + O(h^2) \tag{14}$$

is obtained as

$$v'(t) = A(v(t))v(t) + g, \quad t \in [0, T], \tag{15}$$

with

$$\begin{gathered} A(v) = \frac{1}{2h^2} tridiag\left(\beta_i(v), \alpha_i(v), \gamma_i(v)\right) \\ \alpha_i(v) = -2\hat{\sigma}_i^2(v)S_i^2, \quad \beta_i(v) = \hat{\sigma}_i^2(v)S_i^2, \quad \gamma_i(v) = \hat{\sigma}_i^2(v)S_i^2, \\ \hat{\sigma}_i^2(v) = \frac{\sigma^2}{\left(1 - \rho\lambda(S_i)S_i\Delta_i v\right)^2}, \end{gathered} \tag{16}$$

where $g \in \mathbb{R}^{M-1}$ is the vector, generated by the boundary conditions,

$$g = \frac{1}{2h^2}[\beta_1 f(0), 0, \ldots, 0, \gamma_{M-1} f(b)]^\top$$

and Δ_i is the finite difference operator, corresponding to (14).

2.1 Properties of the semi-discrete nonlinear system

We now discuss the existence of unique solution of the semi-discrete nonlinear system (15). The consistency and stability properties of the scheme are proven. The qualitative behaviour of the semi-discrete solution is investigated by the comparison principle and convergence to the viscosity solution of nonlinear differential problem is obtained.

Lemma 1. *The system* (15) *has unique solution* $v(t) = [v_1(t), \ldots, v_{M-1}(t)]^\top$ *in the convex domain*

$$\mathcal{D} := \{w \in \mathbb{R}^{M-1} : |\Delta_i w| < \frac{1}{\rho\,|\lambda(S_i)S_i|}, \quad i = 1, \ldots, M-1, t \in [0, T]\}$$

Proof. In the following considerations we use the standard notations for the discrete maximum norm

$$\|w\|_\infty = \max_{1 \le i \le M-1} |w_i|\,, \quad \|B\|_\infty = \max_{1 \le j \le M-1} \sum_{k=1}^{M-1} |b_{jk}|$$

for $w \in \mathbb{R}^{M-1}$ and $B \in \mathbb{R}^{(M-1)\times(M-1)}$. Moreover, we also introduce

$$\overline{\delta}_0 := 1 - \rho\lambda(S_i)S_i\Delta_i w$$

so that we have $\overline{\delta}_0 > 0 \ \forall w \in \mathcal{D}$. The Lipschitz continuity of the nonlinear operator $F(w) := A(w)v + g$

$$\|F(\tilde{w}) - F(w)\|_\infty \le \sup_{w \in \mathcal{D}} \|J_F(w)\|_\infty \|\tilde{w} - w\|_\infty \tag{17}$$

is now considered as $J_F(w)$ is the Jacobian matrix of F (note that $\|F(w)\|$ is bounded as $w \in \mathcal{D}$). One computes

$$\begin{aligned}
\frac{\partial F_i}{\partial w_i} &= \frac{1}{2h^2}\left(\alpha_i + \frac{\partial \alpha_i}{\partial w_i} w_i + \frac{\partial \beta_i}{\partial w_i} w_{i-1} + \frac{\partial \gamma_i}{\partial w_i} w_{i+1}\right), \\
\frac{\partial F_i}{\partial w_{i-1}} &= \frac{1}{2h^2}\left(\beta_i + \frac{\partial \alpha_i}{\partial w_{i-1}} w_i + \frac{\partial \beta_i}{\partial w_{i-1}} w_{i-1} + \frac{\partial \gamma_i}{\partial w_{i-1}} w_{i+1}\right), \\
\frac{\partial F_i}{\partial w_{i+1}} &= \frac{1}{2h^2}\left(\gamma_i + \frac{\partial \alpha_i}{\partial w_{i+1}} w_i + \frac{\partial \beta_i}{\partial w_{i+1}} w_{i-1} + \frac{\partial \gamma_i}{\partial w_{i+1}} w_{i+1}\right).
\end{aligned} \tag{18}$$

Further we have

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial w_i} &= \frac{8}{h^2}\frac{\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3}, \\
\frac{\partial \alpha_i}{\partial w_{i+1}} &= \frac{\partial \alpha_i}{\partial w_{i-1}} = \frac{\partial \beta_i}{\partial w_i} = \frac{\partial \gamma_i}{\partial w_i} = -\frac{4}{h^2}\frac{\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3}, \\
\frac{\partial \beta_i}{\partial w_{i+1}} &= \frac{\partial \beta_i}{\partial w_{i-1}} = \frac{\partial \gamma_i}{\partial w_{i+1}} = \frac{\partial \gamma_i}{\partial w_{i-1}} = \frac{2}{h^2}\frac{\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3}.
\end{aligned} \tag{19}$$

so that one derives

$$\begin{aligned}
\frac{\partial F_i}{\partial w_i} &= \frac{1}{2h^2}\left(\alpha_i + \frac{4\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3} \Delta_i w\right), \\
\frac{\partial F_i}{\partial w_{i-1}} &= \frac{1}{2h^2}\left(\beta_i + \frac{2\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3} \Delta_i w\right), \\
\frac{\partial F_i}{\partial w_{i+1}} &= \frac{1}{2h^2}\left(\gamma_i + \frac{2\sigma^2 \rho \lambda(S_i) S_i^3}{\overline{\delta}_0^3} \Delta_i w\right).
\end{aligned}$$

It is now obvious that $F(w)$ is Lipshitz continuous with Lipshitz constant $L = O\left(h^{-2}\right)$ for $w \in \mathcal{D}$ and therefore the assertion follows [13]. □

Remark 2. We comment that the condition

$$|\Delta_i v(0)| \le \frac{1}{\rho\,|\lambda(S_i) S_i|}, \quad i = 1, \ldots, M-1,$$

may be regarded as a semi-discrete equivalent of (13). The condition for existence and uniqueness of semi-discrete solution may be further relaxed to

$$1 - \rho \lambda(S_i) S_i \Delta_i v(t) > 0, \quad i = 1, \ldots, M-1, t \in [0, T].$$

The comparison principle may, however, not be satisfied under this relaxed requirement. Further we need this important monotonicity property to motivate convergence of the obtained discretizations.

Semi-discrete diffusion problems are usually classified as *stiff* problems since the spectral radius of the difference matrix $A(v)$ is proportional to $O(h^{-2})$. While for advection problems this issue might lead to wrong qualitative behaviour such as oscillations and loss of shape, for diffusion problems (4) it is rather harmless.

The next result is a useful standard tool in estimating the growth of functions that satisfy an integral inequality.

Lemma 2. *[11] (Gronwall) Let σ and ρ be continuous real functions with $\sigma \geq 0$. Let c be a non-negative constant. Assume that*

$$\sigma(t) \leq \rho(t) + c\int_0^t \sigma(s)ds \ \ \forall\, t \in [0, T].$$

Then we have the estimate

$$\sigma(t) \leq e^{ct}\rho(t) \quad \forall\, t \in [0, T].$$

Theorem 1. *The semi-discrete difference scheme* (15) *is consistent and stable.*

Proof. We define the spatial truncation error

$$\sigma_h(t) = V_h^{'}(t) - A(V_h(t))V_h(t) - g,$$

where V_h is the projection of the exact PDE solution on the spatial grid. The consistency estimate is subject to similar considerations as given in the paper of Company et al. [6]. We shall now briefly describe the application of these considerations to our problem. The semi-discrete difference scheme (15) is said to be consistent of order q with (4) if we have [14]

$$\|\sigma_h(t)\|_\infty = O(h^q) \ \text{ uniformly for } 0 \leq t \leq T.$$

Starting with the following consideration

$$\begin{aligned}\sigma_h(t) &= V_h^{'}(t) - AV_h(t) - g = V^{'}(t) - \frac{1}{2}\hat{\sigma}(S_i, t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i, t) - g \\ &+ \frac{1}{2}\hat{\sigma}(S_i, t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i, t) - AV_h(t) = \frac{1}{2}\hat{\sigma}(S_i, t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i, t) - AV_h(t)\end{aligned}$$

we introduce the notation $\Delta_i V_h(t) = x + \Delta x$, where $x = \frac{\partial^2 V}{\partial S^2}(S_i, t)$ and $\Delta x = O(h^2)$, according to (14).

If one considers the function $g_S(x) = \frac{x}{1-\rho\lambda(S)Sx}$ for a fixed value of the underlying asset variable S then $g_S(x)$ is well-defined continuously differentiable function in any domain where $1 - \rho\lambda(S)Sx \neq 0$ which corresponds to the well-posedness condition (8). Therefore, we obtain by the mean value theorem

$$\sigma_h(t) = \sigma^2 S_i^2\left(g_{S_i}(x + \Delta x) - g_{S_i}(x)\right) = \sigma^2 S_i^2 g_{S_j}^{'}(x + \theta\Delta x)\Delta x, \ \ 0 < \theta < 1,$$

and since $g'_{S_j} = \frac{1+\rho\lambda(S)Sx}{(1-\rho\lambda(S)Sx)^3}$ is bounded uniformly in $t \in [0,T]$ we derive that

$$\|\sigma_h(t)\|_\infty = O(h^2), \tag{20}$$

i.e. the semi-discrete difference scheme is consistent of order 2 in space.

Next, we proceed with the stability estimate. The solution of (15) for $t \in [0,T]$ is also a solution of the integral equation

$$v(t) = v(0) + \int_0^t F(v(s))ds.$$

A small perturbation in the initial data results in a different solution

$$\tilde{v}(t) = \tilde{v}(0) + \int_0^t F(\tilde{v}(s))ds$$

and we arrive at

$$\begin{aligned}\|v(t)-\tilde{v}(t)\|_\infty &\le \|v(0)-\tilde{v}(0)\|_\infty + \int_0^t \|F(v(s))-F(\tilde{v}(s))\|_\infty ds\\ &\le \|v(0)-\tilde{v}(0)\|_\infty + L\int_0^t \|v(s)-\tilde{v}(s)\|_\infty ds.\end{aligned}$$

By Lemma 2 we obtain

$$\|v(t)-\tilde{v}(t)\|_\infty \le e^{Lt}\|v(0)-\tilde{v}(0)\|_\infty \quad \forall t \in [0,T]. \tag{21}$$

The obtained estimate (21) is, however, sub-optimal for stiff problems (large L). The following improvement is presented in Hairer et al. [13]

$$\|v(t)-\tilde{v}(t)\|_\infty \le e^{0}\|v(0)-\tilde{v}(0)\|_\infty \quad \forall t \in [0,T]. \tag{22}$$

since by (18) and (19) we have

$$\begin{aligned}\mu_\infty(J_F(v)) = \max_{1\le i\le M-1}\Bigg(&\alpha_i+\beta_i+\gamma_i+\left(\frac{\partial\alpha_i}{\partial v_i}+\frac{\partial\beta_i}{\partial v_i}+\frac{\partial\gamma_i}{\partial v_i}\right)v_i\\ +\left(\frac{\partial\alpha_i}{\partial v_{i-1}}+\frac{\partial\beta_i}{\partial v_{i-1}}+\frac{\partial\gamma_i}{\partial v_{i-1}}\right)v_{i-1}&+\left(\frac{\partial\alpha_i}{\partial v_{i+1}}+\frac{\partial\beta_i}{\partial v_{i+1}}+\frac{\partial\gamma_i}{\partial v_{i+1}}\right)v_{i+1}\Bigg)=0,\end{aligned}$$

where $\mu_\infty(\cdot)$ is the logarithmic maximum norm. □

Definition 1. *[14] The system* (15) *is positive (short for "non-negativity preserving") if*

$$v(0) \ge 0 \;\; \textit{implies} \;\; v(t) \ge 0 \;\; \forall t \ge 0.$$

Theorem 2. *[14] Suppose that the nonlinear operator $F(v) = A(v)v + g$ is continuous and satisfies the Lipschitz condition with respect to v. Then the system* (15) *is positive if for any vector $v \in \mathbb{R}^{M-1}$ and $t \ge 0$*

$$v \ge 0, v_i = 0 \;\; \textit{implies} \;\; F_i(v) \ge 0, \quad i = 1,\dots,M-1. \tag{23}$$

Moreover, if also the following property is valid

$$\frac{\partial F_i(v)}{\partial v_j} \geq 0, \quad i \neq j, \quad i,j = 1, \dots, M-1, \tag{24}$$

we also have the comparison principle for the solution of the system (15)*, i.e.*

$$v(0) \leq \tilde{v}(0) \ \textit{ implies } \ v(t) \leq \tilde{v}(t).$$

Proof. Since the off-diagonal elements β_i and γ_i in (16) are non-negative while the diagonal elements α_i are non-positive the requirement (23) is fulfilled.

We now consider the condition (24) as we obtain by (16),(18) and (19)

$$\frac{\partial F_i}{\partial v_{i-1}}(v) = \frac{1}{2h^2} \frac{\sigma^2 S_i^2}{(1-\rho\lambda(S_i)S_i\varDelta_i v)^2} \left(1 + \frac{2\rho\lambda(S_i)S_i}{(1-\rho\lambda(S_i)S_i\varDelta_i v)} \varDelta_i v\right)$$
$$= \frac{1}{2h^2} \frac{\sigma^2 S_i^2}{(1-\rho\lambda(S_i)S_i\varDelta_i v)^2} \left(\frac{1+\rho\lambda(S_i)S_i\varDelta_i v}{1-\rho\lambda(S_i)S_i\varDelta_i v}\right)$$

Recalling that the semi-discrete solution $v \in \mathcal{D}$ we have $|\varDelta_i v| \leq \frac{1}{\rho|\lambda(S_i)S_i|}$ and therefore $\frac{\partial F_i}{\partial v_{i-1}}(v) > 0$. Analogous result also holds for $\frac{\partial F_i}{\partial v_{i+1}}(v) > 0$. $\square$

Following Barles [4] and taking into account theorems 1 and 2 we have the following corrolary.

Corrolary 1 *The semi-discrete solution of* (15) *converges to the viscosity solution of the problem* (4)*.*

2.2 The Picard iteration

We now consider the solution of (15) for $t \in [t_n, t_n+1]$. It is also the solution of the integral equation

$$v(t) = v(t_n) + \int_{t_n}^{t} (A(v(s))v(s) + g)\, ds$$

and will be approximated by the sequence a functions $v^0, v^1, v^2, \dots,$ where $v^0 = v(t_n)$ and

$$v^k(t) = v^0 + \int_{t_n}^{t} \big(A(v^{k-1}(s))v^k(s) + g\big)\, ds. \tag{25}$$

which is called Picard iteration. By similar, yet simplified, considerations as in the nonlinear case we have that v^k exists and is bounded.

On each time level, $t \in [t_n, t_{n+1}]$, the following estimate is valid for $k = 1$

$$\left\| v(t) - v^1 \right\|_\infty \leq \left(\sum_{j=1}^{\infty} \frac{1}{j!} \left(L(t-t_n)\right)^j \right) (t-t_n) \max_{t_n \leq s \leq t} \left\| \frac{1}{2h^2} A(v^0)v^0 \right\|_\infty,$$

where $L > 0$ is the Lipschitz constant, associated with the nonlinear operator $A(v)$. Since $t_{n+1} - t_n = \tau$ we have

$$\left\|v(t) - v^1\right\|_\infty \le \left(L\tau + O\left((L\tau)^2\right)\right)\tau L\left\|v^0\right\|_\infty = L^2\tau^2\left\|v^0\right\|_\infty + O\left((L\tau)^3\right), \quad (26)$$

i.e. second order of convergence in τ on each time level for a fixed $h > 0$.

It is now reasonable to allow the nonlinearities in (15) to lag one step behind and we obtain the following linear system

$$v'(t) = A_n v(t) + g, \quad t \in [t_n, t_{n+1}], \quad (27)$$

$$A_n = \frac{1}{2h^2} tridiag\left(\beta_i^n, \alpha_i^n, \gamma_i^n\right) \quad (28)$$

where the solution of (27) is also the solution of (25). We used the notations

$$\alpha_i^n = -2\hat{\sigma}_{i,n}^2 S_i^2, \;\; \beta_i^n = \hat{\sigma}_{i,n}^2 S_i^2, \;\; \gamma_i^n = \hat{\sigma}_{i,n}^2 S_i^2.$$

with

$$\hat{\sigma}_{i,n}^2 = \frac{\sigma^2}{\left(1 - \rho\lambda(S_i)S_i\varDelta_i v(t_n)\right)^2}.$$

We define the solution of the linearized system at the final time level as the solution, obtained by successive solution of (27) on each time level $[t_n, t_{n+1}],\ i = 0, \ldots, N-1$.

Lemma 3. *The solution of the linearized system* (27) *at the final time level, converges to the solution of the nonlinear system with rate of convergence 2 in* $\tau = t_{n+1} - t_n,\ n = 0, \ldots, N-1$.

Proof. We define $\epsilon_{n+1}(t) = v(t) - v^1(t),\ t \in [t_n, t_{n+1}],\ i = 0, \ldots, N-1$, and therefore, from (26), we have

$$\|\epsilon_1(t)\|_\infty \le L^2\tau^2\|v^0\|_\infty + O(L^3\tau^3) = L^2\tau^2\|f\|_\infty + O(L^3\tau^3).$$

For $\epsilon_2(t) = v(t) - v^1(t),\ t \in [t_1, t_2]$ one obtains

$$\begin{aligned}
\|\epsilon_2(t)\|_\infty &\le L^2\tau^2\|v^1\|_{\infty, t\in[t_0,t_1]} + O(L^3\tau^3) \\
&\le L^2\tau^2\|-v(t) + v(t) + v^1(t)\|_{\infty, t\in[t_0,t_1]} + O(L^3\tau^3) \\
&\le L^2\tau^2\left(\|\epsilon_1(t)\|_{\infty, t\in[t_0,t_1]} + \|v(t)\|_{\infty, t\in[t_0,t_1]}\right) + O(L^3\tau^3) \\
&\le L^2\tau^2\left(L^2\tau^2\|f\|_\infty + O(L^3\tau^3) + \|v(t)\|_{\infty, t\in[t_0,t_1]}\right) + O(L^3\tau^3)
\end{aligned}$$

and therefore we have

$$\|\epsilon_2(t)\|_\infty \le L^2\tau^2\|v(t)\|_{\infty, t\in[t_0,t_1]} + O(L^3\tau^3).$$

Successive application of these considerations results in

$$\|\epsilon_N(t)\|_\infty \le L^2\tau^2\|v(t)\|_{\infty, t\in[t_{N-2},t_{N-1}]} + O(L^3\tau^3). \square$$

Further, we apply the following abuse of notations, see (28),

$$A_n = tridiag\,(\beta_i^n, \alpha_i^n, \gamma_i^n)$$

and the solution of (27) is given by Smith [23]

$$v(t) = -2h^2 A_n^{-1} g + \exp\left(\frac{t - t_n}{2h^2} A_n\right) \left(v(t_n) + 2h^2 A_n^{-1} g\right). \tag{29}$$

3 The LCN time stepping method for the Frey and Patie problem

When considering the numerical analysis of nonlinear problems one has to decide on the type of time-stepping method. While the implicit numerical schemes are a straight-forward choice because of their stability property they have important practical drawbacks that should be checked before deciding. For instance, how to step the iteration, how to evaluate the additional computational cost, resulting from the application of the iterative process in each time step and from the constraints, involved in the convergence conditions of the iterative method.

In the papers by Company et al. [6,7] the authors propose fully explicit finite difference schemes for the PDEs (4) and (7). The presented numerical analysis is detailed as they investigate the non-negativity and the convexity of the numerical solution as well as the stability and consistency of the schemes. However, these schemes are "*convexity-preserving*" when

$$h > \rho m, \quad m = \max\{S\lambda(S) : 0 \leq S \leq b\}$$

and clearly *this is not an acceptable restriction on the space step* h. More importantly, they are *stable only for the severe restriction on the time step* τ

$$\frac{\tau}{h^2} \leq \frac{1}{4L(h)\sigma^2 b^2}, \quad L(h) = \frac{1}{(1 - \rho m/h)^2} > 0. \tag{30}$$

We are now considering the application and the analysis of an alternative, yet also fully explicit, *unconditionally stable* approach to the time semi-discretization under the following assumption.

Assumption 2 *The solution of the problem* (4) *has continuous temporal derivatives up to second order.*

It is well-known that the Crank-Nicolson time-stepping method is based on the following approximation [23]:

$$\exp\left(\frac{\tau}{2h^2} A_n\right) \approx (I - \mu A_n)^{-1}(I + \mu A_n), \tag{31}$$

where $\mu = \frac{\tau}{4h^2}$. We now present the Lie-Trotter product formula:

Lemma 1 *[25] Let the matrix A can be denoted as $A = \sum_{i=1}^{M-1} A_i$. Then*

$$\exp\left(\frac{t}{h^2}A\right) = \lim_{\delta\to\infty}\left(\prod_{i=1}^{M-1}\exp\left(\frac{tA_i}{\delta h^2}\right)\right)^{\delta}, \quad \delta = 1, 2, \ldots .$$

for any h, t.

The Lie-Trotter product formula is a corollary of the Baker-Campbell-Hausdorff formula (BCH) for $A = A_1 + A_2$ [14]

$$\begin{aligned} &\exp(\tau A_2)\exp(\tau A_1) = \exp(\tau\tilde{A}) \ \text{ with} \\ \tilde{A} = A + \frac{1}{2}\tau[A_2, A_1] + &\frac{1}{12}\tau^2\left([A_2,[A_2,A_1]] + [A_1,[A_1,A_2]]\right) + \ldots, \end{aligned} \tag{32}$$

where $[A_2, A_1]$ denotes the commutator of A_2 and A_1. It follows from Lemma 1

$$\exp\left(\frac{\tau}{2h^2}A\right) \approx \prod_{i=1}^{M-1}\exp\left(\frac{\tau A_i}{2h^2}\right), \tag{33}$$

so (33) is a new approximation. In order to use this approximation we split the matrix A in (28) as follows:

$$A_1 = \begin{bmatrix} \alpha_1^n & \gamma_1^n & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix}, \quad A_{M-1} = \begin{bmatrix} 0 & 0 & \ldots & & 0 \\ 0 & 0 & \ldots & & 0 \\ & \ddots & \ddots & & \\ 0 & \ldots & 0 & & 0 \\ 0 & \ldots & \beta_{M-1}^n & & \alpha_{M-1}^n \end{bmatrix},$$

$$A_i = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & \beta_i^n & \alpha_i^n & \gamma_i^n & \ldots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix}.$$

For any $i = 1, 2, \ldots, M-1$ we obtain by (31)

$$\exp\left(\frac{\tau}{2h^2}A_i\right) \approx (I - \mu A_i)^{-1}(I + \mu A_i) \tag{34}$$

and further application of (33) and (34) results in

$$\exp\left(\frac{\tau}{2h^2}A\right) \approx \prod_{i=1}^{M-1}(I - \mu A_i)^{-1}(I + \mu A_i). \tag{35}$$

We now consider the matrix $I - \mu A_i,\ i = 2, \ldots, M-2$ (similar considerations are valid for $i = 1$ and $i = M-1$). The approximation (35) is applicable to the problem (27) iff the inverse matrix $(I - \mu A_{i-1})^{-1}$ exists.

Lemma 4. *The matrix $I - \mu A_{i-1}$ is a M-matrix.*

Proof. By (28) we have that

$$I - \mu A_i = \begin{bmatrix} 1 \dots & \dots & \dots & \dots & \dots\ 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 \dots & -\mu\beta_i^n & 1-\mu\alpha_i^n & -\mu\gamma_i^n & \dots\ 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 \dots & \dots & \dots & \dots & \dots\ 1 \end{bmatrix}$$

so that $I - \mu A_i$ is a M-matrix if the following conditions are present

$$\begin{aligned} \textit{sign condition}: &\quad 1 - \mu\alpha_i^n > 0,\ \beta_i^n \geq 0,\ \gamma_i^n \geq 0, \\ \textit{diagonal dominance}: &\quad 1 - \mu\alpha_i^n - \mu\beta_i^n - \mu\gamma_i^n \geq 0 \end{aligned}$$

All of the above inequalities are trivial to check. □

Consequently, $I - \mu A_{i-1}$ is non-singular and we combine (27) and (35) to derive a new scheme

$$\acute{U}_{n+1} = -2h^2 A_n^{-1} g + \prod_{i=1}^{M-1} (I - \mu A_i)^{-1}(I + \mu A_i)\left(\acute{U}_n + 2h^2 A_n^{-1} g\right). \tag{36}$$

In order to improve the numerical accuracy of (36) we define $B_i = A_{M-i}$. By substituting B_i into (36) we deduce that

$$\grave{U}_{n+1} = -2h^2 A_n^{-1} g + \prod_{i=1}^{M-1} (I - \mu B_i)^{-1}(I + \mu B_i)\left(\grave{U}_n + 2h^2 A_n^{-1} g\right). \tag{37}$$

We take the mean value of (36) and (37) to obtain a more symmetric scheme

$$\begin{aligned} U_{n+1} = \frac{1}{2}\left(\prod_{i=1}^{M-1} (I - \mu A_i)^{-1}(I + \mu A_i) + \prod_{i=1}^{M-1} (I - \mu B_i)^{-1}(I + \mu B_i)\right) \\ \cdot \left(U_n + 2h^2 A_n^{-1} g\right) - 2h^2 A_n^{-1} g. \end{aligned} \tag{38}$$

The presented method is referred to as the local Crank-Nicolson (LCN) method proposed by Abduwali et al. [1,15].

The matrix $(I + \mu A_i)$ can be denoted by a simple form for $i = 2, 3, \dots, M-2$

$$(I + \mu A_i) = \begin{pmatrix} I_{i-2} & & \\ & \bar{R}_i & \\ & & I_{M-i-2} \end{pmatrix},\ \bar{R}_i = \begin{pmatrix} 1 & 0 & 0 \\ \mu\beta_i^n & 1 + \mu\alpha_i^n & \mu\gamma_i^n \\ 0 & 0 & 1 \end{pmatrix}. \tag{39}$$

where I_i is the $i \times i$ identity matrix.

Similar to (39) we derive

$$(I - \mu A_i)^{-1} = \begin{pmatrix} I_{i-2} & & \\ & \hat{R}_i^{-1} & \\ & & I_{M-i-2} \end{pmatrix},\ \hat{R}_i^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\mu\beta_i^n}{1-\mu\alpha_i^n} & \frac{1}{1-\mu\alpha_i^n} & \frac{\mu\gamma_i^n}{1-\mu\alpha_i^n} \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

We obtain an explicit expression of U^{n+1} and, clearly, (38) is an explicit scheme.

3.1 Positivity and stability

We now consider the positivity property of the numerical solution of (38) and the stability of the fully discrete scheme. While stability analysis is a necessary part of the numerical analysis, positivity also has to be considered when solving problems in finance since the prices and costs are always positive and this valuable property should be preserved by the numerical method.

Theorem 3. *The numerical scheme* (38) *is unconditionally stable.*

Proof. Analogously to the considerations in [15] the application of the Gerschgorin theorem [23] to the matrix A_i implies that the non-zero matrix eigenvalues lie in the disc

$$\left|z + 2\sigma_{i,n}^2 S_i^2\right| \leq 2\sigma_{i,n}^2 S_i^2$$

and therefore they are negative. Then, by the spectral mapping theorem,

$$|\eta_i| \leq |I + \mu\zeta_i| \,/\, |I - \mu\zeta_i| \leq 1$$

for any of the eigenvalues η_i of the the matrix $(I - \mu A_i)^{-1}(I + \mu A_i)$, corresponding to the eigenvalues ζ_i of A_i. Further, we have that $\prod_{i=1}^{M-1} |\eta_i| \leq 1$ and $\rho\left(\prod_{i=1}^{M-1}(I - \mu A_i)^{-1}(I + \mu A_i)\right) \leq 1$, where $\rho(A)$ denotes the spectral radius of the matrix A. Stability follows from this estimate as discussed in [23].

Theorem 4. *The solution of* (38) *is positive on each time level* t_{n+1}, $n = 0, \ldots, N-1$, *if* $f(S)$ *is positive and we assume that*

$$\frac{\tau}{2h^2} \leq \frac{1}{\hat{\sigma}_{i,n}^2 S_i^2} = \frac{\overline{\delta}_0^2}{\sigma^2 b^2}. \tag{41}$$

Proof. We analyze the matrix $I + \mu A_i$, $i = 2, \ldots, M-2$, (analogously for $i = 1$ and $i = M-1$):

$$I + \mu A_i = \begin{bmatrix} 1 \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 \ldots & \mu\beta_i^n & 1 + \mu\alpha_i^n & \mu\gamma_i^n & \ldots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 \ldots & \ldots & \ldots & \ldots & \ldots & 1 \end{bmatrix}.$$

All entries of the matrix $I + \mu A_i$ are non-negative by (41). Successful application of this consideration and Lemma 4 (one observes the positive entries of $(I - \mu A_i)^{-1}$ in (40)) yields a non-negative solution U_{n+1} of (38) on each time level, where U_0 is the projection of $f(S)$ on the spatial grid, iff $g = \mathbf{0}$.

If $g \neq \mathbf{0}$ we have that $A_n^{-1}g$ is negative since $-A_n$ is a M-matrix and g is positive. Therefore, since $U_n + 2h^2 A_n^{-1} g$ can be always considered positive for sufficiently small h, we have a positive solution of (38). □

Indeed, we can relate the condition (41) with the assumption A4 in [10]. It enforces slightly relaxed bound on the temporal step τ than the one (30) obtained by Company et al. [6]. However, in our numerical analysis, *it is not a necessary condition for stability* and, as discussed in Section 5, it is solely a sufficient condition for positivity and monotonicity.

3.2 Consistency and convergence

In this section we discuss the consistency, monotonicity and convergence properties of the fully discrete scheme (38).

Lemma 5. *The local Crank-Nicolson method has the second-order approximation in time.*

Proof. We have the following expansion formula

$$\exp\left(\frac{\tau}{2h^2}A_i\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tau}{2h^2}A_i\right)^n.$$

The local Crank-Nicolson approximation (34) can be regarded as

$$(I-\frac{\tau}{2}\frac{1}{2h^2}A_i)^{-1}(I+\frac{\tau}{2}\frac{1}{2h^2}A_i) = I + \frac{\tau}{2h^2}A_i - \frac{\left(\frac{\tau}{2h^2}A_i\right)^2}{2} + \frac{\left(\frac{\tau}{2h^2}A_i\right)^3}{4} - \dots$$

and is a second-order approximation in time to $\exp\left(\frac{\tau}{2h^2}A_i\right)$. Considering (35), one now derives

$$\prod_{i=1}^{M-1}(I-\mu A_i)^{-1}(I+\mu A_i) = \prod_{i=1}^{M-1}\left(\exp\left(\frac{\tau}{2h^2}A_i\right) - O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\right)$$
$$= \prod_{i=1}^{M-1}\exp\left(\frac{\tau}{2h^2}A_i\right) - O\left(\left(\frac{\tau}{2h^2}\right)^2\right) = \exp\left(\frac{\tau}{2h^2}A\right) + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)$$

since $\prod_{i=1}^{M-1}\exp\left(\frac{\tau}{2h^2}A_i\right)$ is a second-order approximation in time to $\exp\left(\frac{\tau}{2h^2}A\right)$.
□

It follows that the LCN time stepping method is first-order consistent in time. This consideration also follows from the BCH formula (32).

Next, we investigate the error $e_n = U_{n+1} - v(t_{n+1})$, where $v(t_{n+1})$ is the solution of linearized system (27). Subtracting (29) from (38) we obtain

$$e_n = \left(\exp\left(\frac{\tau}{2h^2}A_n\right) + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\right)(U_n + 2h^2A_n^{-1}g) - \exp\left(\frac{\tau}{2h^2}A_n\right)$$
$$\cdot\left(v(t_n) + 2h^2A_n^{-1}g\right) = \exp\left(\frac{\tau}{2h^2}A\right)e_{n-1} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_n^{-1}g$$

and therefore

$$
\begin{aligned}
e_N &= \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)e_{N-1} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-1}^{-1}g \\
&= \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)\left(\exp\left(\frac{\tau}{2h^2}A_{N-2}\right)e_{N-2} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-2}^{-1}g\right) \\
&+ O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-1}^{-1}g = \prod_{i=0}^{N-1}\exp\left(\frac{\tau}{2h^2}A_i\right)e_0 \\
&+ O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\left(2h^2A_{N-1}^{-1}g + \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)2h^2A_{N-2}^{-1}g + \dots\right).
\end{aligned}
$$

Finally, since the initial data $f(S)$ can be projected exactly on the grid, i.e. $e_0 = 0$, and by the condition for the temporal step (41) we obtain a first-order convergence in τ of the numerical solution to the solution of the linearized system.

Rewriting the fully-discrete scheme (38) in the following form

$$
U_{i,n+1} = H(U_{i-1,n}, U_{i,n}, U_{i+1,n}), \quad i = 1, \dots, M-1, \tag{42}
$$

we introduce the definition of a monotone scheme, see Grossmann and Roos [12].

Definition 2. *The scheme* (42) *is* monotone *iff* H *is non-decreasing in each argument.*

Theorem 5. *The scheme* (38) *is monotone and it also satisfies the discrete maximum and comparison principles. The numerical solution converges to the viscosity solution of the problem* (4).

Proof. By the condition (41) we deduce that all elements of the matrices (39) and (40) are positive. Therefore H is non-decreasing in each argument and the scheme is monotone. The discrete maximum and comparison principles by Samarskii [22] follow by the monotonicity of the scheme and the diagonal dominance of the matrices (39),(40).

Putting all results together - convergence of the semi-discrete scheme, convergence of the linearized system, consistency, stability and monotonicity of the fully-discrete scheme - *convergence of the solution of fully-discrete scheme to the viscosity solution of* (4) *follows by [4], second order in space and first order in time.* □

Corrolary 2 *A direct consequence of the monotonicity of the fully-discrete scheme is the monotonicity of the numerical solution* U_{n+1} *w.r.t. the spatial variable if* $f(S)$ *is monotone w.r.t. the spatial variable.*

4 The numerical analysis for the Liu and Yong problem

In this section we discuss the application of the LCN time stepping method to the Liu and Yong model (7). The semi-discretization is performed by the second-order approximations

$$\frac{\partial V}{\partial S}(S_i,t) = \frac{V(S_{i+1},t) - V_(S_{i-1},t)}{2h} + O(h^2)$$
$$\frac{\partial^2 V}{\partial S^2}(S_i,t) = \frac{V(S_{i+1},t) - 2V(S_i,t) + V(S_{i-1},t)}{h^2} + O(h^2).$$

The corresponding *nonautonomous* ODEs system for the semi-discrete solution $v(t) = [v_1(t),\ldots,v_{M-1}(t)]^\top$ by the centered-space approximations (14) is obtained as following

$$v'(t) = A(v(t),t)v(t) + g, \tag{43}$$

$$A(v(t),t) = \frac{1}{2h^2}\,tridiag(\beta_i,\alpha_i,\gamma_i) \tag{44}$$

$$\begin{aligned} \alpha_i(v,t) &= -2(\hat{\sigma}_i^2(v,t)S_i^2 + h^2 r),\ \hat{\sigma}_i^2(v,t) = \frac{\sigma^2}{\left(1-\lambda(S_i,t)S_i\Delta_i v\right)^2} \\ \beta_i(v,t) &= \hat{\sigma}_i^2(v,t)S_i^2 - hS_i r,\ \gamma_i(v,t) = \hat{\sigma}_i^2(v,t)S_i^2 + hS_i r. \end{aligned} \tag{45}$$

Before commenting on the analysis of the semi-discretization (43) we assume

$$\frac{\hat{\sigma}_i^2(t)}{r}S_i = \frac{\sigma^2}{r\left(1-\lambda(S_i,t)S_i\Delta_i v\right)^2}S_i \ge h,\ i=1,\ldots,M-1,$$

which is fulfilled when

$$\frac{\sigma^2}{r\left(1-\lambda(S_i,t)S_i\Delta_i v\right)^2} \ge 1,\ i=1,\ldots,M-1, \tag{46}$$

uniformly in $t\in[0,T]$. By (46) it follows that $\beta_i(t)\ge 0$. Indeed, the requirement (46) will not be present when using an upwind discretization for the convection term on the expense of lower spatial consistency order.

The application of similar considerations as presented in Section 2 results in existence and uniqueness of a positive solution of the nonlinear system (43). Moreover, we also have that the semi-discrete monotone difference scheme is consistent [7] and stable since (analogously to the considerations in Section 2)

$$\mu_\infty\left(J_F(v,t)\right) = -r, \tag{47}$$

where $F(v,t)$ is the nonlinear operator in (43). Therefore, we have convergence of the semi-discrete solution to the solution of problem (7).

The Picard iteration is performed by similar considerations as for the Frey and Patie problem (27).

However, the representation of the solution (29) for the time-dependent matrix A *is not valid unless* $[A(t),A(s)]\neq 0$ [13], where $[A,B]$ denotes the commutator of the matrices A and B. One may also refer to the Magnus expansion for the solution of the matrix linear initial value problem [18].

Let us consider in details the matrix (44) (strictly speaking, it's linearized counterpart). Following the Magnus approach, we express the solution of

$$Y'(t) = A(t)Y(t), \;\; Y(t_0) = Y_0$$

by means of a certain $(M-1) \times (M-1)$ matrix function $\Psi(t, t_0)$

$$Y(t) = \exp\left(\Psi(t, t_0)\right) Y_0.$$

We shall further write $\Psi(t)$ instead of $\Psi(t, t_0)$ for $t_0 = 0$. The function $\Psi(t)$ is constructed as a series expansion

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t), \tag{48}$$

where the first three terms of the series are

$$\Psi_1(t) = \int_0^t A(t_1)dt_1, \; \Psi_2(t) = \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \left[A(t_1), A(t_2)\right]$$
$$\Psi_3(t) = \frac{1}{6}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \left(\left[A(t_1), \left[A(t_2), A(t_3)\right]\right] + \left[\left[A(t_3), A(t_2)\right] A(t_1), \right]\right).$$

We now apply these considerations on linearized on each time level (t_n, t_{n+1}) system of nonlinear ODEs (43) i.e. we apply it to (27), where the matrix A_n corresponds to (44), to obtain

$$\Psi_1(t) = \int_{t_n}^{t_{n+1}} A_n(t_1)dt_1, \; \Psi_2(t) = \frac{1}{2}\int_{t_n}^{t_{n+1}} dt_1 \int_{t_n}^{t_{n+1}} dt_2 \left[A_n(t_1), A_n(t_2)\right].$$

Let us note that these considerations are applicable since

$$\int_{t_n}^{t_{n+1}} \left\|A_n(t_1)\right\|_\infty dt_1 \le \pi \tag{49}$$

for sufficiently small $\tau = t_{n+1} - t_n$ (this is the sufficient condition for convergence of the Magnus series). Indeed, by (46) we have that

$$\left\|A_n(t_1)\right\|_\infty \le \frac{4\hat{\sigma}_{i,n}^2 b^2 + h^2 r}{2h^2}$$

with $\hat{\sigma}_{i,n}$ as given in (45). Therefore, we have the bound for τ

$$\frac{\tau}{2h^2} \le \frac{\pi}{4\hat{\sigma}_{i,n}^2 b^2 + h^2 r}. \tag{50}$$

This estimate can be relaxed further (or even disregarded) if one takes into account that $\mu_\infty\left(A_n(t)\right) = -r$.

Simple calculations for the commutator of $A(t)$ and $A(s)$ show that all entries of $[A(t), A(s)]$ are of order $O(rh^{-1})$. Therefore, the error, introduced on each time level by truncating the $\Psi_{k=2,\dots}$ terms in the Magnus series is of order $O\left(r\tau^2 h^{-1}\right)$.

It is non-existent if $r = 0$. We now may proceed to the approximation (29) of the solution of the linearized-on-each-time-level problem (43).

The numerical analysis for problem (7) is further performed analogously to the Frey and Patie problem. The restriction on τ for positivity (and respectively monotonicity of the fully discrete scheme, discrete comparison principle and convergence of the discrete solution) of the numerical solution is

$$\frac{\tau}{2h^2} \leq \frac{\overline{\delta}_0^2}{\sigma^2 b^2 + \overline{\delta}_0^2 h^2 r}. \tag{51}$$

If the condition (51) is fulfilled we also have (50) fulfilled and the truncation error in the Magnus series is negligible. Therefore we also have Theorem 5 valid for the Liu and Yong problem (7) so that the numerical solution converges to the viscosity solution of the differential problem.

5 Numerical experiments

Numerical experiments are presented in this section in order to illustrate the stability and convergence properties of the method. We stress that no smoothing techniques are performed on the terminal condition (6) that exhibits *non-smoothness* at $S = E$. In the numerical experiments the vanilla call option is considered.

We solve numerically the presented Frey and Patie model (FP) (4) and Liu and Yong model (LY) (7) with the terminal condition (6). The parameters are:

1. *(FP)* $\lambda(S) = 1$. The strike price is $E = 100$, the volatility is $\sigma = 0.2$, the maturity date - $T = 0.25$ and the artificial boundary location is $b = 200$ [6].
2. *(LY)* $\lambda(S,t)$ as in (9) with $\beta = 100$, $\gamma = 1$ and $\underline{S} = 20$, $\overline{S} = 80$. The strike price is $E = 50$, the volatility is $\sigma = 0.4$, the interest rate is $r = 0.06$, the maturity date - $T = 0.25$ and the artificial boundary location is $b = 200$ [7].

In the tables below are presented the computed discrete maximum and $RMSE$ (*root mean square error*) norms of the error $E = U - V$, where V is the restriction of the exact solution $V(S,t)$ on the grid, by the formulas

$$\|E\|_\infty = \max_i \|U_i^N - V_i^N\|, \ \|E\|_{RMSE} = \sqrt{\frac{1}{M_{br}} \sum_{i:S_i\in[0.8E;1.2E]} \left(U_i^N - V_i^N\right)^2},$$

where the area of interest to be tracked by the $RMSE$ norm is chosen to be $S_i \in [0.8E, 1.2E]$ (the area of most practical interest) and M_{br} is the number of spatial nodes in this area.

The numerical rate of convergence (RC) is calculated using the double mesh principle

$$RC = \log_2(E^M/E^{2M}), \ \ E^M = \|V^M - U^M\|,$$

where $\|\cdot\|$ is the discrete norm, V^M and U^M are respectively the exact solution and the numerical solution, computed at the mesh with M sub-intervals.

Our numerical experiments are focused on the particular $ratio := \tau/(2h^2)$ that is considered in (41). For the simple case of $\rho = 0$ we have the linear Black-Scholes operator, where the interest rate and dividend rate are equal to 0. The assumption (41) now reads as (with parameters as given in (FP))

$$\frac{\tau}{2h^2} \leq 0.000625. \tag{52}$$

Table 1 displays the convergence results for the discussed simple linear test equation w.r.t. to the solution, generated by the MATLAB function *blsprice(Price, Strike, Rate, Time, Volatility, Yield)*. We observe that even though the assumption (41) is violated there are no stability issues. The accuracy, of course, profits from smaller values of the ratio.

Table 1.

	$ratio = 0.01$				$ratio = 0.001$			
$M \times N$	E^N_∞	RC	$RMSE$	RC	E^N_∞	RC	$RMSE$	RC
160	4.716e-1	-	2.244e-1	-	1.269e-2	-	6.742e-3	-
320	1.287e-1	*1.874*	6.659e-1	*1.753*	3.185e-3	*1.995*	1.704e-3	*1.985*
640	3.195e-2	*2.010*	1.721e-2	*1.952*	7.970e-4	*1.999*	4.278e-4	*1.994*
1280	7.962e-3	*2.005*	4.331e-3	*1.991*	1.993e-4	*1.999*	1.072e-4	*1.997*

Figures 1 and 2 illustrate the numerical solution for $\rho = 0$ and $\rho = 0.06$, respectively, for $ratio = 0.1$ and $M = 1280$. We conclude that there are no stability issues both in the linear and nonlinear case. The conclusion completely corresponds to the *unconditional stability*, obtained in Section 3.1. However,

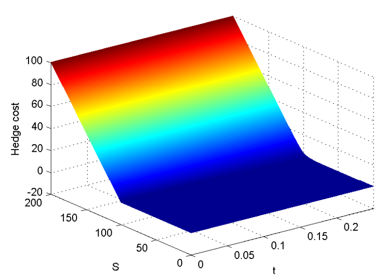

Fig. 1. $\rho = 0$

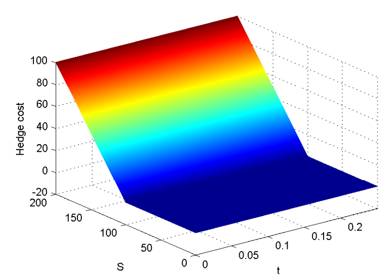

Fig. 2. $\rho = 0.06$

we also observe in Figure 2 that the solution does not show the smoothing-in-time property that characterizes linear parabolic problems and that can be observed in Figure 1. In further support to this consideration we illustrate the corresponding numerical second derivative, the Gamma greek [26], on Figures 3 and 4. The higher the ρ the stronger the nonlinearity in (4) and therefore the smaller δ_0 in the well-posedness condition (5) because of the non-smoothness of the terminal condition.

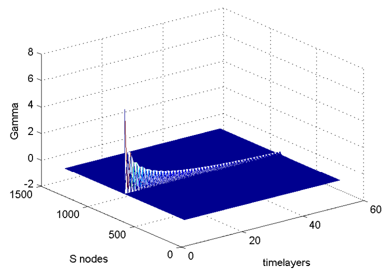

Fig. 3. $\rho = 0$

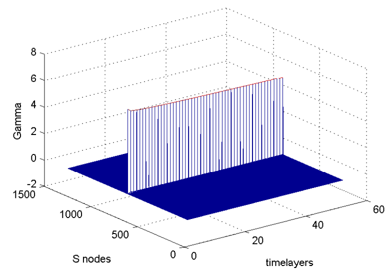

Fig. 4. $\rho = 0.06$

Taking these conclusions into consideration we now present the convergence of the numerical method, applied to (4) with parameters as given in (FP), for $\rho = 0.001$ in Table 2. The error is calculated w.r.t. the numerical solution for $M = 640$ and the size of τ is determined by the value of *ratio*. Clearly, we

Table 2.

	$ratio = 0.001$				$ratio = 0.0001$			
M	E_∞^N	RC	$RMSE$	RC	E_∞^N	RC	$RMSE$	RC
40	9.983e-1	-	6.625e-1	-	1.062e-1	-	5.853e-2	-
80	9.152e-1	*0.126*	6.631e-1	*0.089*	1.875e-2	*2.502*	1.045e-2	*2.486*
160	8.607e-1	*0.089*	5.984e-1	*0.058*	9.647e-3	*0.969*	7.142e-3	*0.548*
320	8.851e-1	*-.040*	6.203e-1	*-.052*	1.144e-3	*3.077*	8.964e-4	*2.994*

observe no convergence for $ratio = 0.001$ because the condition (52) is not even remotely satisfied. By the RMSE behaviour one concludes that issue is the the neighbourhood of $S = E$ where the point of non-smoothness of the terminal condition. Actually, the condition (52) is even stricter in the nonlinear case (41) since $\overline{\delta}_0$ is smaller than 1 and it is getting smaller as $h \to 0$ (the more precisely the numerical second derivative, the Gamma, approximates the $\partial^2 V/\partial S^2$ the smaller $\overline{\delta}_0$ is). Convergence is present for the case $ratio = 0.0001$ however. *These results correspond to Theorem 5 - if* (41) *is violated then the difference scheme is not monotone, no comparison principle exists and therefore it is not convergent.*

We further investigate the influence of the liquidity parameter ρ and the space step h on the numerical solution. In Figures 5 and 6 we present the numerical solution for different values of ρ and h. One deduce that the value of the hedge cost increases as ρ increases as also mentioned in [10]. We do, however, also

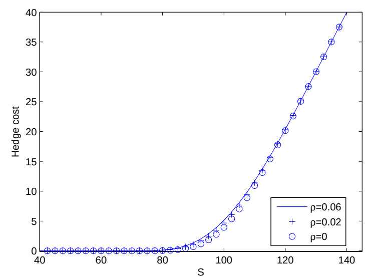

Fig. 5. $M = 80, ratio = 0.001$

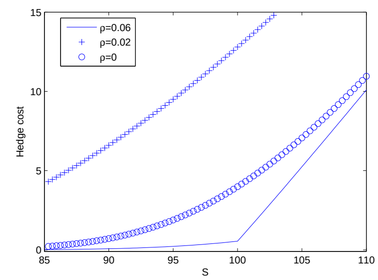

Fig. 6. $M = 640, ratio = 0.001$

confirm that small values of h deteriorate the smoothing-in-time property of the numerical solution as the Gamma copies the behaviour of the second derivative of the terminal condition, the Dirac's δ-function (more precisely, δ-distribution), see Figures 7 and 8.

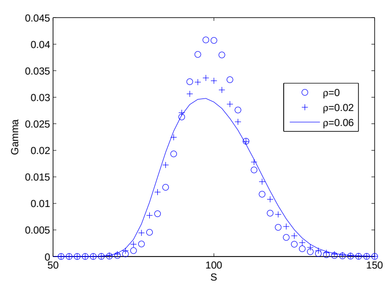

Fig. 7. $M = 80, ratio = 0.001$

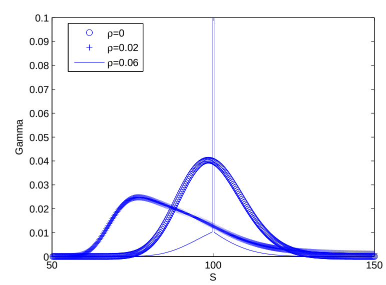

Fig. 8. $M = 640, ratio = 0.001$

We now present the convergence results for the problem (LY), Table 3. The convergence is computed w.r.t. the numerical solution for $M = 1280$ and number of time levels, determined by the value of *ratio*. We observe a solid convergence for the given values of *ratio* although the condition (51) is unlikely to be satisfied. In comparison with the Frey and Patie problem we are now computing convergence on finer mesh without implying serious restriction on the time step τ. It is evident that small values of h do not have any deteriorating impact on the numerical solution even for relatively large *ratio*. This can be related to the smoothing effect that the choice of price impact factor function $\lambda(S, t)$

Table 3.

	ratio = 0.001				ratio = 0.0001			
M	E_∞^N	RC	$RMSE$	RC	E_∞^N	RC	$RMSE$	RC
40	9.988e-2	-	6.685e-2	-	5.662e-1	-	5.334e-2	-
80	4.477e-2	*1.143*	2.890e-2	*1.121*	2.785e-2	*1.023*	2.607e-2	*1.033*
160	1.717e-2	*1.383*	1.288e-2	*1.167*	1.273e-2	*1.130*	1.220e-2	*1.096*
320	6.409e-3	*1.422*	5.387e-3	*1.257*	5.372e-3	*1.244*	5.231e-3	*1.221*
640	1.979e-3	*1.695*	1.728e-3	*1.640*	1.774e-3	*1.599*	1.556e-3	*1.749*

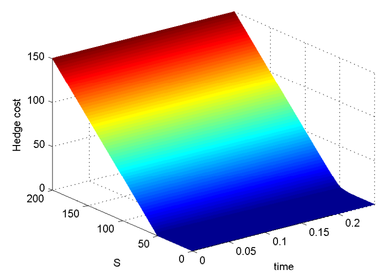

Fig. 9. $M = 1280, ratio = 0.1$

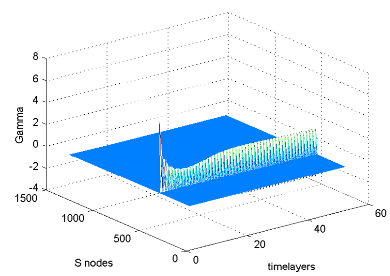

Fig. 10. $M = 1280, ratio = 0.1$

(9) has both on the differential and discrete problems. Our deduction is further supported by Figures 9,10 that one may compare with Figures 2,4.

The impact of the parameter γ on the numerical solution and the Gamma greek is illustrated by Figures 11,12. The parameter's role is analogous to ρ in the Frey and Patie model - the higher the value the more illiquid the market is (and the higher the hedge cost is) and the stronger the nonlinearity in the differential problem is. Unlike the Frey and Patie model we now have smoothing-in-time of the Gamma even for small values of the space step h.

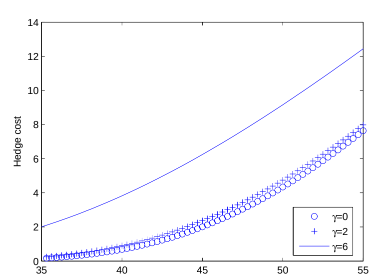

Fig. 11. $M = 640, ratio = 0.001$

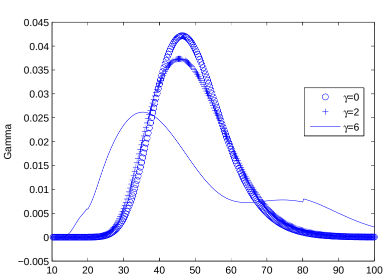

Fig. 12. $M = 640, ratio = 0.001$

6 Conclusion

In this paper we present the numerical analysis of two strongly nonlinear problems, modeling the replication of contingent claims in illiquid markets. The derived discrete scheme is shown to be unconditionally stable, consistent and monotone on semi-discrete and fully-discrete level while the numerical solution is positive and monotone in space so that the qualitative behaviour of the differential solution is preserved by the discretization.

We confirm that the numerical method, based on standard centered-space approximations of the spatial derivatives and a specific *fully explicit time stepping* (the local Crank-Nicolson) technique, is unconditionally stable, regardless of the use of centered-space approximation of the convection term, and the numerical solution converges to the viscosity solution of the nonlinear problem under a general assumption for the ratio of the time and space steps.

Our future intentions of the development of the method include its synchronization with the transparent boundary condition (TBC) method [**?**].

7 Acknowledgement

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