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Long Teng, Cathrin van Emmerich, Matthias Ehrhardt and Michael Günther

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A versatile Approach for Stochastic Correlation using Hyperbolic Functions

L. TENG[a]*, C. VAN EMMERICH[b], M. EHRHARDT[a], M. GÜNTHER[a]

[a] *Lehrstuhl für Angewandte Mathematik und Numerische Analysis, Fachbereich C – Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany*
{teng,ehrhardt,guenther}@math.uni-wuppertal.de
[b] *RWE Supply & Trading, Altenessener Str. 27, 45127 Essen, Germany*
cathrin.van.emmerich@rwe.com

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It is well known that the correlation between financial products, financial institutions, e.g., plays an essential role in pricing and evaluation of financial derivatives. Using simply a constant or deterministic correlation may lead to correlation risk, since market observations give evidence that the correlation is not a deterministic quantity.

In this work, we propose a new approach to model the correlation as a hyperbolic function of a stochastic process. Our general approach provides a stochastic correlation which is much more realistic to model real world phenomena and could be used in many financial application fields. Furthermore, it is very flexible: any mean reverting process (with positive and negative value) can be regarded and no additional parameter restrictions appear which simplifies the calibration procedure. As an example, we compute the price of a quanto applying our new approach. Using our numerical results we discuss concisely the effect of considering stochastic correlation on pricing the quanto.

Keywords: Correlation Risk, Stochastic Correlation, Hyperbolic Functions, Stochastic Process, Ornstein-Uhlenbeck process, Quanto, Fokker-Planck equation, Stochastic Correlation Process.

AMS Subject Classification: 39A50; 62M10; 91G60; 91G80; 97M30

1. Introduction

Correlation is a well established concept for quantifying the relationship between financial products or financial institutions. It plays an essential role in several financial modelling approaches, e.g. the arbitrage pricing model [3] is based on correlation as a measure for the dependence of assets. Also in portfolio credit models, the default correlation is one fundamental factor of risk evaluation, see [2].

A widely used method is to use correlated stochastic processes where the correlation $\rho \in [-1, 1]$ is used as a measure of dependence. Two Brownian motions (BMs) W_1 and W_2 are correlated by the symbolic notion

$$dW_{1,t}dW_{2,t} = \rho dt. \tag{1}$$

*Corresponding author.

Like in the multivariate Black-Scholes model, the correlation of the log-returns is used as a measure of the dependence between assets. An example of coupled stochastic processes is the quantity adjusting option (Quanto) pricing in the Black-Scholes model:

$$\begin{cases} dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S \\ dR_t = \mu_R R_t dt + \sigma_R R_t dW_t^R, \end{cases} \tag{2}$$

with positive constants μ_S, μ_R, σ_S and σ_R. The first stochastic differential equation (SDE) describes the price of the traded asset in a currency A. The second SDE is used to model the exchange rate between currency A and another currency B. Besides, the Brownian motions are assumed to be correlated by a constant correlation $\rho \in [-1, 1]$.

However, it is a well-known fact that the correlation is hardly a fixed constant, see e.g. [13]. It is likely to change over a small time interval as the volatility. To illustrate this statement, we use the following estimator (3) of linear correlation

$$\rho_T(S,R) \approx \hat{\rho}_T = \frac{\sum_{t\in T}(\hat{S}(t) - \frac{1}{n_T}\sum_{t\in T}\hat{S}(t))(\hat{R}(t) - \frac{1}{n_T}\sum_{t\in T}\hat{R}(t))}{\sqrt{\sum_{t\in T}(\hat{S}(t) - \frac{1}{n_T}\sum_{t\in T}\hat{S}(t))^2 \sum_{t\in T}(\hat{R}(t) - \frac{1}{n_T}\sum_{t\in T}\hat{R}(t))^2}}, \tag{3}$$

where $\hat{S}$ and $\hat{R}$ denote the log-return samples, n_T is the number of pairs $(\hat{S}(t), \hat{R}(t))$ with $t \in T$, to compute the historical correlations between S&P 500 and Euro/US-Dollar exchange rate on a daily basis which is displayed in Figure 1.

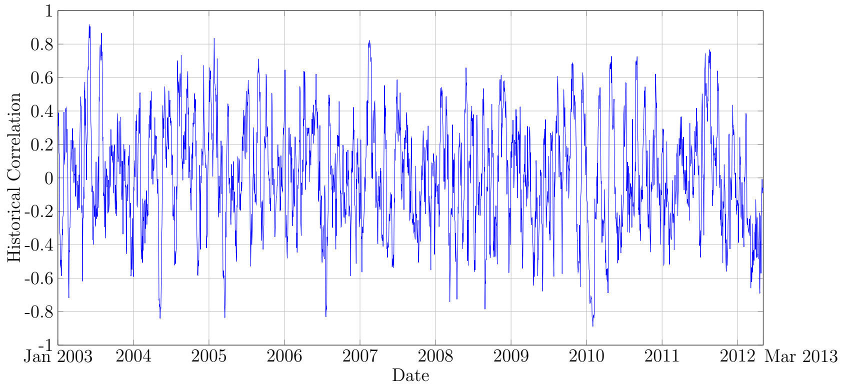

Figure 1. Historical Correlation between S&P 500 and Euro/US-Dollar exchange rate[1].

Figure 1 clearly shows that the correlation changes over time. Some approaches for dynamic correlation have been suggested, like Engle [7] proposed *dynamic conditional correlation (DCC)* which is the *generalized autoregressive conditional heteroskedastic (GARCH)*-type model, see also [8]. A new approach of modelling correlation as a stochastic process was provided by van Emmerich [4], [5] and Ma [10]. However, to ensure that the boundaries −1 and 1 of the correlation process are not attractive and unattainable, one has to restrict the parameter range. We emphasize that in our new approach any mean-reverting process (with positive and negative value) can be considered without facing any additional parameter restrictions.

Obviously, a correlation risk will indeed exist if the correlation is modelled as a constant, which refers to the risk of financial loss due to the usage of wrong correlations. For example in credit risky markets, the inexactly modelled default correlations can lead to

[1] Source of data: yahoo.com

exposure. A reasonable and realistic description of the correlation is a growing need for risk management.

In this work, we propose a new approach to model correlation as a hyperbolic function of a stochastic process. Our approach provides a stochastic correlation which is simple and much more realistic to model real word phenomena and could be used in many financial application fields. The outline of the remaining part is as follows. In the next section we present the stochastic correlation model including its construction and calibration. Section 3 is devoted to the stochastic correlation with an Ornstein-Uhlenbeck process as an example. Finally in Section 4 we compute the price of a particular financial derivative, a quanto, apply our stochastic correlation model and analyze our numerical results.

2. Stochastic Correlation Model

We assume that the usual assumptions about filtration hold throughout this paper, see for example [11].

2.1 *Model Construction*

From Figure 1 we realize that the correlation changes over time. To define further properties of this historical correlation, we draw its empirical density functions in Figure 2 and Figure 3, using different bandwidths. We refer to [1] for the detailed information about the estimation of density function from historical data.

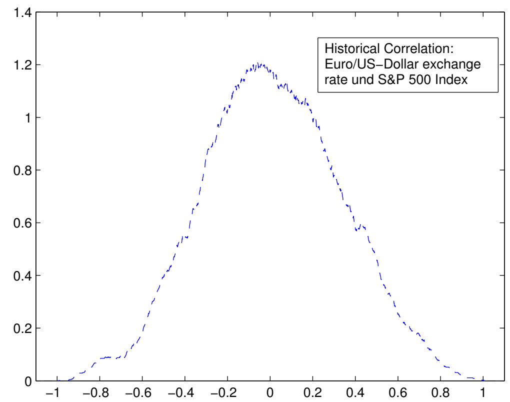

Figure 2. Empirical Density function of the historical correlation between S&P 500 and Euro/US-Dollar exchange rate with bandwidth 1/20.

From studying the empirical density functions we request that the modelled correlation should satisfy the properties as follows:

(i) only takes values in the interval $(-1, 1)$,
(ii) vary around a mean value,
(iii) the probability mass tends to zero at the boundaries $-1, +1$.

For such reasons, based on a mean-reverting stochastic process X_t, like the Ornstein-Uhlenbeck process [14] or the square root diffusion processes (with positive and negative value)

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \qquad t \geq 0,\ X_0 = x_0, \tag{4}$$

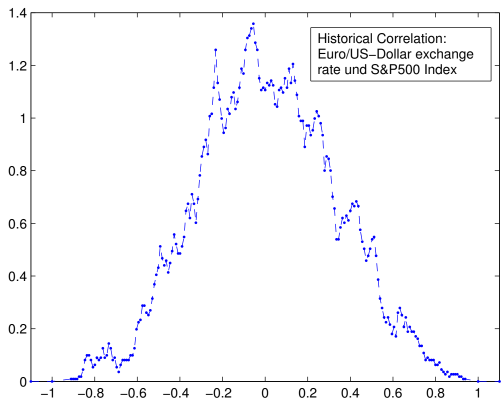

Figure 3. Empirical Density function of the historical correlation between S&P 500 and Euro/US-Dollar exchange rate with bandwidth 1/80.

we propose the correlation to be modelled as the tangens hyperbolicus function of X_t

$$\rho_t = \tanh(X_t), \quad \rho_0 = \tanh(x_0) \in (-1, 1). \tag{5}$$

Obviously, the generated correlation (5) satisfies the desired properties (i)-(iii). Besides, the function tanh is symmetrical and measurable.

Applying *Itô's Lemma* with (5)

$$d\rho_t = d\tanh(X_t) = \frac{\partial \tanh(X_t)}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 \tanh(X_t)}{\partial x^2}(dX_t)^2 \tag{6}$$

we obtain the *stochastic correlation process (SCP)*

$$d\rho_t = (1 - \rho_t^2)\left((\tilde{a} - \rho_t \tilde{b}^2)dt + \tilde{b} dW_t\right), \quad t \geq 0, \tag{7}$$

where $\rho_0 \in (-1, 1), \tilde{a} = a(t, \operatorname{artanh}(\rho_t))$ and $\tilde{b} = b(t, \operatorname{artanh}(\rho_t))$. From (7) we see that there is a suitable number of free parameters to calibrate the model to market data. The free parameters are hidden in the functions a and b, see the example with Orstein-Uhlenbeck process in Section 3.

Although we could intuitively observe that the function $\tanh(x)$ is eminently suitable for correlation modelling, one can still ask whether other functions having values inside the interval $(-1, 1)$, like trigonometric functions or $\frac{2}{\pi}\arctan(\frac{\pi}{2}x), x \in \mathbb{R}$ can also be applied for this purpose? In theory, such functions could be used for the SCP model above. However, the trigonometric function is a periodic function, the arising complex number will complicate further calculations. For the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$, its Itô's formula for (4) is given by

$$\begin{aligned} d\rho_t = d\frac{2}{\pi}\arctan(\frac{\pi}{2}X_t) &= \left(\frac{\tilde{a}}{(1+\tan^2(\frac{\rho_t \pi}{2}))} - \frac{\pi \tilde{b}^2 \tan(\frac{\rho_t \pi}{2})}{2(1+\tan^2(\frac{\rho_t \pi}{2}))^2}\right) dt \\ &+ \frac{\tilde{b}}{(1+\tan^2(\frac{\rho_t \pi}{2}))} dW_t, \end{aligned} \tag{8}$$

which is rather complicated so that the further computation will be tedious. Besides, we take the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$ which is, like $\tanh(x)$ close to the identity in the neighborhood of $x = 0$, see Figure 4. However, compared with $\tanh(x)$ the function $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$ grows much slower up to 1 and down to -1, the estimation of the correlation will thus be worsened, similar to the estimation for the heavy tailed distributions.

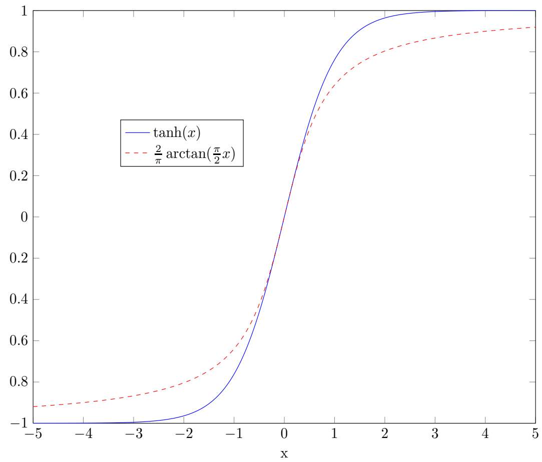

Figure 4. Comparison of $\tanh(x)$ and $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$: the later is less steep having larger tails

2.2 *Model Calibration*

We can estimate the free parameters of (7) using the density function. If we choose for (4) a process which has the known density function, the density function of (5) thus can be derived and used for calibration purpose, see Section 3. Otherwise, we need to determine the transition density with the aid of the *Fokker-Planck equation* [12].

Only for simplicity, we rewrite (7) with the redefined parameters $\hat{a}$ and $\hat{b}$ as

$$d\rho_t = \underbrace{(1-\rho_t^2)(\tilde{a}-\rho_t\tilde{b}^2)}_{:=\hat{a}(t,\rho_t)}\,dt + \underbrace{(1-\rho_t^2)\tilde{b}}_{:=\hat{b}(t,\rho_t)}\,dW_t, \qquad t \geq 0, \tag{9}$$

where $\rho_0 \in (-1, 1)$.

We assume that it possesses a transition density $p(t, \tilde{\rho}|\rho_0)$ which satisfies the *Fokker-Planck equation*

$$\frac{\partial}{\partial t}p(t,\tilde{\rho}) + \frac{\partial}{\partial\tilde{\rho}}(\hat{a}(t,\tilde{\rho})p(t,\tilde{\rho})) - \frac{1}{2}\frac{\partial^2}{\partial\tilde{\rho}^2}(\hat{b}(t,\tilde{\rho})^2p(t,\tilde{\rho})) = 0. \tag{10}$$

For the calibration purpose we consider the stationary density (for $t \to \infty$)

$$p(\tilde{\rho}) := \lim_{t\to\infty} p(t,\tilde{\rho}|\rho_0). \tag{11}$$

With the above construction (7) is also a mean-reverting process, thus one can show that every two solutions of (10) are the same for $t \to \infty$, this is to say that a unique stationary solution $p(\tilde{\rho})$ exists, c.f. [12].

Besides, the following two standard conditions for a density function should be fulfilled by $p(\tilde{\rho})$

$$\int_{-1}^{1} p(\tilde{\rho})d\tilde{\rho} = 1, \tag{12}$$

$$\int_{-1}^{1} \tilde{\rho} \cdot p(\tilde{\rho})d\tilde{\rho} \xrightarrow[t\to\infty]{} \text{mean value.} \tag{13}$$

Up to now, we have just shown our structural idea of SCP. An exact example with the detailed stochastic calculus will be presented in Section 3, the SCP using Ornstein-Uhlenbeck process as basis process.

2.3 *Fully Stochastically correlated BMs*

In this Section we investigate the fully stochastically correlated Brownian motions. Based on two independent Brownian motions $W_{2,t}$ and $W_{3,t}$ we define a new BM by

$$W_{1,t} = \int_0^t \rho_s dW_{2,s} + \int_0^t \sqrt{1-\rho_s^2} dW_{3,s}, \tag{14}$$

where ρ_t is one SCP of type (7), and W_t in (7) is independent of $W_{i,t}$, for $i = 2, 3$. We can easily calculate that $W_{1,t}$ and $W_{2,t}$ satisfies

$$\mathbb{E}\left[W_{1,t} \cdot W_{2,t}\right] = \mathbb{E}\left[\int_0^t \rho_s ds\right]. \tag{15}$$

Thus, we could say that the BMs $W_{1,t}$ and $W_{2,t}$ are correlated by the SCP ρ_t.

One can straightly see that (15) agrees for

$$\mathbb{E}\left[W_{1,t} \cdot W_{2,t}\right] = \rho t, \tag{16}$$

where $W_{1,t}$ and $W_{2,t}$ are correlated by the constant ρ.

3. Stochastic Correlation with Ornstein-Uhlenbeck process

In this section, we specify our SCP model using the Ornstein-Uhlenbeck process. For the basis process (4) we choose the *Ornstein-Uhlenbeck process*

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t, \tag{17}$$

where $\kappa, \sigma > 0$ and $X_0, \mu \in \mathbb{R}$.

Applying *Itô's Lemma* with $\rho_t = \tanh(X_t)$

$$d\rho_t = \frac{\partial \tanh(X_t)}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 \tanh(X_t)}{\partial x^2}\sigma^2 dt \tag{18}$$

gives the **SCP** as

$$d\rho_t = (1-\rho_t^2)\left(\kappa(\mu - \text{artanh}(\rho_t)) - \rho_t\sigma^2\right)dt + (1-\rho_t^2)\sigma dW_t, \tag{19}$$

where $t \geq 0,\ \rho_0 \in (-1,1),\ \kappa, \sigma > 0$ and $\mu \in \mathbb{R}$.

Proof.

$$\begin{aligned}
(18) &= \text{sech}^2(X_t)\kappa(\mu - X_t)dt - \text{sech}^3(X_t)\sinh(X_t)\sigma^2 dt + \text{sech}^2(X_t)\sigma dW_t \\
&= \text{sech}^2(X_t)\kappa(\mu - X_t)dt - \text{sech}^2(X_t)\frac{\sinh(X_t)}{\cosh(X_t)}\sigma dt + \text{sech}^2(X_t)\sigma dW_t \\
&= (1-\rho_t^2)\kappa(\mu - X_t)dt - (1-\rho_t^2)\rho_t\sigma^2 dt + (1-\rho_t^2)\sigma dW_t \\
&= (19).
\end{aligned}$$

□

As mentioned in Section 2.2, we do not really need the transition density of (19) in this case, since the Ornstein-Uhlenbeck process $X_t \sim \mathcal{N}(x_0 e^{-\kappa t} + \mu(1-e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1-e^{-\kappa t}))$ is normal distributed, if the initial value x_0 is given. As $t \to \infty$, then $X_t \sim \mathcal{N}(\mu, \frac{\sigma^2}{2\kappa})$.

Therefore, one can derive the density function for (19) as $t \to \infty$ like

$$f(\tilde{\rho}) = \frac{1}{1-\tilde{\rho}^2} \cdot \frac{\sqrt{\kappa}}{\sigma\sqrt{\pi}} \cdot e^{-\frac{\kappa(\text{artanh}(\tilde{\rho})-\mu)^2}{\sigma^2}}, \tag{20}$$

which can be used to cablibrate the model.

In the following, we still derive the transition density of (19) to show how this approach works. Besides, we want to compare the transition density of (19) to (20). As pointed out in Section 2.2, we assume that (19) possesses a transition density $p(t, \tilde{\rho}|\rho_0)$ which satisfies the following Fokker-Planck equation

$$\frac{\partial}{\partial t}p(t,\tilde{\rho}) + \frac{\partial}{\partial \tilde{\rho}}(\hat{a}(t,\tilde{\rho})p(t,\tilde{\rho})) - \frac{1}{2}\frac{\partial^2}{\partial \tilde{\rho}^2}(\hat{b}(t,\tilde{\rho})^2 p(t,\tilde{\rho})) = 0 \tag{21}$$

with

$$\hat{a}(t,\tilde{\rho}) = (1-\tilde{\rho}^2)\left(\kappa(\mu - \text{artanh}(\tilde{\rho})) - \tilde{\rho}\sigma^2\right), \tag{22}$$

$$\hat{b}(t,\tilde{\rho}) = (1-\tilde{\rho}^2)\sigma. \tag{23}$$

For $t \to \infty$, the stationary density $p(\tilde{\rho})$ can be obtained by solving

$$\frac{\partial}{\partial \tilde{\rho}}((1-\tilde{\rho}^2)\left(\kappa(\mu - \text{artanh}(\tilde{\rho})) - \tilde{\rho}\sigma^2\right)p(\tilde{\rho})) = \frac{1}{2}\frac{\partial^2}{\partial \tilde{\rho}^2}(((1-\tilde{\rho}^2)\sigma)^2 p(\tilde{\rho})) \tag{24}$$

as

$$p(\tilde{\rho}) = \frac{\left(m + n\ \text{erf}\left(\frac{\sqrt{-\kappa}(\text{artanh}(\tilde{\rho})-\mu)}{\sigma}\right)\right) e^{-\frac{\kappa\,\text{artanh}(\tilde{\rho})}{\sigma^2}(\text{artanh}(\tilde{\rho})-2\mu)}}{\tilde{\rho}^2 - 1} \tag{25}$$

with the constants $m,\ n \in \mathbb{R}$.

Now we try to simplify (25). Firstly we can easily observe, n must be zero, so that the condition (13) can be satisfied by (25). We can check this straightly by setting $\mu = 0$. Thus, (25) can be further written as

$$p(\tilde{\rho}) = \frac{m}{\tilde{\rho}^2 - 1} \cdot \mathrm{e}^{-\frac{\kappa \mathrm{artanh}(\tilde{\rho})}{\sigma^2}(\mathrm{artanh}(\tilde{\rho}) - 2\mu)}. \tag{26}$$

In theory we can compute m by solving the condition (12) with (25), but the integration of (25) will be tedious. However, due to the uniqueness of the asymptotic distribution, m can be specified by identifying (20) and (26) as

$$m = -\frac{\sqrt{\kappa}}{\sigma\sqrt{\pi}} e^{\frac{-\mu^2\kappa}{\sigma^2}}. \tag{27}$$

By substituting (27) for m in (26) we can obtain the transition density function which is the same to (20).

As mentioned before, (20) can be used to estimate the parameters of (19). In our case of using Ornstein-Uhlenbeck process $X_t \sim \mathcal{N}(x_0\mathrm{e}^{-\kappa t} + \mu(1 - \mathrm{e}^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-\kappa t}))$ with the (conditional) probability density

$$f_x(x_{s+\Delta t}|x_s, \kappa, \mu, \sigma) = \sqrt{\frac{\kappa}{\pi\sigma^2(1 - \mathrm{e}^{-2\kappa\Delta t})}} \cdot \mathrm{e}^{\frac{-\kappa(x_{s+\Delta t} - x_s\mathrm{e}^{-\kappa\Delta t} - \mu(1-\mathrm{e}^{-\kappa\Delta t}))^2}{\sigma^2(1-\mathrm{e}^{-2\kappa\Delta t})}} \tag{28}$$

from which we derive the density of $\rho_t = \tanh(X_t)$ directly as

$$f_\rho(\tilde{\rho}_{s+\Delta t}|\tilde{\rho}_s, \kappa, \mu, \sigma) = \sqrt{\frac{a}{b}} \cdot \frac{1}{1 - \tilde{\rho}^2_{s+\Delta t}} \cdot \mathrm{e}^{\frac{-\kappa(\mathrm{artanh}(\tilde{\rho}_{s+\Delta t}) - \mathrm{artanh}(\tilde{\rho}_s)\mathrm{e}^{-\kappa\Delta t} - \mu c)^2}{\sigma^2 b}} \tag{29}$$

with

$$a = \frac{k}{\pi\sigma^2}, \quad b = (1 - \mathrm{e}^{-2\kappa\Delta t}) \quad \text{and} \quad c = (1 - \mathrm{e}^{-\kappa\Delta t}). \tag{30}$$

Therefore, we prefer to employ the maximum-likelihood estimation for the historical correlation, see [6]. We use θ to denote the collection of the parameters κ, μ and σ, for the $n+1$ given observed correlations $(\tilde{\rho}_0, \tilde{\rho}_1, \cdots, \tilde{\rho}_t)$. We derive its *log-likelihood* function

$$\begin{aligned}\mathcal{L}(\theta|\tilde{\rho}_0, \tilde{\rho}_1, \cdots, \tilde{\rho}_t) = &\sum_{i=1}^{n} \log\left(\sqrt{\frac{\kappa}{\pi\sigma^2(1 - \mathrm{e}^{-2\kappa(t_i - t_{i-1})})}}\right) + \sum_{i=1}^{n} \log\left(\frac{1}{1 - \tilde{\rho}^2_{t_i}}\right) \\ &+ \sum_{i=1}^{n} \frac{-\kappa(\mathrm{artanh}(\tilde{\rho}_{t_i}) - \mathrm{artanh}(\tilde{\rho}_{t_{i-1}})\mathrm{e}^{-\kappa(t_{i+1}-t_i)} - \mu(1 - \mathrm{e}^{-\kappa(t_{i+1}-t_i)}))^2}{\sigma^2(1 - \mathrm{e}^{-2\kappa(t_{i+1}-t_i)})}.\end{aligned} \tag{31}$$

Furthermore, the parameter estimators $\hat{\theta} = (\hat{\kappa}, \hat{\mu}, \hat{\sigma})$ can be obtained by maximizing (31). This can be done for example by some numerical optimization methods. Besides, we remark that the derivatives of (31) with respect to μ and σ can be found analytically and only tedious with respect to κ. The expressions for $\hat{\mu}$ and $\hat{\sigma}$ can thus be obtained by solving

$$\frac{\partial\mathcal{L}(\theta|\tilde{\rho}_0, \tilde{\rho}_1, \cdots, \tilde{\rho}_t)}{\partial\sigma} = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}(\theta|\tilde{\rho}_0, \tilde{\rho}_1, \cdots, \tilde{\rho}_t)}{\partial\mu} = 0. \tag{32}$$

We write the results as

$$\hat{\mu} = \sum_{i=1}^{n} \frac{\operatorname{artanh}(\tilde{\rho}_{t_i}) - \operatorname{artanh}(\tilde{\rho}_{t_{i-1}}) \mathrm{e}^{-\kappa(t_i - t_{i-1})}}{1 + \mathrm{e}^{-\kappa(t_i - t_{i-1})}} \Big/ \sum_{i=1}^{n} \frac{1 - \mathrm{e}^{-\kappa(t_i - t_{i-1})}}{1 + \mathrm{e}^{-\kappa(t_i - t_{i-1})}} \tag{33}$$

and

$$\hat{\sigma} = \left(\sum_{i=1}^{n} \frac{2\kappa(\operatorname{artanh}(\tilde{\rho}_{t_i}) - \operatorname{artanh}(\tilde{\rho}_{t_{i-1}}) \mathrm{e}^{-\kappa(t_i - t_{i-1})} - \mu(1 - \mathrm{e}^{-\kappa(t_i - t_{i-1})}))^2}{1 - \mathrm{e}^{-\kappa(t_i - t_{i-1})}} \right)^{\frac{1}{2}}. \tag{34}$$

We see that $\hat{\mu}$ has the expression only with respect to κ, as well as $\hat{\sigma}$ by substituting μ in (34) by (33). Hence, we substitute $\hat{\mu}$ and $\hat{\sigma}$ in (31) to gain the *log-likelihood* function only with the parameter κ, which can be computed by maximizing this function. Finally, we can substitute the value of $\hat{\kappa}$ back to (33) and (34) to get values of $\hat{\mu}$ and $\hat{\sigma}$.

As an example we estimated the SCP parameters using the historical correlation in Figure 3. Then, we compared (20) using the estimated parameters and the empirical density function of historical correlation, see Figure 5.

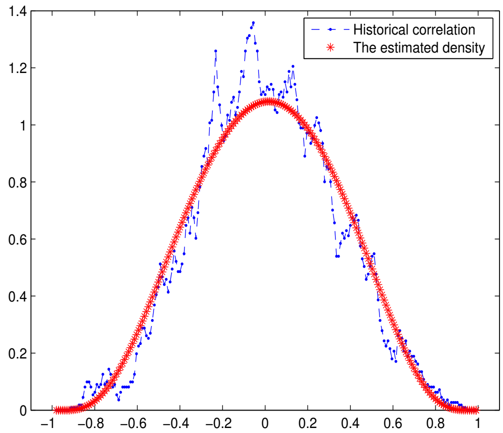

Figure 5. The estimated parameters: $\hat{\kappa} = 32.11$, $\hat{\mu} = 0.012$ and $\hat{\sigma} = 2.96$

4. Pricing Quantos with Stochastic Correlation

In global markets, many financial products e.g. stock index options and futures are traded in the different counteries with different currencies. Hence, trader may have to face the risk due to the uncertainty of exchange rate between trading currency and domestic currency. A Quanto option is one of the financial instruments to manage this risk.

4.1 *Pricing*

As pointed out in the introduction, the correlation between the currency exchange rate and the price of underlying asset in a Quanto option must be considered. In the following, we utilize the extended Black-Scholes formula by using our stochastic correlation model to evaluate the fair price of the quanto options.

As an example we think of a Put-Option on the S&P 500 with payoff in USD

$$(\underbrace{\text{Strike}}_{:=K} - \underbrace{\text{S\&P500}_T}_{:=S_T})^+, \tag{35}$$

where $(\cdot)^+ = \max(0, \cdot)$. Then the payoff of a currency-protected Quanto Put-option in Euro can be written as

$$\underbrace{\text{exchangeRate}_0}_{:=R_0} \cdot (\text{Strike} - \text{S\&P500}_T)^+ \tag{36}$$

where R_0 is the Euro/USD (number of Euro per USD) exchange rate agreed upon at the inception of the contract.

We assume that S_t and R_t follows (2) by

$$\begin{cases} dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S \\ dR_t = \mu_R R_t dt + \sigma_R R_t dW_t^R \end{cases}, \tag{37}$$

where W_t^S and W_t^R are correlated using the SCP (19) as:

$$d\rho_t = (1-\rho_t^2)\left(\kappa(\mu - \operatorname{artanh}(\rho_t)) - \rho_t\sigma^2\right)dt + (1-\rho_t^2)\sigma dW_t. \tag{38}$$

Furthermore, we assume nonzero relationships between the SCP and the price, the exchange rate process, say

$$dW_t dW_t^S = \rho_1 dt \quad \text{and} \quad dW_t dW_t^R = \rho_2 dt. \tag{39}$$

In fact, we are trying to incorporate the SCP (38) in the model (37) exogenously. For this reason we could assume that $\rho_1 = 0$ and $\rho_2 = 0$.

We denote the risk-free interest rate of Euro and US-Dollar respectively by r_e and r_d. If W_t^S and W_t^R are correlated with a constant correlation, we know that the price of a Quanto Put-Option in the Black-Scholes (BS) model with continuous dividend yield is [15]:

$$P_{\text{Quanto}}(S_0, K, r_e, r_d, D, \sigma_S, \sigma_R, T) = R_0\left(K\mathrm{e}^{-r_d T}\mathcal{N}(-d_2) - S_0\mathrm{e}^{-DT}\mathcal{N}(-d_1)\right) \tag{40}$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + ((r_e - D) + \frac{\sigma_S^2}{2})/T}{\sigma_S\sqrt{T}}, \qquad d_2 = d_1 - \sigma_S\sqrt{T} \tag{41}$$

and

$$D = r_e - r_d + \rho\sigma_S\sigma_R. \tag{42}$$

To incorporate the stochasticity of the correlation exogenously in the BS model, we consider the following strategy to obtain the no-arbitrage price: First we think that the expected return of one unit of US-Dollar, exchanged to Euro, risk-free invested in the Euro countries and re-exchanged to US-Dollar must equal the risk-free return on one unit of US-Dollar in US-Dollar countries, which reads

$$\frac{1}{R_0}\exp(r_e T)\mathbb{E}[R_T] = \exp(r_d T). \tag{43}$$

The exchange rate R_t follows a geometric Brownian motion and thus $\mathbb{E}[R_T] = R_0 \exp(\mu_R T)$, which can be substitued into (43) to get

$$\mu_R = r_e - r_d. \tag{44}$$

Besides, the expected value of an investment of one Euro unit into the underlying with price S must be equal to risk-free return on one unit of US-Dollar in US-Dollar counteries, which gives

$$\frac{1}{R_0}\frac{1}{S_0}\mathbb{E}[S_T R_T] = \exp(r_d T). \tag{45}$$

For calculating of $\mathbb{E}[S_T R_T]$, we apply first Itô's lemma to the function $u(t, S_t, E_t) = \ln(S_t R_t)$

$$du(t, S_t, E_t) = d\ln(S_t R_t) = (\mu_S + \mu_R - \frac{1}{2}(\sigma_S^2 + \sigma_R^2))dt + \sigma_S dW_t^S + \sigma_R dW_t^R. \tag{46}$$

Furthermore, the last equation can be rewritten as

$$\ln(S_T R_T) - \ln(S_0 R_0) = (\mu_S + \mu_R - \frac{1}{2}(\sigma_S^2 + \sigma_R^2))T + \sigma_S W_T^S + \sigma_R W_T^R \tag{47}$$

which implies

$$\mathbb{E}[S_T R_T] = S_0 R_0 \exp\left((\mu_S + \mu_R - \frac{1}{2}(\sigma_S^2 + \sigma_R^2))T\right)\mathbb{E}[\exp(\sigma_S W_T^S + \sigma_R W_T^R)]. \tag{48}$$

Now, we set $dX_S = \sigma_S dW_t^S$ and $dX_R = \sigma_R dW_t^R$. A further application of Itô's lemma to the function $f(t, X_S, X_R) = \exp(X_S + X_R)$ leads to

$$\mathbb{E}[\exp(\sigma_S W_T^S + \sigma_R W_T^R)] = \mathbb{E}\left[\exp(\frac{T}{2}(\sigma_S^2 + \sigma_R^2) + \sigma_S\sigma_R \int_0^T \rho_t)dt\right]. \tag{49}$$

We substitue the last equation into (48) to obtain

$$\mathbb{E}[S_T R_T] = S_0 R_0 \mathbb{E}\left[\exp(\mu_S + \mu_R)T + \sigma_S\sigma_R \int_0^T \rho_d dt)\right]. \tag{50}$$

Thus, we can choose

$$\mu_S = r_e - \mu_R - \sigma_S\sigma_R \frac{1}{T}\int_0^T \rho_t dt. \tag{51}$$

so that the no-arbitrage condition (45) can be fulfilled. Besides, in the BS model, we interpret (51) as a return minus the continuous dividend, the dividend can thus be obtained as

$$D(\rho_t) := \mu_R + \sigma_S\sigma_R \frac{1}{T}\int_0^T \rho_t dt. \tag{52}$$

Together with (44) we have

$$D(\rho_t) = r_e - r_d + \sigma_S\sigma_R \frac{1}{T}\int_0^T \rho_t dt. \tag{53}$$

The integral of the stochastic correlation ρ_t can be computed numerically using e.g. the Milstein scheme [9].

Finally, we denote the price of a Quanto Put-Option in the extended Black-Scholes model conditional on the ρ_t path by P_{Quanto} :

$$P_{\text{Quanto}}(S_0, K, r_e, r_d, D(\rho_t), \sigma_S, \sigma_R, T) = R_0\left(K\mathrm{e}^{-r_d T}\mathcal{N}(-d_2) - S_0\mathrm{e}^{-D(\rho_t)T}\mathcal{N}(-d_1)\right) \tag{54}$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + ((r_e - D(\rho_t)) + \frac{\sigma_S^2}{2})/T}{\sigma_S\sqrt{T}}, \qquad d_2 = d_1 - \sigma_S\sqrt{T}. \tag{55}$$

The price of a Quanto Call-Option can be derived easily from the put-call parity.

To keep the stochasticity we use a conditional Monte-Carlo approach. The idea is, we first simulate all the paths of ρ_t^i, for $i \in \{1, 2, ..., M\}$, with each path we can obtain a price P^i_{Quanto} by (54). Then the fair price $\mathbb{P}_0$ is the mean value over all prices as

$$\mathbb{P}_0 = \mathbb{E}\left[\mathbb{E}\left[P_{\text{Quanto}}|\mathcal{F}_t\right]\right] \approx \frac{\sum_{i=1}^M P^i_{\text{Quanto}}}{M}, \tag{56}$$

where $\mathcal{F}_t$ was generated by the SCP. Obviously, the unique BS price with continuous dividend exists always. Now using a conditional Monte-Carlo method to compute the price for each simulated path there exists a unique price, so that (56) is unique.

4.2 *Hedging*

Based on the formula on (40) and (54) we derive the Delta as an example to discuss the effect of stochastic correlation on hedging strategy. For using a constant correlation, the delta is given by

$$\Delta_c = \Phi(d_1) - 1, \tag{57}$$

where Φ is standard normal distribution function and d_1 is defined in (41). Similarly, we have the Delta under stochastic correlation as

$$\Delta_s = \Phi(d_1) - 1, \tag{58}$$

where d_1 is (55). However, we see that $D(\rho_t)$ in (55) is a stochastic process. Thus, we need first a conditional Monte-Carlo approach to compute $D(\rho_t)$ and also Δ_s. Using the same parameters in Figure 8 and taking the maturity $T = 2$ we display Δ_c and Δ_s in

Figure 6. Since the value of μ is closed to the constant correlation, the difference between

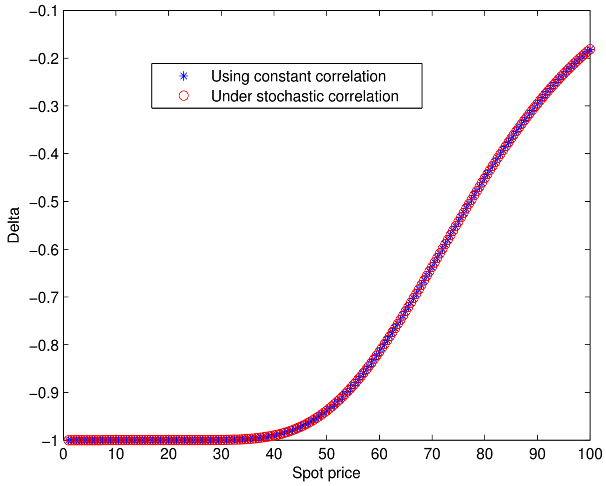

Figure 6. BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.03$, $r_e = 0.05$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, Correlation process parameters: $\kappa = 32.11, \mu = 0.012, \sigma = 2.96$ and $\rho_0 = 0.025$

Δ_c and Δ_s is not apparent. Therefore, in Figure 7, we plot the difference $\Delta_s - \Delta_c$ in Figure 6 and observe that delta values under stochastic correlation are larger than the delta values using constant correlation.

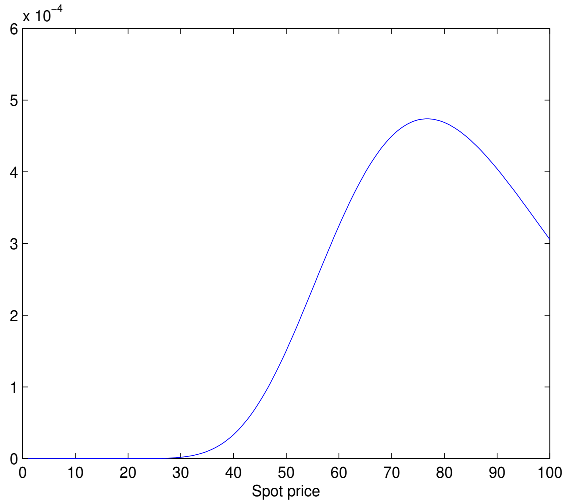

Figure 7. The difference of delta values between using constant and stochastic correlation in Figure 6.

4.3 *Numerical Results*

In Figure 8, 9 and 10, we show our numerical results for pricing a quanto Put-Option and analyze the results centering around correlation risk.

First In Figure 8, we set the parameter for the Black-Scholes model und use the estimated parameter for the SCP model (see Figure 5). Besides, we use the whole historical data (Jan 2003 - Mar 2013) of S&P 500 and Euro/US-Dollar exchange rate but only to estimate the constant correlation which is 0.025. At the same time, we can let the SCP ρ_t starting from this value, $\rho_0 = 0.025$. It is clearly to see, the prices of Put-Option with

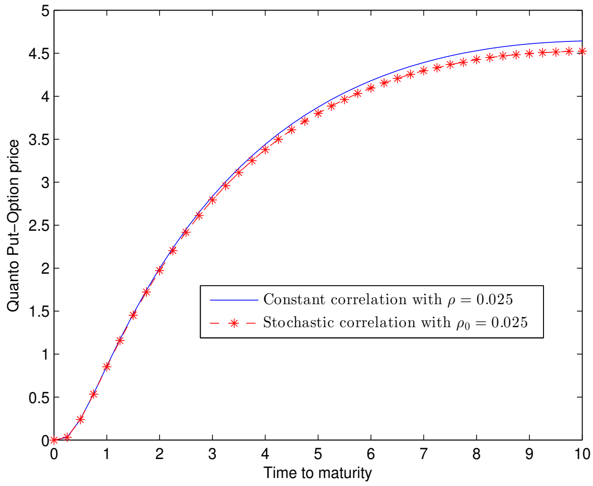

Figure 8. BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.03$, $r_e = 0.05$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, Correlation process parameters: $\kappa = 31.11, \mu = 0.012, \sigma = 2.96$ and $\rho_0 = 0.025$

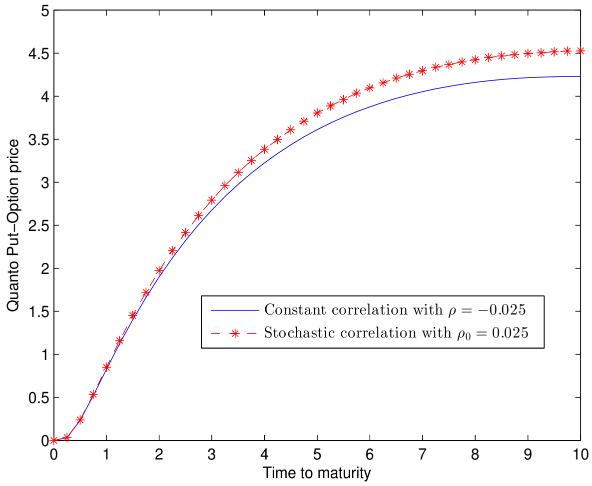

Figure 9. BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.03$, $r_e = 0.05$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, Correlation process parameters: $\kappa = 32.11, \mu = 0.012, \sigma = 2.96$ and $\rho_0 = 0.025$

constant correlation are higher than the prices using the stochastic correlations. The difference is even getting larger with increasing maturity. We consider this difference as the correlation risk using the wrong (constant) correlation.

In Figure 9, we change the constant correlation to -0.025 und keep the other parameter the same as in Figure 8. We observe that the prices with constant correlation could be also lower than the prices applying stochastic correlation. The sign of the price of correlation risk in this case is opposite of that sign in Figure 8.

It is very interesting to see the results in Figure 10, the prices using constant correlation and stochastic correlation are very close, even almost the same for the longer time. We explain the reason for this phenomena as follows. In this case the parameter of Black-Scholes model remain unchanged as the last two examples, but we set here $\kappa = 10, \mu = 0.2$ and $\sigma = 1$. Thus, we can compute numerically the mean value of (38) for these assumed parameter which is 0.1887. We then price the Quanto Put-Option with constant correlation $\rho = 0.1887$ and using stochastic correlation with $\rho_0 = 0$. Because the value of the mean-reverting factor κ is large and the value of σ is small, so that the stochastic

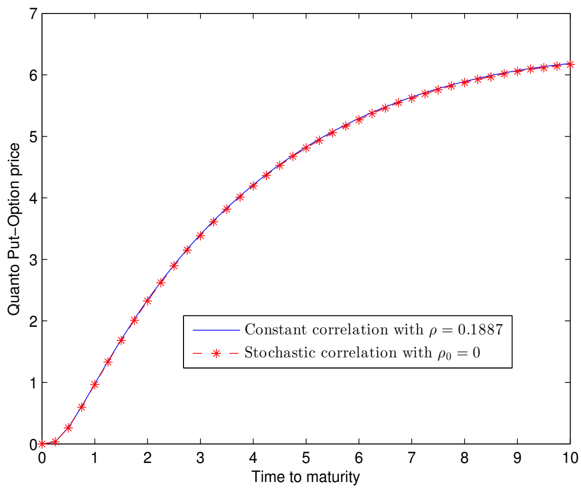

Figure 10. BS parameters: $K = 80$, $S_0 = 100$, $R_0 = 1$, $r_d = 0.03$, $r_e = 0.05$, $\sigma_S = 0.2$, $\sigma_R = 0.4$, Correlation process parameters: $\kappa = 10, \mu = 0.2, \sigma = 1$ and $\rho_0 = 0 \Rightarrow$ Mean value of the SCP model : 0.1887

correlation process concentrate strongly on its mean value, this is why there is no obvious difference between the prices using constant correlation and stochastic correlation in this special case.

5. Conclusion

We proposed a new approach to model the correlation as the hyperbolic function tanh of a mean-reverting process, so that the correlation can be modelled stochastically which is more realistic to model real world phenomena. The detailed construction and calibration of the SCP model using Ornstein-Uhlenbeck process as basis process has been illustrated by an example.

Without any additional parameter restrictions, our new approach can provide the stochastic correlation, which satisfies all the essential properties, that the (observed) estimated correlation in the real market must fulfill. This is a favourable property when estimating model parameters from real market data.

Besides, as an example we compute the price of a Quanto Put-Option with stochastic correlation applying our new approach. Analyzing the numerical results, we find that the correlation risk caused by using wrong (constant) correlation could be priced through applying our new SCP model, which can not be neglected.

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