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Applications of Laplace–Carleson embeddings to admissibility and controllability

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Abstract

It is shown how results on Carleson embeddings induced by the Laplace transform can be use to derive new and more general results concerning the weighted admissibility of control and observation operators for linear semigroup systems with q-Riesz bases of eigenvectors. Next, a new Carleson embedding result is proved, which gives further results on weighted admissibility for analytic semigroups. Finally, controllability by smoother inputs is characterised by means of a new result about weighted interpolation.

Keywords. Semigroup system, controllability, admissibility, Hardy space, weighted Bergman space, Interpolation, Carleson measure.

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1 Introduction

The main purpose of this note is to show how recent results on Carleson embeddings, mostly derived in [11], may be applied to the theory of well-posed linear systems. We shall discuss concepts such as admissibility, controllability and observability, for which some standard references are the books [17, 14] and the survey [7]. Our basic tools will be the theory of a class of function spaces known as Zen spaces, which include the standard Hardy and Bergman spaces.

The structure of the paper is as follows. In Section 2, we review the basics of the theory of admissibility for diagonal semigroups, introduce the key embedding results for Zen spaces, and derive new admissibility results in this context. Section 3 treats the case of analytic semigroups (corresponding to measures supported in a sector), where we are able to move away from the Hilbertian (L^2)

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context and study general L^p spaces. We also derive a new embedding theorem and apply it to weighted admissibility. Finally, in Section 4 we consider notions of controllability, which are linked with interpolation questions.

2 Admissibility for diagonal semigroups

Let A be the infinitesimal generator of a C_0–semigroup $(T(t))_{t\geq 0}$ defined on a Hilbert space H, and consider the system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $u(t) \in \mathbb{C}$ is the input at time t, and $B : \mathbb{C} \to D(A^*)'$, the control operator. Here $D(A^*)'$ is the completion of H with respect to the norm

$$\|x\|_{D(A^*)'} = \|(\beta - A)^{-1}x\|_H,$$

for any $\beta \in \rho(A)$. To ensure that the state $x(t)$ lies in H it is sufficient that $B \in \mathcal{L}(\mathbb{C}, D(A^*)')$ and

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_H \leq m_0\|u\|_{L^2(0,\infty)}$$

for some $m_0 > 0$ (the admissibility condition for B). We note that the C_0–semigroup $(T(t))_{t\geq 0}$ has an extension to $D(A^*)'$.
Dually, we may consider the system

$$\frac{dx(t)}{dt} = Ax(t), \qquad y(t) = Cx(t),$$

with $x(0) = x_0$, say. Here $C : D(A) \to \mathbb{C}$ is an A-bounded observation operator mapping; i.e., for some m_1, $m_2 > 0$,

$$\|Cz\| \leq m_1\|z\| + m_2\|Az\|.$$

C is *admissible*, if there is an $m_0 > 0$ such that $y \in L^2(0, \infty)$ and $\|y\|_2 \leq m_0\|x_0\|$ for all $x_0 \in D(A)$. Note that $y(t) = CT(t)x_0$ for every $x_0 \in D(A)$.

The duality here is that B is an admissible control operator for $(T(t))_{t\geq 0}$ if and only if B^* is an admissible observation operator for the dual semigroup $(T(t)^*)_{t\geq 0}$.

Admissibility is an important concept in the theory of well-posed systems, and we refer to the survey [7] and the book [17] for the basic background to the subject. For diagonal semigroups, admissibility is linked with the theory of Carleson measures as in [6, 18]; namely, supposing that A has a Riesz basis of eigenvectors, with eigenvalues (λ_k), then a scalar control operator corresponding to a sequence (b_k) is admissible if and only if the measure

$$\mu := \sum_k |b_k|^2 \delta_{-\lambda_k}$$

is a Carleson measure for the Hardy space $H^2(\mathbb{C}_+)$ on the right half-plane: this means that the canonical embedding $H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \mu)$ is bounded. In fact, the extension to normal semigroups has also been made [19].
Generalizations to α-admissibility, in which u must lie in $L^2(0, \infty; t^\alpha \, dt)$ for $-1 < \alpha < 0$, were studied by Wynn [20]. The key fact here is that under the Laplace transform the space $L^2(0, \infty; t^\alpha \, dt)$ is mapped to a weighted Bergman space, and for these there are analogues of the Carleson measure theorem available.
The results below enable us to take this generalization further and consider admissibility in the sense of the input lying in much more general spaces $L^2_w(0, \infty) = L^2(0, \infty; w(t) \, dt)$.
In order to state the link between admissibility and embeddings in the greatest generality possible, we assume now that $1 \leq q < \infty$ and the semigroup $(T(t))_{t \geq 0}$ acts on a Banach space X with a q-Riesz basis of eigenvectors (ϕ_k); that is, $T(t)\phi_k = e^{\lambda_k t}\phi_k$ for each k, and (ϕ_k) is a a Schauder basis of X such that for some $C_1, C_2 > 0$ we have

$$C_1 \sum |a_k|^q \leq \| \sum a_k \phi_k \|^q \leq C_2 \sum |a_k|^q$$

for all sequences (a_k) in ℓ^q. In practice, this will mean that without loss of generality we can assume that $X = \ell^q$ and that the eigenvectors of the generator of $(T(t))_{t \geq 0}$, denoted by A, are the canonical basis of ℓ^q. We suppose also that we have a Banach space Z of functions on $(0, \infty)$, which may be an L^p space or a weighted space $L^2(0, \infty; w(t)dt)$, whose dual space Z^* can be regarded, respectively, as either $L^{p'}(0, \infty)$ or $L^2(0, \infty; w(t)^{-1}dt)$ in a natural way. Here, and throughout the paper, p' denotes the conjugate exponent to p, i.e., $p' = p/(p-1)$.
The following result links admissibility and Laplace–Carleson embeddings (that is, Carleson embeddings induced by the Laplace transform).

Theorem 2.1 *Let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$ corresponding to the sequence (b_k). The control operator B is Z-admissible for $(T(t))_{t \geq 0}$, that is, there is a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t) B u(t) \, dt \right\|_X \leq m_0 \|u\|_Z, \quad u \in Z,$$

if and only if the Laplace transform induces a continuous mapping from Z into $L^q(\mathbb{C}_+, d\mu)$, where μ is the measure $\sum |b_k|^q \delta_{-\lambda_k}$.

Proof: Clearly we may suppose without loss of generality that $X = \ell^q$ and that $\phi_k = e_k$, the standard basis of X. We have that

$$\begin{aligned} \left\| \int_0^\infty T(t) B u(t) \, dt \right\|_X &= \left\| \int_0^\infty \sum_k e^{\lambda_k t} b_k e_k u(t) \, dt \right\|_X \\ &= \left(\sum_k |\hat{u}(-\lambda_k)|^q |b_k|^q \right)^{1/q}, \end{aligned}$$

from which the result follows easily. ∎

A duality argument gives the corresponding result for observation operators.

Theorem 2.2 *Let C be a linear bounded map from $D(A)$ to $\mathbb{C}$. The observation operator C is Z-admissible for $(T(t))_{t\geq 0}$, that is, there is a constant $m_0 > 0$ such that*

$$\|CT(\cdot)x\|_Z \leq m_0\|x\|_X$$

for all $x \in D(A)$, if and only if the Laplace transform induces a continuous mapping from Z^* into $L^{q'}(\mathbb{C}_+, d\mu)$, where μ is the measure $\sum |c_k|^{q'}\delta_{-\lambda_k}$ and $c_k := C\phi_k$.

Proof: Again we may suppose without loss of generality that $X = \ell^q$ and that $\phi_k = e_k$, the standard basis of X. Z-admissibility is equivalent to the condition that

$$\sup_{f,x}\left|\int_0^\infty CT(t)xf(t)\,dt\right| < \infty,$$

where we take $f \in Z^*$ and $x = (x_k) \in D(A)$ both of norm 1. Calculating $CT(t)x$, we see that this is equivalent to the condition that

$$\sup_{f,x}|\sum_k c_k\hat{f}(-\lambda_k)x_k| < \infty,$$

and, taking the supremum over the set of x of norm 1 in $D(A)$, which is dense, we obtain

$$\sup_f \|(c_k\hat{f}(-\lambda_k))\|_{q'} < \infty,$$

which is easily seen to be equivalent to the boundedness of the Laplace–Carleson embedding from Z^* into $L^{q'}(\mathbb{C}_+, d\mu)$. ∎

In general we shall state our results in terms of control operators, leaving the interested reader to deduce the corresponding results for observation operators.

Now let $\tilde{\nu}$ be a positive regular Borel measure on $[0,\infty)$ satisfying the following (Δ_2)-condition:

$$R := \sup_{r>0}\frac{\tilde{\nu}[0,2r)}{\tilde{\nu}[0,r)} < \infty. \qquad (\Delta_2)$$

Let ν be the positive regular Borel measure on $\overline{\mathbb{C}_+} = [0,\infty)\times\mathbb{R}$ given by $d\nu = d\tilde{\nu}\otimes d\lambda$, where λ denotes Lebesgue measure. In this case, for $1 \leq p < \infty$, we call

$$A^p_\nu = \left\{f : \mathbb{C}_+ \to \mathbb{C} \text{ analytic} : \sup_{\varepsilon>0}\int_{\overline{\mathbb{C}_+}} |f(z+\varepsilon)|^p d\nu(z) < \infty\right\}$$

a Zen space on $\mathbb{C}_+$. If $\tilde{\nu}(\{0\}) > 0$, then by standard Hardy space theory, f has a well-defined boundary function $\tilde{f} \in L^p(i\mathbb{R})$, and we can give meaning to the expression $\int_{\overline{\mathbb{C}_+}} |f(z)|^p d\nu(z)$. Therefore, we write

$$\|f\|_{A^p_\nu} = \left(\int_{\overline{\mathbb{C}_+}} |f(z)|^p d\nu(z)\right)^{1/p}.$$

Clearly the space A^2_ν is a Hilbert space.
Well-known examples of Zen spaces are Hardy spaces $H^p(\mathbb{C}_+)$, where $\tilde{\nu}$ is the Dirac measure at 0, or the standard weighted Bergman spaces A^p_α, where $d\tilde{\nu}(r) = r^\alpha dr$, $\alpha > -1$. Some further examples constructed from Hardy spaces on shifted half planes were given by Zen Harper in [4, 5].
The following proposition, given in [11], is elementary and appears for special cases in [4, 5]. Partial results are also given in [1, 2].

Proposition 2.3 *(Proposition 2.3 in [11]) Let A^2_ν be a Zen space, and let $w : (0,\infty) \to \mathbb{R}_+$ be given by*

$$w(t) = 2\pi \int_0^\infty e^{-2rt} d\tilde{\nu}(r) \qquad (t > 0).$$

Then the Laplace transform defines an isometric map $\mathcal{L} : L^2_w(0,\infty) \to A^2_\nu$.

Note that the existence of the integral is guaranteed by the (Δ_2)-condition.

We shall require the following Laplace–Carleson Embedding Theorem from [11].

Theorem 2.4 *(Theorem 2.4 in [11]) Let A^2_ν be a Zen space, $\nu = \tilde{\nu} \otimes \lambda$, and let $w : (0,\infty) \to \mathbb{R}_+$ be given by*

$$w(t) = 2\pi \int_0^\infty e^{-2rt} d\tilde{\nu}(r) \qquad (t > 0). \tag{2}$$

Then the following are equivalent:

1. *The Laplace transform $\mathcal{L}$ given by $\mathcal{L}f(z) = \int_0^\infty e^{-tz} f(t)dt$ defines a bounded linear map*
$$\mathcal{L} : L^2_w(0,\infty) \to L^2(\mathbb{C}_+, \mu).$$
2. *For a sufficiently large $N \in \mathbb{N}$, there exists a constant $\kappa > 0$ such that*
$$\int_{\mathbb{C}_+} \left|(\mathcal{L}t^{N-1}e^{-\lambda t})(z)\right|^2 d\mu(z) \le \kappa \int_0^\infty |t^{N-1}e^{-\lambda t}|^2 w(t)dt \text{ for each } \lambda \in \mathbb{C}_+.$$
3. *There exists a constant $\kappa > 0$ such that*
$$\mu(Q_I) \le \kappa\nu(Q_I) \text{ for each Carleson square } Q_I.$$
where Q_I denotes the Carleson square $Q_I = \{z = x + iy \in \mathbb{C}_+ : iy \in I, 0 < x < |I|\}$.

From this we may deduce a result characterizing admissibility for normal semigroups in the sense of $L^2_w(0,\infty)$.

Theorem 2.5 *Suppose that $A : D(A) \subset H \to H$ has a Riesz basis (ϕ_k) of eigenvectors with eigenvalues (λ_k) satisfying $Re\,\lambda_k < 0$ and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$ given by the sequence (b_k). Let A^2_ν be a Zen space and let w be given by (2). Then the following statements are equivalent.*

1. *B is an admissible control operator with respect to $L^2_w(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_H \le m_0 \|u\|_{L^2_w(0,\infty)}$$

for every $u \in L^2_w(0,\infty)$.

2. *For a sufficiently large $N \in \mathbb{N}$, there exists a constant $\kappa > 0$ such that*

$$\|(\lambda - A)^{-N} B\|^2 \le \kappa \int_0^\infty |t^{N-1} e^{-\lambda t}|^2 w(t)\,dt, \qquad (\lambda \in \mathbb{C}_+).$$

3. *There exists a constant $\kappa > 0$ such that*

$$\mu(Q_I) \le \kappa \nu(Q_I) \text{ for each Carleson square } Q_I,$$

where $\mu = \sum_k |b_k|^2 \delta_{-\lambda_k}$.

Proof: This follows from Theorem 2.1 and Theorem 2.4, taking $q = 2$ and $Z = L^2_w(0,\infty)$. Note that the resolvent condition follows because

$$\|(\lambda - A)^{-N} B\|^2 = \sum_k |\lambda - \lambda_k|^{-2N} |b_k|^2 = \int_{\mathbb{C}_+} \frac{d\mu(z)}{|\lambda + z|^{2N}},$$

and the Laplace transform of $t^{N-1}e^{-\lambda t}$ is a constant multiple of $(\lambda + z)^{-N}$. ∎

Remark 2.6 *Theorem 2.5 in the case that $\tilde{\nu}$ equals the Dirac measure in 0, that is $A^2_\nu = H^2(\mathbb{C}_+)$, is due to Ho and Russell [6], and Weiss [18]. In the case $d\tilde{\nu} = r^\alpha dr$, $\alpha \in (-1,0)$ the result is due to Wynn [20]. Using Theorem 2.5 any $\alpha > -1$ can now be considered.*

In [3], Haak applied Carleson measure theory to find conditions for admissibility of the system (1) above, with A generating a diagonal semigroup defined on ℓ^q, and inputs lying in the space $L^p(0,\infty)$. Using the Hausdorff–Young inequality, it is possible to characterize admissibility in the situation $p \le 2$ and $1 < p' \le q < \infty$.

Theorem 2.7 *Let $p \le 2$ and $1 < p' \le q < \infty$. Suppose that $A : D(A) \subset \ell^q \to \ell^q$ is a diagonal operator with eigenvalues (λ_k) satisfying $Re\,\lambda_k < 0$, and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$ corresponding to the sequence (b_k). Then the following statements are equivalent.*

1. *B is an admissible control operator with respect to $L^p(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \le m_0 \|u\|_{L^p(0,\infty)}$$

for every $u \in L^p(0,\infty)$.

2. *There exists a constant $\kappa > 0$ such that*

$$\mu(Q_I) \leq \kappa |I|^{q/p'}$$

for all intervals in $I \subset i\mathbb{R}$, where $\mu = \sum_k |b_k|^q \delta_{-\lambda_k}$.

3. *There exists a constant $\kappa > 0$ such that*

$$\|(\lambda - A)^{-1} B\|_{\ell^q} \leq \kappa |\operatorname{Re} \lambda|^{-1/p}$$

for all $\lambda \in \mathbb{C}_+$.

The theorem is a corollary of Theorem 2.1 combined with [11, Thm. 3.2]. Moreover the equivalence of Part 1 and 2 can be found in [3].

3 Admissibility for analytic semigroups

There is no general characterization of admissibility of the system (1) above, with A generating a diagonal semigroup defined on ℓ^q, and inputs lying in the space $L^p(0, \infty)$, $p > 2$, as there is no known full characterization of boundedness of Laplace–Carleson embeddings

$$L^p(0, \infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f = \int_0^\infty e^{-t\cdot} f(t) dt.$$

However, characterizations are possible in some cases with additional information on the support on the measure.
If the measure μ is supported on a sector $S(\theta) = \{z \in \mathbb{C}_+ : |\arg z| < \theta\}$ for some $0 < \theta < \frac{\pi}{2}$, then the oscillatory part of the Laplace transform can be discounted, and a full characterization of boundedness can be achieved (see also [3], Theorem 3.2 for an alternative characterization by means of a different measure).

Theorem 3.1 *(Theorem 3.3 in [11]) Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$, and let $q \geq p > 1$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{L} : L^p(0, \infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

2. *There exists a constant $\kappa > 0$ such that $\mu(Q_I) \leq \kappa |I|^{q/p'}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0.*

3. *There exists a constant $\kappa > 0$ such that $\|\mathcal{L} e^{-\cdot z}\|_{L^q_\mu} \leq \kappa \|e^{-\cdot z}\|_{L^p}$ for all $z \in \mathbb{R}_+$.*

From this we may deduce a result characterizing admissibility for analytic semigroups with respect to $L^p(0, \infty)$.

Theorem 3.2 *Let $1 < p \le q < \infty$. Suppose that $A : D(A) \subset \ell^q \to \ell^q$ is a diagonal operator with eigenvalues (λ_k) satisfying $Re\,\lambda_k < 0$ and $(-\lambda_k) \subset S(\theta)$ for some $\theta \in (0, \frac{\pi}{2})$, and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$ given by the sequence (b_k). Then the following statements are equivalent.*

1. *B is an admissible control operator with respect to $L^p(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \le m_0 \|u\|_{L^p(0,\infty)}$$

for every $u \in L^p(0,\infty)$.

2. *There exists a constant $\kappa > 0$ such that*

$$\mu(Q_I) \le \kappa\nu |I|^{q/p'}$$

for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0, where $\mu = \sum_k |b_k|^q \delta_{-\lambda_k}$.

3. *There exists a constant $\kappa > 0$ such that*

$$\|(z-A)^{-1}B\|_{\ell^q} \le \kappa z^{-1/p}$$

for all $z \in \mathbb{R}_+$.

Remark 3.3 *Let μ, θ, p and q be as in Theorem 3.2. In [3], Theorem 3.2, the equivalence of the following statements is shown:*

1. *B is an admissible control operator with respect to $L^p(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \le m_0 \|u\|_{L^p(0,\infty)}$$

for every $u \in L^p(0,\infty)$.

2. *There exists a constant $\kappa > 0$ such that $\tilde{\mu}(Q_I) \le \kappa |I|^{q/p}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0, where $d\tilde{\mu}(z) = |z|^q d\mu(\frac{1}{z})$.*

Now let us consider the case $p > q$ for sectorial measures μ. In [11] a condition in terms of the balayage S_μ of μ has been obtained. Recall that the balayage S_μ of a positive Borel measure μ on $\mathbb{C}_+$ is given by $S_\mu(t) = \int_{\mathbb{C}_+} p_z(t)d\mu(z)$, where p_z denotes the Poisson kernel of the right half plane. Let $S_n := \{z \in \mathbb{C} \mid 2^{n-1} < \operatorname{Re} z \le 2^n\}$.

Theorem 3.4 *(Theorem 3.5 in [11]) Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$ and let $1 \le q < p < \infty$. Then the following are equivalent:*

1. *The embedding*

$$\mathcal{L} : L^p(\mathbb{R}_+) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

   is well-defined and bounded.

2. *The sequence* $(2^{-nq/p'}\mu(S_n))$ *is in* $\ell^{p/(p-q)}(\mathbb{Z})$.

3. *The sequence* $(2^{n/p}\|\mathcal{L}e^{-2^n}\|_{L^q_\mu})$ *is in* $\ell^{qp/(p-q)}(\mathbb{Z})$.

If $p' < q$, *then the above is also equivalent to*

4. $t^{q(2-p)/p}S_\mu \in L^{p/(p-q)}(\mathbb{R})$.

As a corollary we obtain the following result.

Theorem 3.5 *Let* $1 \leq q < p < \infty$. *Suppose that* $A : D(A) \subset \ell^q \to \ell^q$ *is a diagonal operator with eigenvalues* (λ_k) *satisfying* $\mathrm{Re}\,\lambda_k < 0$ *and* $(-\lambda_k) \subset S(\theta)$ *for some* $\theta \in (0, \frac{\pi}{2})$, *and let* B *be a linear bounded map from* $\mathbb{C}$ *to* $D(A^*)'$ *given by the sequence* (b_k). *Then the following statements are equivalent.*

1. B *is an admissible control operator with respect to* $L^p(0,\infty)$, *that is, there exists a constant* $m_0 > 0$ *such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \leq m_0\|u\|_{L^p(0,\infty)}$$

   for every $u \in L^p(0,\infty)$.

2. *The sequence* $(2^{-nq/p'}\mu(S_n))$ *is in* $\ell^{p/(p-q)}(\mathbb{Z})$.

3. *The sequence* $(2^{n/p}\|(2^n - A)^{-1}B\|_{\ell^q})$ *is in* $\ell^{qp/(p-q)}(\mathbb{Z})$.

If $p' < q$, *then the above is also equivalent to*

4. $t^{q(2-p)/p}S_\mu \in L^{p/(p-q)}(\mathbb{R})$.

Here $\mu = \sum_k |b_k|^q \delta_{-\lambda_k}$.

Example 3.6 *We study the one-dimensional heat equation on the interval* $[0,1]$ *which is given by*

$$\begin{aligned}
\frac{\partial z}{\partial t}(\zeta,t) &= \frac{\partial^2 z}{\partial \zeta^2}(\zeta,t), \qquad \zeta \in (0,1), t \geq 0,\\
\frac{\partial z}{\partial \zeta}(0,t) &= 0, \quad \frac{\partial z}{\partial \zeta}(1,t) = u(t), \quad t \geq 0,\\
z(\zeta,0) &= z_0(\zeta), \quad \zeta \in (0,1).
\end{aligned}$$

This PDE can be written equivalently in the form (1) with $X = \ell^2$, $Ae_n = -n^2\pi^2 e_n$, *and* b_n *defined by* $b_n = 1$ *for each* n. *By Theorem 3.2 (for* $1 \leq p \leq 2$*) and Theorem 3.5 (for* $2 \leq p < \infty$*), the operator* B *is an admissible control operator with respect to* $L^p(0,\infty)$ *if and only if* $p \geq 4/3$.

Control problems involving smoother, Sobolev–space valued, controls are related to embeddings of the form

$$\mathcal{H}^p_{\beta,w}(0,\infty) \to L^q(\mathbb{C}_+,\mu), \quad f \mapsto \mathcal{L}f = \int_0^\infty e^{-t\cdot} f(t)dt,$$

given by the Laplace transform $\mathcal{L}$. Here, for $\beta > 0$ the space $\mathcal{H}^p_\beta(0,\infty)$ is given by

$$\begin{aligned} \mathcal{H}^p_\beta(0,\infty) &= \left\{ f \in L^p(\mathbb{R}_+) : \int_0^\infty \left| \left(\frac{d}{dx}\right)^\beta f(t) \right|^p dt < \infty \right\}, \\ \|f\|^p_{\mathcal{H}^p_\beta} &= \|f\|^p_p + \left\| \left(\frac{d}{dx}\right)^\beta f \right\|^p_p . \end{aligned}$$

Here $(\frac{d}{dx})^\beta f$ is defined as a fractional derivative via the Fourier transform. We call $\mathcal{H}^p_{\beta,w}$ the Sobolev space of index β and weight w.
In [11, Corollaries 3.7 and 3.8] the following characterisations of the boundedness of the Laplace–Carleson embedding has been proved.

Proposition 3.7 *(Corollary 3.7 in [11]) Let μ be a positive Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$, and let $q \geq p > 1$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{L} : \mathcal{H}^p_\beta(0,\infty) \to L^q(\mathbb{C}_+,\mu), \quad f \mapsto \mathcal{L}f,$$

   is well-defined and bounded.

2. *There exists a constant $\kappa > 0$ such that $\mu_{q,\beta}(Q_I) \leq \kappa |I|^{q/p'}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0. Here, $d\mu_{q,\beta}(z) = (1 + \frac{1}{|z|^{q\beta}}) d\mu(z)$.*

3. *There exists a constant $\kappa > 0$ such that $\|\mathcal{L}e^{-\cdot z}\|_{L^q_\mu} \leq \kappa \|e^{-\cdot z}\|_{\mathcal{H}^p_\beta}$ for all $z \in \mathbb{R}_+$.*

Proposition 3.8 *(Corollary 3.8 in [11]) Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$ and let $1 \leq q < p$, $\beta \geq 0$. Suppose that $S_{\mu_{\beta,q}} \in L^{p/(p-q)}$. Then the embedding*

$$\mathcal{L} : \mathcal{H}^p_\beta(0,\infty) \to L^q(\mathbb{C}_+,\mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

As an application we are able to characterize admissibility with respect to Sobolev space valued control functions.

Theorem 3.9 *Let $1 < p < q < \infty$. Suppose that $A : D(A) \subset \ell^q \to \ell^q$ is a diagonal operator with eigenvalues (λ_k) satisfying $\mathrm{Re}\,\lambda_k < 0$ and $(-\lambda_k) \subset S(\theta)$ for some $\theta \in (0, \frac{\pi}{2})$, and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$. Write $\mu = \sum |b_k|^q \delta_{-\lambda_k}$. Then the following statements are equivalent.*

1. *B is an admissible control operator with respect to $\mathcal{H}^p_\beta(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \leq m_0 \|u\|_{\mathcal{H}^p_\beta(0,\infty)},$$

   for every $u \in \mathcal{H}^p_\beta(0,\infty)$.

2. *There exists a constant $\kappa > 0$ such that $\mu_{q,\beta}(Q_I) \leq \kappa |I|^{q/p'}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0. Here, $d\mu_{q,\beta}(z) = (1 + \frac{1}{|z|^{q\beta}})d\mu(z)$.*

3. *There exists a constant $\kappa > 0$ such that $\|(z-A)^{-1}B\|_{\ell^q} \leq \kappa \|e^{-\cdot z}\|_{\mathcal{H}^p_\beta}$ for all $z \in \mathbb{R}_+$.*

Theorem 3.10 *Let $1 < q < p < \infty$. Suppose that $A : D(A) \subset \ell^q \to \ell^q$ is a diagonal operator with eigenvalues (λ_k) satisfying $Re\,\lambda_k < 0$ and $(-\lambda_k) \subset S(\theta)$ for some $\theta \in (0, \frac{\pi}{2})$, and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$. Suppose that $S_{\tilde{\mu}_{\beta,q}} \in L^{p/(p-q)}$. Then B is an admissible control operator with respect to $\mathcal{H}^p_\beta(0,\infty)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^q} \leq m_0 \|u\|_{\mathcal{H}^p_\beta(0,\infty)},$$

for every $u \in \mathcal{H}^p_\beta(0,\infty)$.

In the setting of analytic diagonal semigroups, it is also possible to characterise $L^2((0,\infty), t^\alpha dt)$-admissibility of control operators and $L^2((0,\infty), t^{-\alpha} dt)$-admissibility of observation operators in the range $0 < \alpha < 1$ in terms of a Carleson-type condition or a resolvent condition (compare this with the counterexample by Wynn in [21] for a diagonal semigroup).
Generally, the difficulty in the case $\alpha > 0$ stems from the well-known fact that the boundedness of Carleson embeddings on Dirichlet space cannot be tested on reproducing kernels or by means of a simple Carleson-type condition [15]. In the case of sectorial measures, however, such a characterisation is possible, at least for $0 < \alpha < 1$. For a interval $I \subset i\mathbb{R}$, let T_I denote the right half of the Carleson square Q_I.

Theorem 3.11 *Let μ be a positive Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$, and let $0 < \alpha < 1$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{L} : L^2((0,\infty), t^\alpha dt) \to L^2(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

   is well-defined and bounded.

2. *There exists a constant $\gamma > 0$ such that*

$$\mu(T_I) \leq \gamma |I|^{1-\alpha}$$

   for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0.

3. There exists a constant $\kappa > 0$ such that

$$\|\mathcal{L}t^{-\alpha}e^{-tz}\|_{L^2(\mathbb{C}_+,\mu)} \le \kappa \|t^{-\alpha}e^{-tz}\|_{L^2(t^\alpha dt)}$$

for all $z \in \mathbb{R}_+$.

Proof: The implication (1) $\Rightarrow$ (3) is immediate. For (3) $\Rightarrow$ (2), let $z = z_I = |I|/2$ denote the centre of the Carleson square Q_I over an intervals $I \subset i\mathbb{R}$ which is symmetric about 0. Then the modulus of the function

$$(\mathcal{L}t^{-\alpha}e^{-tz_I})(s) = \frac{\Gamma(1-\alpha)}{(z_I+s)^{-\alpha+1}} \tag{3}$$

is bounded below by $\Gamma(1-\alpha)\frac{1}{(2z_I)^{-\alpha+1}}$ on T_I, and therefore

$$\begin{aligned}\mu(T_I) &\le \frac{(2z_I)^{2-2\alpha}}{\Gamma(1-\alpha)^2}\int_{\mathbb{C}_+}|(\mathcal{L}t^{-\alpha}e^{-tz_I})(s)|^2 d\mu(s)\\ &\le \kappa^2\frac{(2z_I)^{2-2\alpha}}{\Gamma(1-\alpha)^2}\|t^{-\alpha}e^{-tz_I}\|^2_{L^2(t^\alpha dt)} = \frac{\kappa^2}{\Gamma(1-\alpha)}|I|^{1-\alpha}.\end{aligned}$$

Let us now consider (2) $\Rightarrow$ (1). We use the argument from [11], Thm 3.3. For $n \in \mathbb{Z}$, let

$$T_n = \{x+iy \in \mathbb{C}_+ : 2^{n-1} < x \le 2^n, -2^{n-1} < y \le 2^{n-1}\}.$$

That is, T_n is the right half of the Carleson square Q_{I_n} over the interval $I_n = \{y \in i\mathbb{R}, |y| \le 2^{n-1}\}$. The T_n are obviously pairwise disjoint.
Without loss of generality we assume $0 < \theta < \arctan(\frac{1}{2})$, in which case $S(\theta) \subseteq \bigcup_{n=-\infty}^{\infty} T_n$ (otherwise, we also have to use finitely many translates $T_{n,k}$ of each T_n, for which the same estimates apply) .
Now let $z \in T_n$ for some $n \in \mathbb{Z}$. Then we obtain, for $f \in L^2((0,\infty), t^\alpha dt)$,

$$\begin{aligned}|\mathcal{L}f(z)| \le \int_0^\infty |e^{-zt}||f(t)|dt &\le \int_0^\infty |t^{-\alpha/2}e^{-2^{n-1}t}||f(t)t^{\alpha/2}|dt\\ &\le C_\alpha 2^{(-n+1)(1-\alpha/2)}(M(t^{\alpha/2}f))(2^{-n+1}),\end{aligned}$$

where $C_\alpha > 0$ is a constant dependent only on the L^1-integration kernel $\phi_\alpha(t) = \chi_{[0,\infty)}(t+1)(t+1)^{\alpha/2}e^{-t-1}$, and Mf is the Hardy–Littlewood maximal function. We refer to e.g. [16], page 57, equation (16) for a pointwise estimate between the maximal function induced by the kernel ϕ_α, and M. Note that we can easily dominate ϕ_α by a positive, radial, decreasing L^1 function here. Consequently,

$$\begin{aligned}\int_{S(\theta)}|\mathcal{L}f(z)|^2 d\mu(z) &\le C_\alpha^2 \sum_{n=-\infty}^{\infty} 2^{(-n+1)(2-\alpha)}(M(t^{\alpha/2}f)(2^{-n+1}))^2\mu(T_n)\\ &\le \gamma C_\alpha^2 \sum_{n=-\infty}^{\infty} 2^{(-n+1)(2-\alpha)}2^{n(1-\alpha)}(M(t^{\alpha/2}f)(2^{-n+1}))^2\\ &= \gamma C_\alpha^2 2^{1-\alpha}\sum_{n=-\infty}^{\infty} 2^{-n+1}M(t^{\alpha/2}f)(2^{-n+1})^2\\ &\lesssim \|f\|^2_{L^2(t^\alpha dt)}.\end{aligned}$$

$\blacksquare$

We therefore have the following corollary of Theorem 3.11 and Theorem 2.1.

Corollary 3.12 *Let $0 < \alpha < 1$. Suppose that $A : D(A) \subset \ell^2 \to \ell^2$ is a diagonal operator with eigenvalues (λ_k) satisfying $Re\,\lambda_k < 0$ and $(-\lambda_k) \subset S(\theta)$ for some $\theta \in (0, \frac{\pi}{2})$, and let B be a linear bounded map from $\mathbb{C}$ to $D(A^*)'$. Write $\mu = \sum |b_k|^2 \delta_{-\lambda_k}$. Then the following statements are equivalent.*

1. *B is an admissible control operator with respect to $L^2((0,\infty), t^\alpha dt)$, that is, there exists a constant $m_0 > 0$ such that*

$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\|_{\ell^2} \leq m_0 \|u\|_{L^2((0,\infty),t^\alpha dt)},$$

   for every $u \in L^2((0,\infty), t^\alpha dt)$.

2. *There exists a constant $\kappa > 0$ such that $\mu(T_I) \leq \kappa |I|^{1-\alpha}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0.*

3. *There exists a constant $\kappa > 0$ such that*

$$\|\mathcal{L}t^{-\alpha}e^{-\lambda t}\|_{L^2(\mathbb{C}_+,\mu)} \leq \kappa \|t^{-\alpha}e^{-\lambda t}\|_{L^2(t^\alpha dt)}$$

   for all $\lambda \in \mathbb{R}_+$.

4. *There exists a constant $\kappa > 0$ such that*

$$\|(\lambda - A)^{\alpha-1}B\| \leq \kappa \lambda^{(\alpha-1)/2}$$

   for all $\lambda \in \mathbb{R}_+$.

Proof: It is enough to check that conditions 3 and 4 are equivalent. Note that for $\lambda \in \mathbb{R}_+$

$$\|(\lambda - A)^{\alpha-1}B\|^2 = \sum_k |b_k|^2 |\lambda - \lambda_k|^{2\alpha-2},$$

which is a constant multiple of $\|\mathcal{L}t^{-\alpha}e^{-\lambda t}\|^2_{L^2(\mathbb{C}_+,\mu)}$ (cf. (3)) . Similarly, we have that $\|t^{-\alpha}e^{-\lambda t}\|^2_{L^2(t^\alpha dt)}$ is a constant multiple of $\lambda^{\alpha-1}$. $\blacksquare$

Note that the corollary is also valid in the limiting case $\alpha = 0$.

4 Exact controllability for diagonal systems

We consider the equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \tag{4}$$

with solution

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\,ds,$$

suitably interpreted. Again we consider an exponentially stable C_0 semigroup $(T(t))_{t\geq 0}$, on a Hilbert space H, i.e.,

$$\|T(t)\| \leq Me^{-\lambda t}, \qquad (t \geq 0),$$

for some $M > 0$ and $\lambda > 0$. Suppose first that B is admissible. Then we have a bounded operator $\mathcal{B}_\infty : L^2(0,\infty;U) \to H$, defined by

$$\mathcal{B}_\infty u = \int_0^\infty T(t)Bu(t)\,dt.$$

The system is *exactly controllable*, if its range $R(\mathcal{B}_\infty)$ equals H. If B is not admissible, then the operator $\mathcal{B}_\infty$ is commonly defined as a mapping into a larger (extrapolation) space and exact controllability requires that its image contains H.
In [8] exact controllability for diagonal systems with scalar inputs was characterised in terms of Carleson measures (a version for multivariable inputs was given in [9]). In particular if A has a Riesz basis of eigenvectors, with eigenvalues (λ_n), then with a control operator corresponding to a sequence (b_n) the system is exactly controllable if and only if

$$\nu_\lambda := \sum_n \frac{|\operatorname{Re}\lambda_n|^2}{|b_n|^2 \prod_{k\neq n} p(\lambda_n,\lambda_k)^2}\delta_{-\lambda_n}$$

is a Carleson measure. Here $p(\lambda_n,\lambda_k)$ is the pseudo-hyperbolic metric, i.e.,

$$p(\lambda_n,\lambda_k) = \left|\frac{\lambda_n - \lambda_k}{\lambda_n + \overline{\lambda}_k}\right|.$$

Exact controllability by inputs in Sobolev spaces $\mathcal{H}^2_\beta$ with $0 < \beta < 1/2$ was characterised in [10, Thm. 3.8]. In [11, Thm. 3.9] the following result was proved, which enable us to dispense with the restriction on β.

Theorem 4.1 *Let μ be a positive Borel measure on the right half plane $\mathbb{C}_+$ and let $\beta > 0$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{H}^2_\beta(0,\infty) \to L^2(\mathbb{C}_+,\mu)$$

   is bounded.

2. *The measure $|1+z|^{-2\beta}d\mu(z)$ is a Carleson measure on $\mathbb{C}_+$.*

In [13], [12], the following theorem was proved:

Theorem 4.2 *Let $(g_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers and let $(z_k)_{k\in\mathbb{N}}$ be a Blaschke sequence in $\mathbb{C}_+$. Write $b_{\infty,k} = \prod_{j\neq k} p(z_j,z_k)$.*
Let $m_2 = \sup_{(a_k)\in\ell^2, \|(a_k)\|_2=1} \inf_{f\in H^2(\mathbb{C}_+), g_k f(z_k)=a_k} \|f\|_{H^2}$.
Then

$$m_2 = \|\mathcal{J}_{\mu_2}\|$$

where

$$\mu_2 = \sum_{k=1}^{\infty} \frac{|2\operatorname{Re} z_k|^2}{|b_{\infty,k} g_k|^2} \delta_{z_k},$$

and $\mathcal{J}_{\mu_2}$ *is the Carleson embedding*

$$\mathcal{J}_{\mu_2} : H^2(\mathbb{C}_+) \to L^2(\mathbb{C}_+, \mu_2).$$

Using Theorem 4.1, we obtain

Corollary 4.3 *Let* $(g_k)_{k\in\mathbb{N}}$ *be a sequence of nonzero complex numbers and let* $(z_k)_{k\in\mathbb{N}}$ *be a Blaschke sequence in* $\mathbb{C}_+$. *Write* $b_{\infty,k} = \prod_{j\neq k} p(z_j, z_k)$. *Let*

$$m_\beta = \sup_{(a_k)\in\ell^2, \|(a_k)=1\|} \inf_{f\in\mathcal{H}^2_\beta} \{\|f\|_{\mathcal{H}^2_\beta} : g_k \mathcal{L} f(z_k) = a_k\}.$$

Then $m_\beta < \infty$, *if and only if there is a constant* $\kappa > 0$ *such that*

$$\sum_{z_k\in Q_I} \frac{|2\operatorname{Re} z_k|^2 |1+z_k|^{2\beta}}{|b_{\infty,k}|^2 |g_k|^2} \leq \kappa |I| \text{ for all intervals } I \subset i\mathbb{R}.$$

Finally, Corollary 4.3 enables us to characterize controllability by inputs in Sobolev spaces $\mathcal{H}^2_\beta$.

Theorem 4.4 *Let* $\beta > 0$. *Suppose that* $A : D(A) \subset H \to H$ *has a Riesz basis* (ϕ_k) *of eigenvectors with eigenvalues* (λ_k) *satisfying* $Re\,\lambda_k < 0$ *and let* B *be a linear bounded map from* $\mathbb{C}$ *to* $D(A^*)'$. *Write* $b_{\infty,k} = \prod_{j\neq k} p(\lambda_j, \lambda_k)$. *Then the following statements are equivalent.*

1. *System (4) is exactly controllable with respect to* $\mathcal{H}^2_\beta$, *that is,*

$$H \subset \mathcal{B}_\infty(\mathcal{H}^2_\beta).$$

2. *there is a constant* $\kappa > 0$ *such that*

$$\sum_{-\lambda_k\in Q_I} \frac{|2\operatorname{Re}\lambda_k|^2 |1-\lambda_k|^{2\beta}}{|b_{\infty,k}|^2 |g_k|^2} \leq \kappa |I| \text{ for all intervals } I \subset i\mathbb{R}.$$

Similar results can be proved for null-controllability, along the lines of [8].

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