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Abstract

In this note we derive using the subequation method a new nonstandard finite difference scheme (NSFD) for a class of convection-diffusion equations. Despite the fact that this scheme has nonlinear denominator functions of the step sizes (even for linear PDEs), it has a couple of favourable properties: it is explicit and due to its construction it reproduces important properties of the solution of the parabolic PDE. This proposed method conserves the positivity of the solution and hence it is perfectly suited for solving e.g. air pollution problems or the Black–Scholes equation for the valuation of Asian options, since it avoids negative values for the calculated prices. Finally, we illustrate the usefulness of this newly proposed method on a test example from the literature.

1 Introduction

In the process of modelling and simulation of physical phenomena, it is desirable that numerical quantities reflect the original physical behaviour, e.g. in electrical engineering where the charge transport in semiconductor devices, is usually described by a convection-diffusion equation. Thereby the charge transport is described in terms of charge carrier densities, which – by their physical – meaning should be nonnegative. Similar reasoning holds for the simulation of the pollutant density in air pollution problems [4]. Also, in financial applications, e.g. the computation of the fair price of an option, one is interested in numerical methods that guarantees the positivity of the solution.

Hence, it is of tremendous importance to construct these *positivity preserving schemes* [3, 7] that avoid unrealistic negative values for the solution and, as a side-effect, are stable with respect to the maximum norm. One possibility are so-called *nonstandard finite difference (NSFD) methods*. These methods are tailor made special schemes for the numerical integration of differential equations in order to preserve certain properties (positivity, asymptotic behaviour, etc.) of the analytic solution on the discrete level. The general basic rules to construct NSFDs [11, 12] are the following:

- The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives appearing in the differential equations.
- Discrete representations for derivatives must, in general, have nontrivial denominator functions.
- Nonlinear terms should, in general, be replaced by nonlocal discrete representations.
- Special conditions that hold for either the differential equation and/or its solutions should also hold for the difference equation model and/or its solutions.

In this work we will demonstrate how to construct a NSFD scheme for a general convection-diffusion type PDE of the form

$$u_\tau + b\, u_x = a\, u_{xx} - c\, u, \tag{1.1}$$

by using the *subequation method*. The idea is to start from an explicit upwind discretization, construct suitable denominator functions and then develop a strategy for choosing the time step appropriately.

Hereby we focus on the financial application, especially the Asian option pricing problem.

This work is organized as follows: first we introduce in §2 the Black–Scholes equation, recall the standard transformation to a forward-in-time heat equation and present the convection-diffusion equation of the form (1.1) that results from the pricing problem for Asian options. In §3 we present the *subequation method* which is the basic tool in deriving the NSFD scheme. Afterwards we analyze in §4 the properties of the resulting numerical scheme by the modified equation technique. Finally, we illustrate in §5 the accuracy and efficiency of the new method with a numerical test example and compare it to classical finite difference schemes.

2 The Black–Scholes Equation

The famous Black–Scholes equation is an effective model for option pricing. It was named after the pioneers Black, Scholes and Merton who suggested it 1973 [2, 10] and received in 1997 the Nobel Prize in Economics for their discovery.Mathematically it is a final value problem for a backward-in-time second order parabolic equation. A concise derivation of the Black–Scholes equation can be found in [17].

An option is a contract that admits the owner the right (not the duty) to buy (*'call option'*) or to sell (*'put option'*) an asset (typically a stock or a parcel of shares of a company) for a prespecified price E (*'strike price'*) by the date T to receive some payoffs. The basic problem here is to specify a fair price to charge for permitting these rights. A closely related question is how to hedge the risks that arises when selling these options. 'European' options can only be exercised at the expiration date T. For 'American' options exercise is permitted at any time until the expiry date. A third type are the Asian options that define a payoff dependending on the temporal average of the price of the underlying. The notion European, American or Asian are not meant geographically, they just declare the type of option.

In general, closed–form solutions do not exist (especially for American options) and the solution has to be computed numerically. The standard approach for solving the Black–Scholes equation for pricing options consists in transforming the original equation to a convection-diffusion equation posed on a semi-unbounded domain [14, 17]. Often finite differences are used to discretize this convection-diffusion equation and *artificial boundary conditions* (ABCs) [5] are introduced in order to confine the computational domain appropriately and retain the accuracy and stability properties of the underlying scheme.

Next we present one standard example, the European put, and later we turn to the Asian option pricing problem, that will lead us to a convection-diffusion equation of the form (1.1).

2.1 The European Call Option

Here we focus on *European Call options*; the treatment of *European Put options* is analogous. The *value of a Call option* is denoted by V and depends on the current market price of the underlying asset, S, (the letter 'S' symbolizes *stocks*) and the remaining time t until the option expires: $V = V(S,t)$. The *Black–Scholes equation* is a backward–in–time parabolic equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad S > 0,\ 0 \le t < T, \quad (2.1)$$

where σ denotes the annual *volatility* of the asset price, r the risk–free interest rate and T is the *expiry date* ($t = 0$ means 'today').

The *final condition* ('payoff condition') at the expiry $t = T$ reads

$$V(S,T) = (S-K)^+, \qquad S \ge 0, \quad (2.2)$$

with the notation $f^+ = \max(f,0)$. Here $K > 0$ denotes the previously agreed *exercise price* or 'strike', of the contract.

The 'spatial' or asset–price *boundary conditions* for European Call options at $S = 0$, and $S \to \infty$ are

$$V(0,t) = 0, \qquad 0 \le t \le T, \quad (2.3a)$$

$$V(S,t) \sim S - Ke^{-r(T-t)} \qquad \text{as } S \to \infty, \quad 0 \le t \le T. \quad (2.3b)$$

i.e. at $S = 0$ the option is worthless. Correspondingly, we have for a European Put the terminal condition

$$V(S,T) = (K-S)^+, \qquad S \ge 0, \quad (2.4)$$

and the boundary conditions

$$V(0,t) = 0, \qquad \text{as} \quad S \to \infty, \quad 0 \le t \le T, \quad (2.5a)$$

$$V(S,t) = Ke^{-r(T-t)} - S \qquad \text{for} \quad S \approx 0, \quad 0 \le t \le T. \quad (2.5b)$$

The focus of this paper is the numerical solution of the problem, which is achieved by initially analytically approaching the solution for the European Call by transforming (2.1) with (2.2), (2.3) into a forward-in-time parabolic problem. In the section thereafter both a classical and a NSFD scheme will be specified and used to solve the transformed problem.

2.2 The Transformation to the Heat Equation

Next we review how to transform (2.1) into a pure diffusion equation (cf. [17, § 5.4]). It is convenient to apply a time reversal and transform (2.1) to a *forward–in–time equation* by the change $t = T - 2\tau/\sigma^2$. The new time variable τ stands for (up to the scaling by $\sigma^2/2$) the *remaining life time* of the option. We denote the new variables by:

$$\widetilde{V}(S,\tau) = V(S,t) = V\Big(S, T - \frac{2\tau}{\sigma^2}\Big), \qquad \tilde{r} = \frac{2}{\sigma^2} r, \quad \widetilde{T} = \frac{\sigma^2}{2} T.$$

The resulting forward–in–time equation then reads:

$$\frac{\partial \widetilde{V}}{\partial \tau} = S^2 \frac{\partial^2 \widetilde{V}}{\partial S^2} + \tilde{r} S \frac{\partial \widetilde{V}}{\partial S} - \tilde{r}\, \widetilde{V}, \qquad S > 0, \quad 0 \le \tau < \widetilde{T}, \tag{2.6}$$

with the initial condition

$$\widetilde{V}(S,0) = (S-K)^+, \qquad S \ge 0, \tag{2.7}$$

and the boundary conditions

$$\lim_{S \to 0} \widetilde{V}(S,\tau) = 0, \qquad 0 \le \tau \le \widetilde{T}, \tag{2.8}$$

$$\widetilde{V}(S,\tau) \sim S - K e^{-r(2\tau/\sigma^2)} \qquad \text{as } S \to \infty, \quad 0 \le \tau \le \widetilde{T}. \tag{2.9}$$

The right hand side of (2.6) is a well–known *Euler's differential equation* and therefore it is standard practice (cf. [14]) to transform (2.6) to the heat equation. To do so, we let $\alpha = -(\tilde{r}-1)/2$, $\lambda = -\alpha^2 - \tilde{r}$, and use the change of variables

$$S = E e^x, \quad \widetilde{V}(S,\tau) = E\, e^{\alpha x + \lambda \tau} v(x,\tau).$$

Then problem (2.6)–(2.9) is equivalent to the initial boundary value problem for the heat equation:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}, \qquad x \in \mathbb{R}, \quad 0 \le \tau < \widetilde{T}. \tag{2.10}$$

The equation (2.10) is supplied with the initial condition

$$v(x,0) = \left(e^{\frac{1}{2}(\tilde{r}+1)x} - e^{\frac{1}{2}(\tilde{r}-1)x}\right)^+, \qquad x \in \mathbb{R}, \tag{2.11}$$

with the asymptotic boundary conditions for $0 \le \tau \le \widetilde{T}$:

$$v(x,\tau) = 0 \qquad \text{for } x \to -\infty, \tag{2.12}$$

$$v(x,\tau) = \exp\Big(\frac{\tilde{r}+1}{2}x + \frac{\tilde{r}+1)^2}{4}\tau\Big) \qquad \text{for } x \to \infty. \tag{2.13}$$

The problem how to appropriately confine the spatial domain for solving the whole space problem (2.10) posed on $x \in \mathbb{R}$ by (discrete) artificial boundary conditions that preserve the stability, accuracy and computational effort of the interior scheme, was discussed concisely in [5]. Thus, in the sequel we will restrict ourselves to the task of constructing a suitable interior NSFD method.

Next, we briefly sketch one application background for a convection diffusion equation.

2.3 Asian Options

We consider a backward–in–time 2D parabolic PDE for $V = V(S, A, t)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial A} - rV = 0, \tag{2.14}$$

on the domain $S > 0$, $A > 0$, $0 \le t < T$, that is simililar to the standard Black-Scholes equation (2.1). The new term in (2.14) is $f(S,t)\partial V/\partial A$ that destroys the Euler structure of the spatial operator and makes the standard transformation of §2.2 impossible.

There exist a couple of approaches for reducing the PDE (2.14) to one-dimensional convection diffusion problems, cf. [13]. Here, we restrict ourselves to one strategy and briefly review from [14] the case $f(S,t) = S$ (arithmetic average case). Let us focus on a European *arithmetic average strike call* with a payoff

$$V(S,A,T) = \Big(S - \frac{A}{T}\Big)^+ = S\Big(1 - \frac{1}{TS}\int_0^T S_\theta \, d\theta\Big)^+. \tag{2.15}$$

We introduce the auxiliary variable

$$R = \frac{1}{S}\int_0^T S_\theta \, d\theta, \tag{2.16}$$

and use the *separation of variables* (motivated by (2.15))

$$V(S, A, t) = S \cdot H(R, t) \tag{2.17}$$

to obtain the 1D parabolic PDE

$$\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR) \frac{\partial H}{\partial R} = 0, \qquad R > 0,\ t > 0, \tag{2.18}$$

supplied with the boundary conditions

$$\begin{aligned} \frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} &= 0, \quad \text{for } R = 0, \\ H &= 0, \quad \text{for } R \to \infty, \end{aligned} \tag{2.19}$$

and the terminal payoff condition

$$H(R, T) = \Big(1 - \frac{R}{T}\Big)^+. \tag{2.20}$$

This final boundary value problem is solved numerically for $H(R, t)$ and by (2.17) we get the value for V. For the monitoring of the average at discrete time points we refer the reader to [14, Section 6.2.4].

Let us recall that the *cell Péclet number* Pe (ratio of convection and diffusion term, multiplied by a typical discretization scale) is an indicator about the possible appearance of oscillations in the numerical solution. For the classical Black-Scholes equation (2.1) this ratio is

$$\mathrm{Pe} = \frac{2r}{\sigma^2} \frac{\Delta S}{S}, \tag{2.21}$$

and for the Asian option case (2.18) this number reads

$$\mathrm{Pe} = \frac{2(1 - rR)}{\sigma^2 R} \frac{\Delta R}{R}. \tag{2.22}$$

The difficulty associated with (2.22) (in contrast to the Péclet number (2.22)) is that a small volatility σ cannot be compensated by a small interest rate r.

We turn to the main part of this work, the subequation method.

3 The Subequation Method

In this section we will explain step by step the subequation method to obtain a NSFD for the convection-diffusion PDE. To start, let us consider the pure heat equation for the unknown $u = u(x, \tau)$

$$u_\tau = a\, u_{xx}, \qquad x \in \mathbb{R},\ \tau > 0. \tag{3.1}$$

Then, the standard (explicit) finite difference discretization reads

$$\frac{u_j^{n+1} - u_j^n}{\Delta\tau} = a\,\frac{u_{j+1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}, \quad j \in \mathbb{Z},\ n \in \mathbb{N}_0, \tag{3.2}$$

i.e.

$$u_j^{n+1} = a\,\mu\,(u_{j+1}^n + u_{j+1}^n) + (1 - 2a\mu)\,u_j^n, \tag{3.3}$$

with the *parabolic mesh ratio* $\mu = \Delta\tau/(\Delta x)^2$ and the pointwise approximation $u_j^n \approx u(x_j, t_n)$, where $x_j = j\Delta x$, $t_n = n\Delta t$, $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$.

Now *positivity* requires

$$1 - 2a\mu \geq 0. \tag{3.4}$$

Here, positivity means that solutions of this finite difference scheme obey a *discrete maximum principle*

$$\max_j |u_j^{n+1}| \leq \max_j |u_j^n|, \qquad n \in \mathbb{N}_0. \tag{3.5}$$

A way to ensure this condition (3.4) is to use the ansatz

$$1 - 2a\mu = \gamma a\mu, \qquad \gamma \geq 0, \tag{3.6}$$

or

$$\Delta\tau = \frac{(\Delta x)^2}{(2+\gamma)\,a}. \tag{3.7}$$

We have found that $\gamma = 2$ gives good stability and smoothness behavior for the solutions.

3.1 An illustrative Example

We want to outline the method of subequations for NSFD schemes using as an illustrative example the equation

$$u_\tau + u_x = u(1-u) \qquad x \in \mathbb{R},\ \tau > 0. \tag{3.8}$$

A *subequation* of (3.8) is an ordinary differential equation (ODE) or partial differential equation (PDE) obtained by dropping one or more terms appearing in the full equation (3.8). Thus, we have from (3.8) the following three subequations

$$u_\tau = u(1-u), \qquad \text{an ODE} \tag{3.9a}$$

$$u_x = u(1-u), \qquad \text{an ODE} \tag{3.9b}$$

$$u_\tau + u_x = 0, \qquad \text{a PDE} \tag{3.9c}$$

All three subequations have known exact schemes, cf. [11, 12]. They are respectively

$$\frac{u^{n+1} - u^n}{\Phi_1(\Delta\tau)} = u^n - u^n u^{n+1}, \qquad \Phi_1(\Delta\tau) = e^{\Delta\tau} - 1, \tag{3.10a}$$

$$\frac{u_{j+1} - u_j}{\Phi_2(\Delta x)} = u_j - u_j u_{j+1}, \qquad \Phi_2(\Delta x) = e^{\Delta x} - 1, \tag{3.10b}$$

$$\frac{u_j^{n+1} - u_j^n}{\Psi(\Delta\tau)} + \frac{u_j^n - u_{j-1}^n}{\Psi(\Delta x)} = 0, \qquad \Delta\tau = \Delta x. \tag{3.10c}$$

Note that for $u_\tau + u_x = 0$, we have a relationship between the step sizes, i.e. $\Delta\tau = \Delta x$. Further, any function $\Psi(z)$ can be used as a *denominator function*, as long as

$$\Psi(z) = z + \mathrm{O}(z^2). \tag{3.11}$$

The issue is to combine the equations (3.10) into a scheme for (3.8). For the above three subequations (3.10), the only way to do this results in the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Phi(h)} + \frac{u_j^n - u_{j-1}^n}{\Phi(h)} = u_{j-1}^n - u_{j-1}^n u_j^{n+1}, \tag{3.12}$$

where

$$\Phi(h) = e^h - 1, \qquad h = \Delta\tau = \Delta x. \tag{3.13}$$

In fact, equation (3.12) can be shown to be the *exact NSFD scheme* for equation (3.8).

We briefly check this result. First we consider the space independent equation (3.9a) and drop the j-dependence in (3.12) to obtain

$$\frac{u^{n+1} - u^n}{\Phi(\Delta\tau)} = u^n - u^n u^{n+1}, \qquad \Phi(\Delta\tau) = e^{\Delta\tau} - 1. \tag{3.14}$$

For the time-independent equation (3.9b) we drop the n-dependence in (3.12) to get after an index shift $j \to j + 1$

$$\frac{u_{j+1} - u_j}{\Phi(\Delta x)} = u_j - u_j u_{j+1}, \qquad \Phi(\Delta x) = e^{\Delta x} - 1. \tag{3.15}$$

Finally, dropping the terms on the right hand side of (3.12) gives

$$\frac{u_j^{n+1} - u_j^n}{\Phi(h)} + \frac{u_j^n - u_{j-1}^n}{\Phi(h)} = 0, \qquad \Phi(h) = e^h - 1, \tag{3.16}$$

with $h = \Delta\tau = \Delta x$.

3.2 The Nonstandard Difference Scheme for the Convection-Diffusion Equation

We consider a convection-diffusion PDE for the unknown $u = u(x, \tau)$

$$u_\tau + b\,u_x = a\,u_{xx} - c\,u, \qquad x \in \mathbb{R}, \tau \geq 0 \tag{3.17}$$

In general, the diffusion coefficient $a > 0$ and the reaction rate $c > 0$, but the convection speed b may have either sign. In this work we assume for simplicity $b > 0$, but the other case where $b < 0$ can be treated analogously.

In the preceeding subsection 3.1 we used three subequations to construct a NSFD scheme for the original PDE (3.8). For the PDE (3.17) there are six subequations, but, in general we cannot use all of them. The NSFD scheme for the general convection-diffusion equation (3.17) proposed here used the following four subequations

$$u_\tau = -c\,u, \qquad \text{an ODE} \tag{3.18a}$$

$$0 = a\,u_{xx} - c\,u, \qquad \text{an ODE} \tag{3.18b}$$

$$u_\tau = b\,u_x, \qquad \text{a PDE} \tag{3.18c}$$

$$0 = b\,u_x - c\,u, \qquad \text{a ODE} \tag{3.18d}$$

All four subequations have known exact schemes, cf. [11].

3.3 The case without Reaction Rate

Let us first consider the special case without reaction rate, i.e. we consider (3.17) with $c = 0$:

$$u_\tau + b\,u_x = a\,u_{xx}, \qquad x \in \mathbb{R}, \tau \geq 0. \tag{3.19}$$

The explicit (upwind) scheme for (3.17) is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta\tau} + b\frac{u_j^n - u_{j-1}^n}{\Delta x} = a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \qquad j \in \mathbb{Z}. \tag{3.20}$$

$n \in \mathbb{N}_0$. Now define the parabolic and hyperbolic mesh ratios as

$$\mu = \frac{\Delta\tau}{\Delta x^2}, \qquad \lambda = \frac{\Delta\tau}{\Delta x}, \qquad \text{with } \Delta\tau = \Delta\tau(a, b, \gamma_{1,2}), \tag{3.21}$$

then (3.20) becomes

$$u_j^{n+1} = a\mu u_{j+1}^n + (b\lambda + a\mu)u_{j-1}^n + (1 - b\lambda - 2a\mu)u_j^n. \tag{3.22}$$

Now, positivity requires $1 - b\lambda - 2a\mu \geq 0$. To proceed, we introduce two parameters $\gamma_1 \geq 0$, $\gamma_2 \geq 0$. The following two cases are allowed

Case A

$$1 - b\lambda - 2a\mu = \gamma_1 a\mu \tag{3.23}$$

Then we have the scheme

$$u_j^{n+1} = a\mu u_{j+1}^n + (b\lambda + a\mu)u_{j-1}^n + \gamma_1 a\mu u_j^n. \tag{3.24}$$

with

$$\Delta\tau = \frac{\Delta x^2}{(2+\gamma_1)a + b\Delta x} \tag{3.25}$$

$$\lambda = \frac{\Delta x}{(2+\gamma_1)a + b\Delta x}, \quad \mu = \frac{1}{(2+\gamma_1)a + b\Delta x}. \tag{3.26}$$

Case B

$$1 - b\lambda - 2a\mu = \gamma_2 b\lambda \tag{3.27}$$

Then we have the scheme

$$u_j^{n+1} = a\mu u_{j+1}^n + (b\lambda + a\mu)u_{j-1}^n + \gamma_2 b\lambda u_j^n. \tag{3.28}$$

with

$$\Delta\tau = \frac{\Delta x^2}{2a + (1+\gamma_2)b\Delta x} \tag{3.29}$$

$$\lambda = \frac{\Delta x}{2a + (1+\gamma_2)b\Delta x}, \quad \mu = \frac{1}{2a + (1+\gamma_2)b\Delta x}. \tag{3.30}$$

For both cases A and B one can show that

$$u_j^n \geq u_j^{n+1} \geq 0, \tag{3.31}$$

i.e. bounded positive input gives bounded positive numerical solutions.

3.4 Subequation Technique for the full PDE

We have the following six *subequations* of the full PDE (3.17):

$$u_\tau = -c\,u, \quad \text{an ODE} \tag{3.32a}$$
$$0 = a\,u_{xx} - c\,u, \quad \text{an ODE} \tag{3.32b}$$
$$u_\tau + b\,u_x = 0, \quad \text{a PDE} \tag{3.32c}$$
$$b\,u_x = -c\,u, \quad \text{an ODE} \tag{3.32d}$$
$$b\,u_x = a\,u_{xx}, \quad \text{an ODE} \tag{3.32e}$$
$$u_\tau = a\,u_{xx}, \quad \text{a PDE} \tag{3.32f}$$

The first five of (3.32) have exact schemes, cf. [11]. We will now give a novel scheme that incorporates the exact schemes of the first four equations. The resulting NSFD scheme for (3.17) is:

$$\frac{u_j^{n+1}-u_j^n}{\frac{1-e^{-c\Delta\tau}}{c}} + b\left\{\frac{u_j^n - u_{j-1}^n}{\frac{e^{\frac{c\Delta x}{b}}-1}{\frac{c}{b}}}\right\} = a\left\{\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\frac{4\sinh^2[\sqrt{\frac{c}{a}}\frac{\Delta x}{2}]}{\frac{c}{a}}}\right\} - c\,u_j^n, \tag{3.33}$$

with $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$.

In the sequel we will give some more insight to (3.33) and define the three *denominator functions*

$$\Phi_1(\Delta\tau, c) = \frac{1-e^{-c\Delta\tau}}{c}, \tag{3.34a}$$

$$\Phi_2(\Delta x, a, c) = \frac{4a}{c}\sinh^2\Big[\sqrt{\frac{c}{a}}\frac{\Delta x}{2}\Big], \tag{3.34b}$$

$$\Phi_3(\Delta x, b, c) = \frac{e^{\frac{c\Delta x}{b}}-1}{\frac{c}{b}}, \tag{3.34c}$$

and write the NSFD scheme (3.33) in the form

$$\frac{u_j^{n+1}-u_j^n}{\Phi_1(\Delta\tau, c)} + b\frac{u_j^n - u_{j-1}^n}{\Phi_3(\Delta x, b, c)} = a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Phi_2(\Delta x, a, c)} - c\,u_j^n. \tag{3.35}$$

Furthermore, let

$$\begin{aligned} \mu(\Delta\tau, \Delta x, a, c) &\equiv \frac{\Phi_1(\Delta\tau, c)}{\Phi_2(\Delta x, a, c)}, \\ \lambda(\Delta\tau, \Delta x, b, c) &\equiv \frac{\Phi_1(\Delta\tau, c)}{\Phi_3(\Delta x, b, c)}, \end{aligned} \tag{3.36}$$

and we rewrite the *explicit scheme* (3.35) as

$$u_j^{n+1} = a\mu u_{j+1}^n + (b\lambda + a\mu)u_{j-1}^n + \big(1 - b\lambda - 2a\mu - c\Phi_1\big)u_j^n. \tag{3.37}$$

Now, for *positivity* we require that in (3.37) the coefficient

$$1 - b\lambda - 2a\mu - c\Phi_1 = e^{-c\Delta\tau} - 2a\mu - b\lambda \geq 0 \tag{3.38}$$

must be positive. To ensure this, we set this coefficient equal to $\gamma(2a\mu + b\lambda)$ and immediately obtain for the parameter γ

$$\gamma = \frac{e^{-c\Delta\tau}}{2a\mu + b\lambda} - 1. \tag{3.39}$$

Hence, finally the NSFD scheme (3.37) reads

$$u_j^{n+1} = a\mu u_{j+1}^n + \gamma\big[2a\mu + b\lambda\big]u_j^n + \big[b\lambda + a\mu\big]u_{j-1}^n, \tag{3.40}$$

with

$$\Delta\tau = \frac{1}{c}\ln\left\{\frac{1}{1+\gamma}\frac{1}{2a\mu + b\lambda}\right\}, \tag{3.41a}$$

$$\text{with} \quad 2a\mu = \frac{1}{2}\frac{1-e^{-c\Delta\tau}}{\sinh^2\big[\sqrt{\frac{c}{a}}\frac{\Delta x}{2}\big]}, \tag{3.41b}$$

$$b\lambda = \frac{1-e^{-c\Delta\tau}}{e^{\frac{c\Delta x}{b}} - 1}. \tag{3.41c}$$

Since the non-constant mesh ratios μ and λ depend on $\Delta\tau$ and Δx, we have a functional relationship between them. An elementary calculation gives now

$$\Delta\tau = \frac{1}{c}\ln\left\{1 + \frac{1}{1+\gamma}\frac{2\big(e^{\frac{c\Delta x}{b}} - 1\big)\sinh^2\big[\sqrt{\frac{c}{a}}\frac{\Delta x}{2}\big]}{e^{\frac{c\Delta x}{b}} - 1 + 2\sinh^2\big[\sqrt{\frac{c}{a}}\frac{\Delta x}{2}\big]}\right\}, \tag{3.42}$$

which is a nonlinear equation to determine the time step since γ depends on $\Delta\tau$, cf. (3.39).

We conclude this section by summarizing the resulting algorithm

Algorithm.

1. Input parameters: a, b, c.
2. Select a value for $\gamma \geq 0$.
3. Choose the spatial step size Δx.
4. Calculate the time step $\Delta\tau$ from (3.42).
5. Calculate the ratios μ and λ from (3.36) using (3.34).
6. Using the initial and boundary-value data, use the explicit finite difference scheme (3.40) to determine a numerical solution; if $c = 0$, then use the scheme from Section 3.3 instead.

Remark. Another way of deriving a (different) NSFD scheme for the full PDE (3.17) is based on the classical *separation of variables* approach that leads to the solution of two ODEs having well-known NSFD schemes, cf. [11, 12]. However, here we prefer the subequation methods, since it involves PDEs.

4 The Modified Equation Analysis

In this section we want to study the effects of the discretization to the dispersion and dissipation by the *modified equation technique* [9, 16]. The idea is to describe the qualitative behaviour of the solutions to the finite difference approximation, i.e. mainly the dissipation and dispersion errors, by an analytic PDE with mesh dependent coefficents for which the finite difference method (FDM) has a higher order of consistency.

4.1 The Effect of the Spatial Discretization

4.1.1 The Centered-in-Space Scheme

We start with considering a standard spatial discretization of (3.17) using a central difference quotient for the convection term:

$$\partial_\tau u_j(\tau) + b\frac{u_{j+1}(\tau) - u_{j-1}(\tau)}{2\Delta x} = a\frac{u_{j+1}(\tau) - 2u_j(\tau) + u_{j-1}(\tau)}{\Delta x^2} - c\, u_j(\tau), \quad (4.1)$$

$j \in \mathbb{Z}$, $\tau \geq 0$, where $u_j(\tau) \approx u(j\Delta x, \tau)$. Next, assume the existence of a smooth solution $u(x,\tau)$ to the PDE with the values $u(x_j,\tau) = u_j(\tau)$ at the spatial grid points x_j and expand the solution $u_j(\tau)$ in Taylor series around $x_j = j\Delta x$ to obtain the modified PDE

$$u_\tau + b\, u_x + b\frac{\Delta x^2}{6}u_{xxx} = a\, u_{xx} + a\frac{\Delta x^2}{12}u_{xxxx} - c\, u + \mathrm{O}(\Delta x^4), \quad (4.2)$$

$x \in \mathbb{R}$, $\tau \geq 0$. We insert an appropriate Fourier-type solution of the form:

$$u(x,\tau) = \exp\{-(A + c)\tau\}\, \exp\{ik(x - B\tau)\}, \quad (4.3)$$

that consist of an exponential modelling the dissipation (+ reaction) effects and a second exponential that represents the convection, and solve for A and B:

$$A = a\Big[1 - \frac{\Delta x^2}{12}k^2\Big]k^2, \qquad B = b\Big[1 - \frac{\Delta x^2}{6}k^2\Big] \quad (4.4)$$

and one can easily study in (4.4) how the deviations in dissipation and convection speed depending on the spatial grid size Δx and the wave number k. In this case of centered differences the discretization leads to a reduction of the diffusion and convection.

4.1.2 The Upwind Scheme

Analogously, one gets for the upwind discretization

$$\begin{aligned} u_\tau + b\,u_x + b\frac{\Delta x^2}{6} u_{xxx} \\ = \Big[a + b\frac{\Delta x}{2}\Big] u_{xx} + \Big[a\frac{\Delta x^2}{12} + b\frac{\Delta x^3}{24}\Big] u_{xxxx} - c\,u + \mathrm{O}(\Delta x^4), \quad (4.5) \end{aligned}$$

having a Fourier-type solution (4.3) with

$$A = \Big[a + b\frac{\Delta x}{2}\Big]\Big[1 - \frac{\Delta x^2}{12}k^2\Big]k^2, \qquad B = b\Big[1 - \frac{\Delta x^2}{6}k^2\Big], \quad (4.6)$$

i.e. the upwind discretization yields some additional artifical dissipation of magnitude $b\Delta x/2$.

4.1.3 The NSFD Scheme

Next, we consider the time-continuous version of the NSFD scheme (3.35) with the two denominator functions Φ_2, Φ_3, given in (3.34)

$$\begin{aligned} \partial_\tau u_j(\tau) + b\frac{u_j(\tau) - u_{j-1}(\tau)}{\Phi_3(\Delta x, b, c)} \\ = a\frac{u_{j+1}(\tau) - 2u_j(\tau) + u_{j-1}(\tau)}{\Phi_2(\Delta x, a, c)} - c\,u_j(\tau), \quad (4.7) \end{aligned}$$

with the expansion

$$\Phi_2(\Delta x, a, c) = \Delta x^2 + \frac{c}{a}\frac{\Delta x^4}{12} + \mathrm{O}(\Delta x^6).$$

Now, a tedious, but elementary calculation yields

$$\begin{aligned} u_\tau + b\Big[1 - \frac{c}{b}\frac{\Delta x}{2} + \Big(\frac{c}{b}\Big)^2\frac{\Delta x^2}{12} + \Big(\frac{c}{b}\Big)^2\frac{\Delta x^3}{24}\Big] u_x \\ + b\frac{\Delta x^2}{6}\Big[1 - \frac{c}{b}\frac{\Delta x}{2}\Big] u_{xxx} \\ = \Big[a\Big(1 - \frac{c}{a}\frac{\Delta x^2}{12}\Big) + b\frac{\Delta x}{2}\Big(1 - \frac{c}{b}\frac{\Delta x}{2} + \Big(\frac{c}{b}\Big)^2\frac{\Delta x^2}{12}\Big)\Big] u_{xx} \\ + \Big[a\frac{\Delta x^2}{12} + b\frac{\Delta x^3}{24}\Big] u_{xxxx} - c\,u + \mathrm{O}(\Delta x^4), \end{aligned} \quad (4.8)$$

cf. (4.5), having a Fourier-type solution (4.3) with

$$
\begin{aligned}
A &= \Big[a + b\frac{\Delta x}{2}\Big]\Big[1 - \frac{\Delta x^2}{12}k^2\Big]k^2 - c\frac{\Delta x^2}{3}\Big[1 - \frac{c}{b}\frac{\Delta x}{8}\Big]k^2, \\
B &= b\Big[1 - \frac{c}{b}\frac{\Delta x}{2} + \Big(\frac{c}{b}\Big)^2\frac{\Delta x^2}{12} + \Big(\frac{c}{b}\Big)^2\frac{\Delta x^3}{24}\Big] \\
&\quad - b\frac{\Delta x^2}{6}\Big[1 - \frac{c}{b}\frac{\Delta x}{2}\Big].
\end{aligned} \tag{4.9}
$$

Compared to (4.6) we observe that both the dissipation and convection coefficients in the Fourier solution are reduced by correction terms that depend on the reaction rate c, especially the ratio of reaction to convection c/b.

The effect of the time discretization is analyzed analogously.

Remark (Financial Interpretation). If the PDE (3.17) is considered in a financial context, e.g. for the pricing of Asian options, than the above analysis shows that the centered difference discretization leads to a lower volatility and a lower drift. Both perturbations are proportional to the square of the spatial frequency (wave number) k, i.e. non-smooth pay-offs (2.2) will stimulates these distortions, since high spatial frequencies are induced.

4.2 Analysis of the Fully Discrete Scheme

4.2.1 The Upwind Scheme

First we start with explaining the method using the upwind scheme, cf. (3.20). We use Taylor series around the point (x_j, t_n) and insert it into the difference scheme

$$
\begin{aligned}
&\Big(\frac{\partial u}{\partial \tau}\Big)_j^n + \frac{\Delta\tau}{2}\Big(\frac{\partial^2 u}{\partial \tau^2}\Big)_j^n + \frac{\Delta\tau^2}{6}\Big(\frac{\partial^3 u}{\partial \tau^3}\Big)_j^n + \mathrm{O}(\Delta\tau^3) \\
&\qquad + b\Big(\frac{\partial u}{\partial x}\Big)_j^n - b\frac{\Delta x}{2}\Big(\frac{\partial^2 u}{\partial x^2}\Big)_j^n + b\frac{\Delta x^2}{6}\Big(\frac{\partial^3 u}{\partial x^3}\Big)_j^n \\
&\quad = a\Big(\frac{\partial^2 u}{\partial x^2}\Big)_j^n + a\frac{\Delta x^2}{12}\Big(\frac{\partial^4 u}{\partial x^4}\Big)_j^n - c(u)_j^n + \mathrm{O}(\Delta x^3),
\end{aligned} \tag{4.10}
$$

This modified equation consists of the original PDE plus the terms stemming from the truncation error of the scheme.

The next (tedious) step is to replace all time derivatives with order higher than one in (4.10) by spatial derivatives. To do so, we differentiate (4.10) with respect to τ

$$\begin{aligned}
&\left(\frac{\partial^2 u}{\partial \tau^2}\right)_j^n + \frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial \tau^3}\right)_j^n + \mathrm{O}(\Delta\tau^2) \\
&\quad + b\left(\frac{\partial^2 u}{\partial x \partial \tau}\right)_j^n - b\frac{\Delta x}{2}\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n + b\frac{\Delta x^2}{6}\left(\frac{\partial^4 u}{\partial x^3 \partial \tau}\right)_j^n \\
&\quad = a\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n + a\frac{\Delta x^2}{12}\left(\frac{\partial^5 u}{\partial x^4 \partial \tau}\right)_j^n - c\left(\frac{\partial u}{\partial \tau}\right)_j^n + \mathrm{O}(\Delta x^3),
\end{aligned} \tag{4.11}$$

and differentiate (4.10) with respect to x and multiply by b

$$\begin{aligned}
&b\left(\frac{\partial u^2}{\partial x \partial \tau}\right)_j^n + b\frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial x \partial \tau^2}\right)_j^n + \mathrm{O}(\Delta\tau^2) \\
&\qquad + b^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n - b^2\frac{\Delta x}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + b^2\frac{\Delta x^2}{6}\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n \\
&\quad = ab\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + ab\frac{\Delta x^2}{12}\left(\frac{\partial^5 u}{\partial x^5}\right)_j^n - bc\left(\frac{\partial u}{\partial x}\right)_j^n + \mathrm{O}(\Delta x^3).
\end{aligned} \tag{4.12}$$

Now subtracting (4.12) from (4.11) and replacing the first time derivative by (4.10) gives

$$\begin{aligned}
&\left[1 - c\frac{\Delta\tau}{2}\right]\left(\frac{\partial^2 u}{\partial \tau^2}\right)_j^n = 2bc\left(\frac{\partial u}{\partial x}\right)_j^n + (b^2 - ac)\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n \\
&\quad - ab\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + c^2 (u)_j^n + a\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n \\
&\quad + \frac{\Delta\tau}{2}\left[b\left(\frac{\partial^3 u}{\partial x \partial \tau^2}\right)_j^n - \left(\frac{\partial^3 u}{\partial \tau^3}\right)_j^n\right] + \mathrm{O}(\Delta\tau^2) \\
&\quad + b\frac{\Delta x}{2}\left[\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n - b\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n - c\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n\right] + \mathrm{O}(\Delta x^2).
\end{aligned} \tag{4.13}$$

The next step is to substitute any term in (4.13) involving a time derivative, i.e.

$$\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n, \qquad \left(\frac{\partial^3 u}{\partial x \partial \tau^2}\right)_j^n, \qquad \left(\frac{\partial^3 u}{\partial \tau^3}\right)_j^n,$$

E.g. the first term can be expressed by differentiating (4.12) with respect to x:

$$\left(\frac{\partial^3 u}{\partial x^2 \partial \tau}\right)_j^n = a\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n - b\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n - c\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n - \frac{\Delta\tau}{2}\left(\frac{\partial^4 u}{\partial x^2 \partial \tau^2}\right)_j^n + b\frac{\Delta x}{2}\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n + \mathrm{O}(\Delta\tau^2) + \mathrm{O}(\Delta x^2) \tag{4.14}$$

and inserting into (4.13) yields up to first order

$$\left(\frac{\partial^2 u}{\partial \tau^2}\right)_j^n = a^2\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n + 2bc\left(\frac{\partial u}{\partial x}\right)_j^n + (b^2 - 2ac)\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n - 2ab\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + c^2\left(u\right)_j^n + \mathrm{O}(\Delta\tau) + \mathrm{O}(\Delta x) \tag{4.15}$$

and finally substituted in (4.10) gives the *modified equation*

$$\left(\frac{\partial u}{\partial \tau}\right)_j^n = \left[a + (2ac - b^2)\frac{\Delta\tau}{2} + b\frac{\Delta x}{2}\right]\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n - b\left[1 + c\Delta\tau\right]\left(\frac{\partial u}{\partial x}\right)_j^n - c\left[1 + c\frac{\Delta\tau}{2}\right]\left(u\right)_j^n + \mathrm{O}(\Delta x^2) + ab\Delta\tau\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n - a^2\frac{\Delta\tau}{2}\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n + \mathrm{O}(\Delta\tau^2), \tag{4.16}$$

that allows us to study how numerical diffusion and numerical dispersion is introduced by the upwind scheme. To investigate higher order effects, higher order time derivatives in (4.13) must be eliminated.

4.2.2 The NSFD Scheme

Secondly, if we consider the NSFD scheme (3.35) we obtain instead of (4.10) the equation

$$\begin{aligned}
&\left[1+c\frac{\Delta\tau}{2}+c^2\frac{\Delta\tau^2}{12}\right]\left(\frac{\partial u}{\partial\tau}\right)_j^n \\
&+\frac{\Delta\tau}{2}\left[1+c\frac{\Delta\tau}{2}\right]\left(\frac{\partial^2 u}{\partial\tau^2}\right)_j^n+\frac{\Delta\tau^2}{6}\left(\frac{\partial^3 u}{\partial\tau^3}\right)_j^n+\mathrm{O}(\Delta\tau^3) \\
&+b\Big[1-\frac{c}{b}\frac{\Delta x}{2}+\Big(\frac{c}{b}\Big)^2\frac{\Delta x^2}{12}+\Big(\frac{c}{b}\Big)^2\frac{\Delta x^3}{24}\Big]\left(\frac{\partial u}{\partial x}\right)_j^n \\
&\quad+b\frac{\Delta x^2}{6}\Big[1-\frac{c}{b}\frac{\Delta x}{2}\Big]\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n \\
&=\Big[a\Big(1-\frac{c}{a}\frac{\Delta x^2}{12}\Big)+b\frac{\Delta x}{2}\Big(1-\frac{c}{b}\frac{\Delta x}{2}+\Big(\frac{c}{b}\Big)^2\frac{\Delta x^2}{12}\Big)\Big]\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n \\
&\quad+\Big[a\frac{\Delta x^2}{12}+b\frac{\Delta x^3}{24}\Big]\left(\frac{\partial^4 u}{\partial x^4}\right)_j^n-c\big(u\big)_j^n+\mathrm{O}(\Delta x^4),
\end{aligned} \tag{4.17}$$

where the time derivatives with order higher than one can be eliminated following exactly the same strategy as before, i.e. we replace all time derivatives with order higher than one in (4.17) by spatial derivatives. First, we differentiate (4.17) with respect to the time τ

$$\begin{aligned}
&\left[1+c\frac{\Delta\tau}{2}\right]\left(\frac{\partial^2 u}{\partial\tau^2}\right)_j^n+\frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial\tau^3}\right)_j^n+\mathrm{O}(\Delta\tau^2) \\
&\qquad+b\Big[1-\frac{c}{b}\frac{\Delta x}{2}\Big]\left(\frac{\partial^2 u}{\partial x\partial\tau}\right)_j^n \\
&=\Big[a+b\frac{\Delta x}{2}\Big]\left(\frac{\partial^3 u}{\partial x^2\partial\tau}\right)_j^n-c\left(\frac{\partial u}{\partial\tau}\right)_j^n+\mathrm{O}(\Delta x^2).
\end{aligned} \tag{4.18}$$

To eliminate the mixed derivative we and differentiate (4.17) with respect to x and multiply by the *perturbed convection speed*

$$\tilde{b}=b\,\frac{1-\frac{c}{b}\frac{\Delta x}{2}}{1+c\frac{\Delta\tau}{2}},$$

yielding

$$\begin{aligned}
&b\Big[1-\frac{c}{b}\frac{\Delta x}{2}\Big]\left(\frac{\partial^2 u}{\partial x\partial\tau}\right)_j^n+\tilde{b}\frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial x\partial\tau^2}\right)_j^n+\mathrm{O}(\Delta\tau^2)\\
&\qquad+\tilde{b}b\Big[1-\frac{c}{b}\frac{\Delta x}{2}\Big]\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n\\
&\quad=\tilde{b}\Big[a+b\frac{\Delta x}{2}\Big]\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n-\tilde{b}c\left(\frac{\partial u}{\partial x}\right)_j^n+\mathrm{O}(\Delta x^2).
\end{aligned}\tag{4.19}$$

Next, subtracting (4.19) from (4.18) gives

$$\begin{aligned}
\Big[1+c\frac{\Delta\tau}{2}\Big]\left(\frac{\partial^2 u}{\partial\tau^2}\right)_j^n&=-\frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial\tau^3}\right)_j^n+\tilde{b}\frac{\Delta\tau}{2}\left(\frac{\partial^3 u}{\partial x\partial\tau^2}\right)_j^n+\mathrm{O}(\Delta\tau^2)\\
&+\tilde{b}b\Big[1-\frac{c}{b}\frac{\Delta x}{2}\Big]\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n+\Big[a+b\frac{\Delta x}{2}\Big]\left(\frac{\partial^3 u}{\partial x^2\partial\tau}\right)_j^n\\
&-c\left(\frac{\partial u}{\partial\tau}\right)_j^n-\tilde{b}\Big[a+b\frac{\Delta x}{2}\Big]\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n+\tilde{b}c\left(\frac{\partial u}{\partial x}\right)_j^n+\mathrm{O}(\Delta x^2),
\end{aligned}$$

multiplying by $1+c\Delta\tau/2$ and replacing the first time derivative by (4.17) leads to

$$\begin{aligned}
\Big[1+c\frac{\Delta\tau}{2}\Big]^2\left(\frac{\partial^2 u}{\partial\tau^2}\right)_j^n&=\Big[\Big(b-c\frac{\Delta x}{2}\Big)^2-c\Big(a+b\frac{\Delta x}{2}\Big)\Big]\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n\\
&-\frac{\Delta\tau}{2}\Big[1+c\frac{\Delta\tau}{2}\Big]\left(\frac{\partial^3 u}{\partial\tau^3}\right)_j^n+\frac{\Delta\tau}{2}\Big(b-c\frac{\Delta x}{2}\Big)\left(\frac{\partial^3 u}{\partial x\partial\tau^2}\right)_j^n\\
&+\Big[a+b\frac{\Delta x}{2}\Big]\Big[1+c\frac{\Delta\tau}{2}\Big]\left(\frac{\partial^3 u}{\partial x^2\partial\tau}\right)_j^n+\mathrm{O}(\Delta\tau^2)\\
&-\Big[a+b\frac{\Delta x}{2}\Big]\Big(b-c\frac{\Delta x}{2}\Big)\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n+c^2(u)_j^n\\
&+2c\Big[b-c\frac{\Delta x}{2}\Big]\left(\frac{\partial u}{\partial x}\right)_j^n+\mathrm{O}(\Delta x^2).
\end{aligned}\tag{4.20}$$

Now we replace the mixed term (4.20)

$$\Big[1+c\frac{\Delta\tau}{2}\Big]\left(\frac{\partial^3 u}{\partial x^2\partial\tau}\right)_j^n$$

by differentiating (4.17) two times with respect to x. This procedure yields up to first order

$$\begin{aligned}\Big[1+c\frac{\Delta\tau}{2}\Big]^2\Big(\frac{\partial^2 u}{\partial\tau^2}\Big)_j^n &= \Big[\Big(b-c\frac{\Delta x}{2}\Big)^2 - c\Big(a+b\frac{\Delta x}{2}\Big)\Big]\Big(\frac{\partial^2 u}{\partial x^2}\Big)_j^n \\ &- \frac{\Delta\tau}{2}\Big[1+c\frac{\Delta\tau}{2}\Big]\Big(\frac{\partial^3 u}{\partial\tau^3}\Big)_j^n + \frac{\Delta\tau}{2}\Big(b-c\frac{\Delta x}{2}\Big)\Big(\frac{\partial^3 u}{\partial x\partial\tau^2}\Big)_j^n \\ &+ \Big[a+b\frac{\Delta x}{2}\Big]^2\Big(\frac{\partial^4 u}{\partial x^4}\Big)_j^n + \frac{\Delta\tau}{2}\Big[a+b\frac{\Delta x}{2}\Big]\Big(\frac{\partial^4 u}{\partial x^2\partial\tau^2}\Big)_j^n \\ &- 2\Big[a+b\frac{\Delta x}{2}\Big]\Big(b-c\frac{\Delta x}{2}\Big)\Big(\frac{\partial^3 u}{\partial x^3}\Big)_j^n + c^2\big(u\big)_j^n + \mathrm{O}(\Delta\tau^2) \\ &+ 2c\Big[b-c\frac{\Delta x}{2}\Big]\Big(\frac{\partial u}{\partial x}\Big)_j^n + \mathrm{O}(\Delta x^2).\end{aligned} \tag{4.21}$$

Finally we substitute (4.21) in (4.17), multiplied by $1+c\Delta\tau/2$, and obtain the *modified equation for the NSFD scheme*

$$\begin{aligned}\Big(\frac{\partial u}{\partial\tau}\Big)_j^n &= \Big[a+\frac{b}{2}\frac{\Delta x-\Delta\tau}{1+c\Delta t}\Big]\Big(\frac{\partial^2 u}{\partial x^2}\Big)_j^n - b\Big[1-\frac{c}{2b}\frac{\Delta x}{1+c\Delta t}\Big]\Big(\frac{\partial u}{\partial x}\Big)_j^n \\ &- c\big(u\big)_j^n + ab\frac{\Delta\tau}{1+c\Delta t}\Big(\frac{\partial^3 u}{\partial x^3}\Big)_j^n - a^2\frac{\Delta\tau}{2(1+c\Delta t)}\Big(\frac{\partial^4 u}{\partial x^4}\Big)_j^n \\ &+ \mathrm{O}(\Delta\tau^2) + \mathrm{O}(\Delta x^2),\end{aligned} \tag{4.22}$$

with $\Delta\tau$ implicitly given by the nonlinear equation (3.42).

Next, one can compare the two modified equations of the upwind scheme (4.16) and of the NSFD scheme (4.22) to study the different qualitative behavior by observing the numerical diffusion and dispersion errors. E.g. it can be realized that the NSFD method reproduces better the effect of the reaction rate c.

Remark. Alternatively, following the work of Junk and Yang [8], one can use asymptotic methods, such as (discrete) multiscale expansions to analyse the properties of the finite difference methods, e.g. an upwind discretization is investigated in [8, Section 2.4].

5 Numerical Results

In this section we compare the results of our new proposed NSFD scheme with the two classical explicit finite difference methods presented earlier. For a numerical test we consider the case $c = 0$, i.e.

$$u_\tau + b\, u_x = a\, u_{xx}, \tag{5.1}$$

with $a > 0$ and positive convection $b \geq 0$, which was considered in §3.3. For the comparison we use two classical explicit schemes: the *centered-in-space discretization*

$$u_j^{n+1} = u_j^n - \frac{b\lambda}{2}(u_{j+1}^n - u_{j-1}^n) + a\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{5.2}$$

i.e.

$$u_j^{n+1} = a\mu[1 + \mathrm{Pe}]u_{j-1}^n + [1 - 2a\mu]u_j^n + a\mu[1 - \mathrm{Pe}]\mu u_{j+1}^n, \tag{5.3}$$

with the cell Péclet number

$$\mathrm{Pe} := \frac{b\lambda}{2a\mu} = \frac{b\Delta x}{2a}. \tag{5.4}$$

Secondly, we consider the *upwind differencing* of the convection term

$$u_j^{n+1} = u_j^n - b\lambda(u_j^n - u_{j-1}^n) + a\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{5.5}$$

i.e.

$$u_j^{n+1} = [a\mu + b\lambda]u_{j-1}^n + [1 - 2a\mu - b\lambda]u_j^n + a\mu u_{j+1}^n, \tag{5.6}$$

Remark. We note that the upwind scheme (5.5) can be written as a centered-in-space scheme with added *artificial viscosity* $a\,\mathrm{Pe}\,u_{xx}$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta\tau} + b\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = a(1 + \mathrm{Pe})\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}. \tag{5.7}$$

A von Neumann analysis [14] yields the *stability criterion* for (5.2)

$$a\mu \leq \frac{1}{2}, \qquad \frac{b^2}{a}\Delta t \leq 2 \tag{5.8}$$

and for the upwind scheme (5.5)

$$a\mu \leq \left(2 + \left|\frac{b\Delta x}{a}\right|\right)^{-1}, \tag{5.9}$$

that is valid for both signs of b.

Furthermore, one can easily observe that we need in (5.3) $\text{Pe} \leq 1$ and in (5.6) $2a\mu + b\lambda \leq 1$ for the *positivity* of the difference scheme. These conditions must be fulfilled such that a discrete maximum principle holds preventing oscillations in the numerical solution.

Example 5.1 ([15, p. 159]). *We choose for the coefficients $a = 0.1$, convection speed $b = 10$, and the spatial step sizes $\Delta x = 1/20$ and $\Delta x = 1/30$ Δt is given by fixing $\mu = 1$, i.e. the cell Péclet number* (5.4) *has the value* $\text{Pe} := 50\Delta x$. *As initial data we use the tent function between -0.5 and 0.5, i.e.*

$$u(x,0) = \begin{cases} 2x+1, & -0.5 < x < 0 \\ -2x+1, & 0 \leq x < 0.5, \end{cases}$$

which serves as a model for a non-smooth payoff-function like (2.2) *that stimulates high spatial frequency parts in the solution. For Péclet numbers greater than one it is expected that none of the two standard schemes yield reasonable results, cf. [15].*

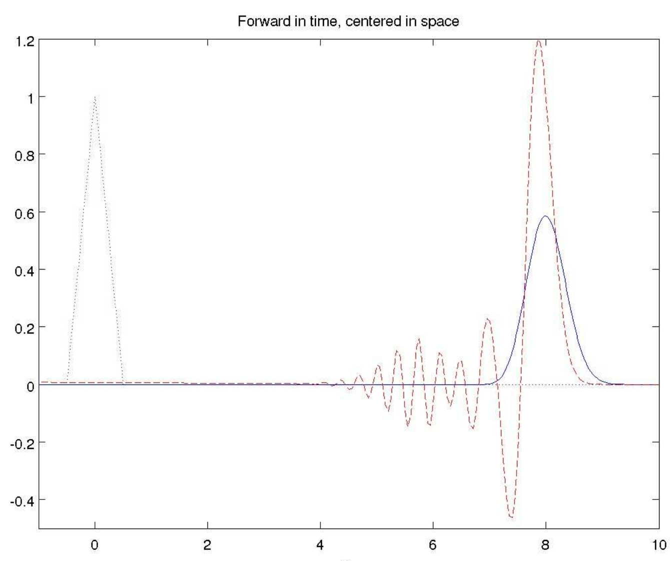

Figure 1: Forward in time centered in space.

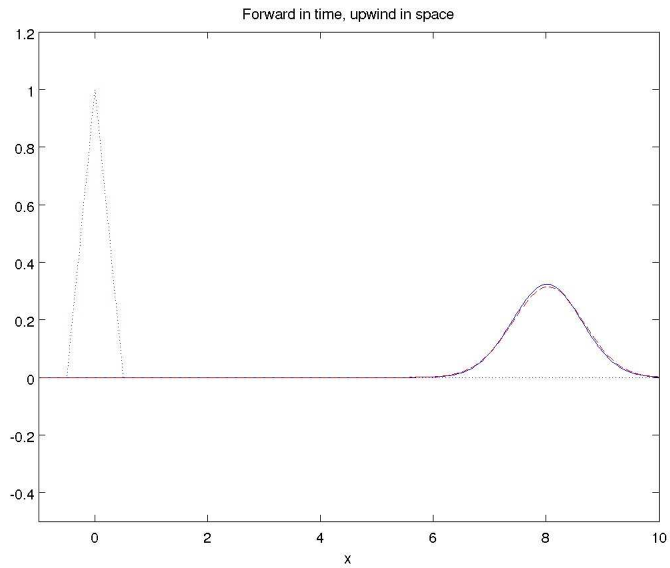

Figure 2: Forward in time upwind in space.

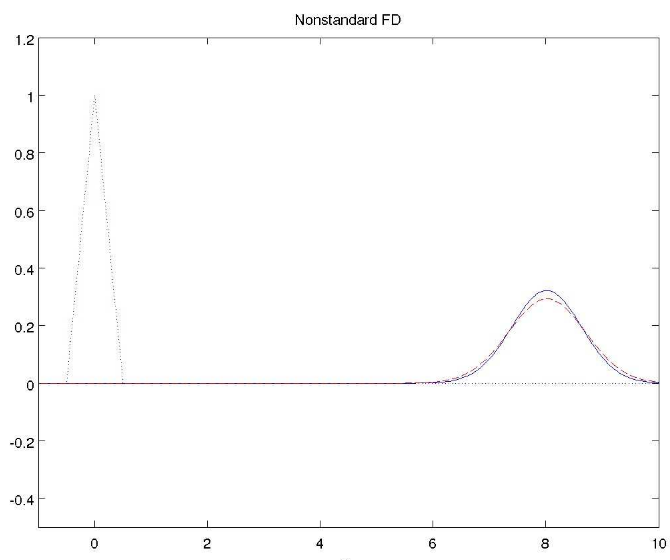

Figure 3: Nonstandard FD scheme.

In the three figures we plot for each of the three schemes the initial function and the solution at the final time $t = 0.8$ for the spatial step sizes $\Delta x = 1/30$ (blue solid line) and $\Delta x = 1/20$ (red dashed line). The FTCS scheme yields for $\Delta x = 1/20$ an oscillatory solution and for the finer grid the solution is smooth but the values are significantly larger than the exact solution. The upwind and the NSFD scheme show similar results: they eliminate the oscillations, but the solution at $t = 0.8$ is significantly smaller than the exact solution (with a maximum value ≈ 0.44). However, we recall that the NSFD scheme additionally preserves the positivity of the solution.

Conclusion and Outlook

In this paper we have presented a novel nonstandard finite difference (NSFD) method for the solution of a general convection-diffusion PDE that can be used for the linear Black-Scholes equation for pricing Asian options.

Future work will focus on the construction of NSFD schemes for spatially twodimensional PDEs and for nonlinear Black–Scholes equations [1, 6] especially for equations with a nonlinear convective term, cf. [18]. Also, we will derive a NSFD scheme for the original Black-Scholes equation based on a finite volume formulation, since often a standard transformation is not possible.

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References

[1] J. Ankudinova and M. Ehrhardt, *On the numerical solution of nonlinear Black-Scholes equations*, Comput. Math. Appl. **56** (2008), 799–812.

[2] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Econ. **81** (1973), 637–659.

[3] C. Bolley and M. Crouzeix, *Conservation de la positivité lors de la discrétisation des problèmes d'évolution paraboliques*, R.A.I.R.O. Analyse numérique, **12** (1978), 81–88.

[4] Dang Quang A and M. Ehrhardt, *Adequate Numerical Solution of Air Pollution problems by Positive Difference schemes on unbounded domains*, Math. Comput. Modelling, **44** (2006), 834–856.

[5] M. Ehrhardt and R.E. Mickens, *A fast, stable and accurate numerical method for the Black-Scholes equation of American options*, Int. J. Theoret. Appl. Finance **11** (2008), 471–501.

[6] M. Ehrhardt (ed.), *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, Nova Science Publishers, Inc., Hauppauge, NY 11788, 2008.

[7] W. Hundsdorfer, B. Koren, M. van Loon and J.G. Verwer, *A Positive Finite-Difference Advection Scheme*, J. Comput. Phys. **117** (1994), 35–46.

[8] M. Junk and Z. Yang, *Asymptotic analysis of finite difference methods*, Appl. Math. Comput. **158** (2004), 267–301.

[9] M.B. Kalinowska and P.M Rowiński, *Truncation errors of selected finite difference methods for two-dimensional advection-diffusion equation with mixed derivatives*, Acta Geophys. **55** (2007), 104-118.

[10] R.C. Merton, *Theory of rational option pricing*, Bell J. Econ. Manag. Sci. **4** (1973), 141–183.

[11] R.E. Mickens, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.

[12] R.E. Mickens, *Nonstandard Finite Difference Schemes for Differential Equations*, J. Diff. Eqs. Appl. **8** (2002), 823–847.

[13] G.I. Ramírez-Espinoza and M. Ehrhardt, *Conservative and Finite Volume Methods for the convection-dominated Pricing Problem*, Preprint 12/07, University of Wuppertal, February 2012.

[14] R. Seydel, *Tools for Computational Finance*, Fourth ed., Springer, 2009.

[15] J.C. Strikwerda, *Finite difference schemes and partial differential equations*, Mathematics Series, Wadsworth & Brooks/Cole, 2nd edition, 2004.

[16] R.F. Warming and B.J. Hyett, *The Modified Equation Approach to the Stability and Accuracy Analysis of Finite-Difference Methods*, J. Comput. Phys. **14** (1974), 159–179.

[17] P. Wilmott, S. Howison and J. Dewynne, *The Mathematics of Financial Derivatives, A Student Introduction*, Cambridge University Press, 2002.

[18] H. Windcliff, J. Wang, P.A. Forsyth and K.R. Vetzal, *Hedging with a correlated asset: Solution of a nonlinear pricing PDE*, J. Comput. Appl. Math, 200 (2007), 86–115.

[19] R. Zvan, P.A. Forsyth and K. Vetzal, *Robust Numerical Methods for PDE Models of Asian Options*, J. Comput. Finance **1** (1998) 39–78.